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CONTRIBUTIONS TO QUATERNIONIC OPERATOR THEORY AND APPLICATIONS

Doctoral Dissertation of:
Jonathan Gantner

Supervisor:
Prof. Fabrizio Colombo

Tutor:
Prof. Roberto Lucchetti

The Chair of the Doctoral Program:
Prof. Irene Sabadini

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To my father,
who was the first to show me the beauty of mathematics

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Contents

CHAPTER 1

Introduction

Birkhoff and von Neumann showed 1936 in their celebrated paper [15] with the title "The Logic of Quantum Mechanics" that the logical structure of a quantum system, its propositional calculus, can be modelled by the lattice of orthogonal projections on a real, complex or quaternionic Hilbert space. Quantum mechanics on a real Hilbert space was soon discarded because Stueckelberg argued in [79, 80] that this approach is equivalent to the classical approach on a complex Hilbert space. Several researchers showed however interest in a quaternionic formulation of quantum mechanics, which seemed different from the standard formulation. Their efforts were summarised by Adler in the monograph [1]. However, developing the mathematical foundations of such formulation of quantum mechanics, a sound spectral theory for quaternionic linear operators, turned out to be much more difficult than expected. It caused mathematicians severe difficulties and it was only about ten years ago, seventy years after the first suggestion of quaternionic quantum mechanics, that the fundamental concepts of such theory were understood [36].

The difficulties arise from the algebraic structure of the quaternions $\mathbb{H}$. This number system extends the complex numbers, but since its multiplication is not commutative, it does not form a field, but a skew-field. Even the notion of eigenvalues of an operator is therefore ambiguous. If T is a quaternionic right linear operator, that is $T(\mathbf{v}a+\mathbf{u}) = T(\mathbf{v})a + T(\mathbf{u})$ for all vectors $\mathbf{v}, \mathbf{u}$ and all scalars $a \in \mathbb{H}$, then we can consider either left or right eigenvalues, which satisfy

$$T\mathbf{v} = s\mathbf{v} \qquad \text{resp.} \qquad T\mathbf{v} = \mathbf{v}s$$

for some vector $\mathbf{v} \neq \mathbf{0}$.

Since the operator T is via the linearity condition related with the right multiplication on the vector space, the notion of right eigenvalues seems to be the natural one.

The set of right-eigenvalues $\sigma_R(T)$ of T is however accompanied by several strange phenomena. If $\mathbf{v}$ is a right eigenvector of T associated with s, that is $T\mathbf{v} = \mathbf{v}s$, and $a \in \mathbb{H} \setminus \{0\}$ is an arbitrary quaternionic scalar, then

$$T(\mathbf{v}a) = (T\mathbf{v})a = \mathbf{v}sa = (\mathbf{v}a)a^{-1}sa. \tag{1.1}$$

If s and a do not commute, then $a^{-1}sa \neq s$ and so $\mathbf{v}a$ is not an eigenvector associated with s, but an eigenvector associated with the eigenvalue $a^{-1}sa$. The set of eigenvectors associated with a single eigenvalue does therefore in general not form a quaternionic right linear eigenspace of the operator T. The set of right eigenvalues $\sigma_R(T)$ on the other hand satisfies a certain symmetry. Whenever s is an eigenvalue of T, any quaternion in the equivalence class

$$[s] = \left\{a^{-1}sa : a \in \mathbb{H} \setminus \{0\}\right\} \tag{1.2}$$

is also an eigenvalue of T. Since the operator $\mathbf{v} \mapsto T\mathbf{v} - \mathbf{v}a$ of the right-eigenvalue equation is because of (1.1) not right linear it is moreover not suitable for generalising the concept of right eigenvalues to a meaningful notion of spectrum for quaternionic right linear operators. If A is a complex linear operator on a complex Banach space $V_{\mathbb{C}}$, then the resolvent set $\rho(A)$ of A is the set of all complex numbers λ such that the operator $\lambda\mathcal{I} - A$ has a bounded inverse, that is

$$\rho(A) = \left\{ \lambda \in \mathbb{C} : (\lambda\mathcal{I} - A)^{-1} \in \mathcal{B}(V_{\mathbb{C}})\right\},$$

where $\mathcal{B}(V_{\mathbb{C}})$ denotes the set of bounded linear operator on $V_{\mathbb{C}}$, and its spectrum is the set

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

For any $\lambda \in \rho(A)$, the operator $R_\lambda(A) := (\lambda\mathcal{I} - A)^{-1}$ is called the resolvent of A at λ and it plays a fundamental role in complex operator theory. Obviously it is the inverse of the operator of the classical eigenvalue equation. If $V_{\mathbb{C}}$ has finite dimension, then the invertibility of an operator is equivalent to its injectivity, and so the resolvent of A exists in this case at every λ with $\ker(\lambda\mathcal{I} - A) = \{\mathbf{0}\}$, that is at every λ that is not an eigenvalue of A. The spectrum of an operator on a finite dimensional space consequently coincides with the set of its eigenvalues. Generalising the set of right eigenvalues of a quaternionic linear operator in a similar way using the operator of the right eigenvalue equation is not possible because this operator is not quaternionic right linear.

Despite the technical difficulties that come along with the notion of right eigenvalues, this type of eigenvalues proved to be useful both in applications and the mathematical theory. They have an interpretation within quaternionic quantum mechanics [1, 41] and they allow to prove the spectral theorem for matrices with quaternionic entries [40].

The set of left eigenvalues $\sigma_L(T)$ of a quaternionic right linear operator T does not cause these technical difficulties since the operator of the left eigenvalue equation is quaternionic right linear. Left eigenvalues did however not seem to have any meaningful application neither in the mathematical theory nor in applications.

This paradoxical situation left mathematicians helpless and the confusion was even increased by a second problem, namely that it was not clear on which function theory

one could build a proper theory of quaternionic linear operators. Complex linear operator theory is based on the theory of holomorphic functions. The fact that the resolvent $\lambda \mapsto R_\lambda(A)$ of an operator A is holomorphic is for instance fundamental in this theory. The most successful notion of generalized holomorphicity for functions defined on the quaternions was Fueter regularity, introduced by Fueter in the 1930s in [42, 43]. It allowed to reproduce most of the important results of the theory of holomorphic functions (an overview on the theory can be for instance found in [56]) and hence mathematicians tried to build a theory of operators by introducing functional calculi for functions of this kind [60, 64]. (The different approaches are also explained and compared in the survey paper [23].) These results were however unsatisfactory: the set of Fueter regular functions does not contain power series expansions in the pure quaternionic variable. Such functions can therefore only be expanded into power series containing the real coordinates of a quaternion. Hence, the operator $f(T)$ is defined as a function of the $\mathbb{R}$-linear components of the operator T and not as a function of the operator itself. Moreover, the integral kernels of these functional calculi were expressed in terms of power series and closed forms for these series, which could serve for defining a notion of quaternionic resolvent and in turn a notion of quaternionic spectrum, were only found under additional assumptions such as commutativity of the components of the operator. Despite its success in function theory, Fueter regularity did therefore not seem to be the notion of generalized holomorphicity on which a proper theory of quaternionic linear operators could be based.

Due to the problems described above, little progress was made until the introduction of the S-spectrum and the S-functional calculus for slice hyperholomorphic functions about ten years ago, almost seventy years after quaternionic quantum mechanics was suggested for the first time [36]. Any quaternion $x \in \mathbb{H}$ is of the form $x = x_0 + \underline{x} = \xi_0 + \sum_{\ell=1}^{3} \xi_\ell e_\ell$ with $\xi_\ell \in \mathbb{R}$, where $e_\ell^2 = -1$ and $e_\ell e_\kappa = -e_\kappa e_\ell$ for $\ell, \kappa \in \{1,2,3\}$ with $\ell \neq \kappa$. If we set $x_1 = |\underline{x}| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$ and define $\mathsf{i}_x := \underline{x}/|\underline{x}|$, then

$$x = x_0 + \mathsf{i}_x x_1.$$

The quaternion i_x satisfies $\mathsf{i}_x^2 = -1$ and is therefore called an imaginary unit. Furthermore, a set U is called axially symmetric if $x_0 + \mathsf{i} x_1 \in U$ for any imaginary unit i whenever $x \in U$.

A function $f : U \subset \mathbb{H} \to \mathbb{H}$ that is defined on an axially symmetric open set U is now called (left) slice hyperholomorphic if it is of the form

$$f(x) = \alpha(x_0, x_1) + \mathsf{i}_x \beta(x_0, x_1) \tag{1.3}$$

with quaternion-valued functions α and β such that α is even and β is odd in the second variable, that is $\alpha(x_0, -x_1) = \alpha(x_0, x_1)$ and $\beta(x_0, -x_1) = -\beta(x_0, x_1)$, and such that α and β satisfy the Cauchy-Riemann-equations. (Note that any quaternion can be represented using the two imaginary units i_x and $-\mathsf{i}_x$ as

$$x = x_0 + \mathsf{i}_x x_1 = x_0 + (-\mathsf{i}_x)(-x_1).$$

For $x \in \mathbb{R}$, we have $x_1 = 0$ so that we can even choose i_x to be any arbitrary imaginary unit. The even-odd-condition for α and β guarantees that the choice of i_x is irrelevant

in (1.3) since

$$\begin{aligned} f(x_0 + (-\mathsf{i}_x)(-x_1)) &= \alpha(x_0, -x_1) + (-\mathsf{i}_x)\beta(x_0, -x_1) \\ &= \alpha(x_0, x_1) + \mathsf{i}_x\beta(x_0, x_1) = f(x_0 + \mathsf{i}_x x_1). \end{aligned}$$

In particular $\alpha(x_0, -x_1)$ and $\beta(x_0, -x_1)$ are defined whenever $\alpha(x_0, x_1)$ and $\beta(x_0, x_1)$ are defined.) Similarly, a function f is called right slice hyperholomorphic, if it is of the form

$$f(x) = \alpha(x_0, x_1) + \beta(x_0, x_1)\mathsf{i}_x$$

with α and β as before. Since the equivalence classes of right eigenvalues in (1.2) are of the form

$$[s] = \{\, s_0 + \mathsf{i}s_1 : \mathsf{i} \in \mathbb{S}\},$$

where $\mathbb{S} = \{\, \mathsf{i} \in \mathbb{H} : \mathsf{i}^2 = -1\}$ denotes the set of all imaginary units, these functions do obviously respect the symmetry of the set of right eigenvalues. Even more important, power series in a quaternionic variable of the form $\sum_{n=0}^{+\infty} x^n a_n$ with $a_n \in \mathbb{H}$ are left slice hyperholomorphic. Furthermore, slice hyperholomorphic functions satisfy a generalized Cauchy formula. The left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s, x)$ is given by

$$S_L^{-1}(s, x) = \sum_{n=0}^{+\infty} x^n s^{-(n+1)} = (x^2 - 2s_0 x + |s|^2)^{-1}(\overline{s} - x) \tag{1.4}$$

with $\overline{s} = s_0 - \underline{s} = s_0 - \mathsf{i}_s s_1$. The series expansion holds for $|x| < |s|$ and the closed form is defined for any $x \notin [s]$. Any left slice hyperholomorphic function f satisfies now

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_\mathsf{i})} S_L^{-1}(s, x)\, ds_\mathsf{i}\, f(s), \qquad x \in U, \tag{1.5}$$

where U is a suitable open set contained in the domain $\mathcal{D}(f)$ of f, where i is an arbitrary imaginary unit and $\mathbb{C}_\mathsf{i} = \{x_0 + \mathsf{i}x_1 : x_0, x_1 \in \mathbb{R}\}$ and where $ds_\mathsf{i} := ds(-\mathsf{i})$.

The series expansion and the closed form of $S_L^{-1}(s, x)$ both contain only the pure quaternionic variable x and not only its real coordinates as it is the case power series expansions of Fueter regular functions. One can therefore hope to formally replace the variable x by the operator T and indeed, if T is a bounded quaternionic right linear operator on a two-sided quaternionic Banach space V, then

$$\sum_{n=0}^{+\infty} T^n s^{-(n+1)} = (T^2 - 2s_0 T + |s|^2\mathcal{I})^{-1}(\overline{s}\mathcal{I} - T), \qquad \|T\| < |s| \tag{1.6}$$

where $\mathcal{I}$ denotes the identity operator on V. Observe that also here only the pure operator T appears and none of its $\mathbb{R}$-linear components. This identity can furthermore be shown without any additional assumptions on the operator such as that its components commute. The identity (1.6) and its correspondence to (1.4) then motivate the following definitions. For $s \in \mathbb{H}$, we define the operator

$$\mathcal{Q}_s(T) := T^2 - 2s_0 T + |s|^2\mathcal{I}$$

and we define the S-resolvent set of T as

$$\rho_S(T) := \{s \in \mathbb{H} : \mathcal{Q}_s(T)^{-1} \in \mathcal{B}(V)\},$$

where $\mathcal{B}(V)$ denotes the set of bounded right linear operators on V, and the S-spectrum of T as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

The S-spectrum of T has properties that are analogue to those of the classical spectrum of a complex linear operator. Furthermore, it generalizes the set $\sigma_R(T)$ of right eigenvalues of T in the same way as the spectrum of a complex linear operator generalizes its set of eigenvalues. Indeed, the operator $\mathcal{Q}_s(T)$ is actually the operator of a generalized eigenvalue equation. It is however not associated with the single eigenvalue s but with the entire equivalence class $[s]$ and its kernel is exactly the quaternionic eigenspace associated with $[s]$. Since the S-spectrum generalizes the set of right eigenvalues, it is suitable for applications in quaternionic quantum mechanics.

For $s \in \rho_S(T)$, we furthermore define in accordance with (1.4) the left S-resolvent operator of T at s as

$$S_L^{-1}(s,T) = \mathcal{Q}_s(T)^{-1}(\overline{s}\mathcal{I} - T). \tag{1.7}$$

This yields a $\mathcal{B}(V)$-valued slice hyperholomorphic function in the variable s so that we can furthermore define for any function f that is left slice hyperholomorphic on a suitable set U with $\sigma_S(T) \subset U$ the operator

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s). \tag{1.8}$$

The result is the S-functional calculus, the generalisation of the Riesz-Dunford-functional calculus for holomorphic functions to the quaternionic setting, which has exactly the same properties as its classical prototype. A similar procedure starting from the right slice hyperholomorphic Cauchy kernel $S_R^{-1}(s,x)$ leads to the definition of the right S-resolvent operator $S_R^{-1}(s,T)$ and the S-functional calculus for right slice hyperholomorphic functions.

After the fundamental principles of the theory of quaternionic linear operators were understood, its development gathered speed and the literature exploded. The techniques described above were gathered in the monograph [36] and the techniques therein allowed the development of a theory of strongly continuous quaternionic groups and semigroups [9, 32] and the extension of the H^∞-functional calculus to quaternionic linear sectorial operators in [8]. The continuous functional calculus for normal operators on quaternionic Hilbert spaces was developed in [49] and two highlights of the theory were the proofs of the spectral theorems for unitary and for unbounded normal operators on quaternionic Hilbert spaces based on the S-spectrum in [5, 6]. Slice hyperholomorphic functions were investigated by several authors and the results of these efforts were presented in the monographs [35, 48].

The theory is furthermore not limited to quaternionic linear operators. Using a Clifford algebra approach, the S-functional calculus was in [29, 34] also defined for n-tuples of noncommuting operators. Slice hyperholomorphicity can even be defined for functions on real alternative algebras [52] and this allowed to develop a theory of semigroups over real alternative $*$-algebras in [53]. Based on the Fueter mapping theorem in integral form, it was possible to develop a functional calculus of a quaternionic linear operator with commuting components that applies to functions that are Fueter regular and, similarly, a functional calculus for n-tuples of commuting operators that applies

to monogenic Clifford-valued functions. Both of them are based on the S-spectrum and the notion of slice hyperholomorphicity [20, 30, 33]. A similar functional calculus based on the W-transform was defined in [25]. Finally, the S-resolvents play a crucial role in slice hyperholomorphic Schur analysis, which was developed in [7, 10, 11] and the monograph [12].

This thesis aims at further expanding the understanding of quaternionic linear operators. It extends several known techniques of complex operator theory to this setting and discusses their possible applications in the field of fractional evolution equations and some of their consequences for quaternionic operator quantum mechanics. After recalling the essential foundational results in quaternionic operator theory in Chapter 2, it is divided into three parts: the first part generalizes several holomorphic functional calculi from the complex to the quaternionic setting, the second part develops a concise theory of spectral integration in the quaternionic setting and studies quaternionic spectral operators and the final third part discusses the aforementioned applications of quaternionic operator theory in the fields of fractional evolution and quaternionic quantum mechanics.

I) Chapters 3 to 8 form the first part of this thesis. In this part, we define several slice hyperholomorphic functional calculi, which are the counterparts of holomorphic functional calculi in the complex setting, and we investigate their properties. Such functional calculus is always based on an integral representation for a certain class of slice hyperholomorphic functions. If such integral representation holds for a certain class of functions that is slice hyperholomorphic on the S-spectrum of a quaternionic linear operator T, then we can define functions of T by formally replacing the scalar variable in the respective integral kernel by the operator (provided that this is possible). A typical example is the S-functional calculus, where the variable x in (1.5) is replaced by the operator T in order to obtain (1.8). Depending on the properties of the operator T, one can however use other integral representations than the slice hyperholomorphic Cauchy formula in order to enlarge the class of admissible functions. Precisely, we present the following results:

- Chapter 3 contains some preparatory results published in [21]. Precisely, we show that the S-resolvents of a closed operator are slice hyperholomorphic. For bounded operators this follows by a simple computation and hence it has been assumed to hold true also for the S-resolvents of an unbounded operator. In this case, the proof is however quite involved and actually requires the introduction of a new series expansion of the pseudo-resolvent of T. We close this gap in the theory and furthermore study the behavior of the S-resolvents close to the S-spectrum of T.
- In Chapter 4 we develop the S-functional calculus for closed and possibly unbounded quaternionic linear operators. The S-functional calculus for unbounded closed operators has already been introduced in [24], where the unbounded operator and the function were transformed suitably so that the S-functional calculus for bounded operators could be applied. This strategy is standard in the complex case, but in the quaternionic case it has the disadvantage that it requires that $\rho_S(T) \cap \mathbb{R} \neq \emptyset$, which is not always true. The nabla operator on $L^2(\mathbb{R}, \mathbb{H})$, one of the most important quaternionic linear

operators, does for instance not satisfy this condition.

We therefore define the S-functional calculus for closed operators in Chapter 4 directly via a slice hyperholomorphic Cauchy integral. If T is a closed operator with nonempty S-resolvent set and f is a function that is left slice hyperholomorphic on a suitable set U with $\sigma_S(T) \subset U$ that contains a neighbourhood of ∞, then we can write

$$f(x) = f(\infty) + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,x)\, ds_{\mathsf{i}}\, f(s), \qquad x \subset U.$$

Formally replacing x by T yields

$$f(T) := f(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s), \qquad \sigma_S(T) \subset U.$$

This functional calculus is well-defined and its properties agree with those of the Riesz-Dunford-functional calculus for closed complex linear operators. We investigate these properties in detail. In particular we discuss the product rule and show that this functional calculus is compatible with intrinsic polynomials of T although these polynomials do not belong to the class of admissible functions because they are not slice hyperholomorphic at infinity. Furthermore, we discuss the relation between the S-functional calculus for left and the S-functional calculus for right slice hyperholomorphic functions, we prove the spectral mapping theorem and we show that the functional calculus is capable of generating Riesz-projectors onto invariant subspaces. These results were published in [45].

In Section 4.6, we finally present the Taylor series expansion of the S-functional calculus, a result published in [22]. If T is bounded and N is a bounded operator that commutes with T such that $\|N\| < \varepsilon$ for some $\varepsilon > 0$, then $\sigma_S(T+N) \subset C_\varepsilon(\sigma_S(T)) = \{s \in \mathbb{H} : \operatorname{dist}(s, \sigma_S(T)) \leq \varepsilon\}$ and for any f that is left slice hyperholomorphic on $C_\varepsilon(\sigma_S(T))$ the operator $f(T+N)$ can be expressed as the formal Taylor series of f at T, namely as

$$f(T+N) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!} (\partial_S{}^n f)(T),$$

where $\partial_S f$ denotes the slice derivative of f.

- If the operator T is the generator of a strongly continuous group of quaternionic linear operators, then one can define a slice hyperholomorphic functional calculus via the quaternionic Laplace-Stieltjes-transform. This so-called Phillips functional calculus applies to a larger class of functions than the S-functional calculus because it does not require slice hyperholomorphicity at infinity. For the quaternionic setting, it was developed in [3] and these results are presented in Chapter 5.

  If T is the infinitesimal generator of a strongly continuous group $(\mathcal{U}_T(t))_{t\in\mathbb{R}}$ with growth bound $\omega > 0$, that is

$$\sigma_S(T) \subset \{s \in \mathbb{H} : -\omega \leq \operatorname{Re}(s) \leq \omega\} \tag{1.9}$$

and

$$\|\mathcal{U}_T(t)\| \leq Me^{-|t|\omega} \tag{1.10}$$

for some constant $M > 0$, then we consider the subset $\mathbf{S}(T)$ of all quaternion-valued measures on $\mathbb{R}$ given by

$$\mathbf{S}(T) := \left\{ \mu \in \mathcal{M}(\mathbb{R}, \mathbb{H}) : \int_{\mathbb{R}} e^{-|t|(\omega+\varepsilon)}\, d|\mu|(t) < +\infty \right\}, \tag{1.11}$$

where $\varepsilon > 0$ might depend on the measure μ. The quaternionic Laplace-Stieltjes-transform

$$f(x) := \int_{\mathbb{R}} d\mu(t)\, e^{-tx}, \quad -(\omega + \varepsilon) < \mathrm{Re}(x) < \omega + \varepsilon$$

is then a right slice hyperholomorphic function and we can define

$$f(T) := \int_{\mathbb{R}} d\mu(t)\, \mathcal{U}_T(-t).$$

This procedure is meaningful and the integral converges due to (1.9), (1.10), and (1.11). We discuss the quaternionic Laplace-Stieltjes-transform and show that this functional calculus is well-defined. We discuss its algebraic properties and show its compatibility with the S-functional calculus defined in Chapter 4. Finally, we conclude with showing how to invert the operator $f(T)$ for intrinsic f using an inverting sequence of polynomials.

- Chapter 6 introduces the famous H^∞-functional calculus invented by McIntosh in [67, 69] for sectorial quaternionic operators. This functional calculus has already been considered in [8] under the assumption that T is injective and densely defined. We introduce it in its full generality following the strategy of [59]. Any quaternion can be written as $s = |s|e^{\mathsf{i}_s \arg(s)}$ with a unique angle $\arg(s) \in [0, \pi]$. A quaternionic right linear operator is called sectorial if its S-spectrum is contained in the closure of a symmetric sector around the negative real axis of the form

$$\Sigma_\omega = \{s \in \mathbb{H} : \arg(s) \in [0, \omega)\}$$

with $\omega \in (0, \pi)$ and for any $\varphi \in (\omega, \pi)$ there exists a constant $C > 0$ such that

$$\left\|S_L^{-1}(s,T)\right\| \leq \frac{C}{|s|} \quad \text{and} \quad \left\| S_R^{-1}(s,T)\right\| \leq \frac{C}{|s|}$$

for all $s \in \mathbb{H} \setminus \Sigma_\varphi$. If f is left slice hyperholomorphic on a sector Σ_φ for some $\varphi \in (\omega, \pi)$ and has polynomial limit 0 at 0 and infinity, then we can choose $\varphi' \in (\omega, \varphi)$ and define $f(T)$ by a Cauchy integral of the form (1.8) as

$$f(T) := \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi'} \cap \mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s). \tag{1.12}$$

If f is left slice hyperholomorphic on Σ_φ and has finite polynomial limits at 0 and infinity, then it is of the form

$$f(x) = \tilde{f}(x) \, + a + (1 + x)^{-1} b \tag{1.13}$$

with $a, b \in \mathbb{H}$ and $\tilde{f}$ admissible for (1.12). Since $-1 \in \rho_S(T)$, the operator $-S_L^{-1}(s,T) = (\mathcal{I} + T)^{-1}$ exists, and we can define for such functions

$$f(T) := \tilde{f}(T) + \mathcal{I}a + (\mathcal{I} + T)^{-1}b, \tag{1.14}$$

where $\tilde{f}(T)$ is intended in the sense of (1.12). We denote the class of functions of the form (1.13) by $\mathcal{E}_L(\Sigma_\varphi)$ and the class of intrinsic functions of the form (1.13) by $\mathcal{E}(\Sigma_\varphi)$.

Finally, the class of admissible functions can be extended even further, which yields the H^∞-functional calculus. A regulariser for a left slice meromorphic function f on Σ_φ is a function $e \in \mathcal{E}(\Sigma_\varphi)$ such that $ef \in \mathcal{E}_L(\Sigma_\varphi)$ and such that $e(T)$ is injective. If such a regulariser exists for f, then we define

$$f(T) := e(T)^{-1}(ef)(T),$$

where $e(T)$ and $(ef)(T)$ are intended in the sense of (1.14). This operator is not necessarily bounded, because $e(T)^{-1}$ can be unbounded.

We define this functional calculus precisely and discuss its properties. Since the H^∞-functional calculus was already introduced in [8], we focus in particular on the composition rule and the spectral mapping theorem. These results were not shown in [8] and several technical difficulties have to be overcome when generalising them from the complex to the quaternionic setting.

- Fractional powers of operators are an important topic in operator theory, that has been studied extensively since they were introduced in the 1960s in the papers [14, 63, 65, 89]. There exist several approaches for defining fractional powers of complex linear operators and we generalise three of them to the quaternionic setting in Chapter 7. These results are essential for the possible application of quaternionic techniques in the field of fractional diffusion discussed in Chapter 11.

  We first follow [39] and define fractional powers of sectorial operators with bounded inverse directly by the slice hyperholomorphic Cauchy integral

  $$T^{-\alpha} := \frac{1}{2\pi}\int_\Gamma s^{-\alpha}\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)$$

  where Γ is a path that goes from $-\infty e^{\mathsf{i}\theta}$ to $\infty e^{-\mathsf{i}\theta}$ in the set $\mathbb{C}_{\mathsf{i}} \setminus (\Sigma_\varphi \cup B_\varepsilon(0))$ for sufficiently small $\varepsilon > 0$, sufficiently large $\theta \in (0,\pi)$ and arbitrary $\mathsf{i} \in \mathbb{S}$ and avoids the negative real axis. We then discuss the properties of these fractional powers, in particular we prove several integral representations and the semigroup property. The presented results are part of [21].

  In the second approach, we follow [59] and define fractional powers of positive exponent via the H^∞-functional calculus. We present results from [19], in which we discuss the properties of the fractional powers obtained via this approach.

  The third approach that we consider follows the ideas of [63], where fractional powers of an operator were introduced indirectly. We first define for

$\alpha \in (0,1)$ the operator-valued function

$$F_\alpha(p,T) := \frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} t^\alpha \left(p^2 - 2pt^\alpha \cos(\alpha\pi) + t^{2\alpha}\right)^{-1} S_R^{-1}(-t,T)\,dt,$$

which corresponds to an integral representation of $S_R^{-1}(p, T^\alpha)$ of the form (1.12), in which we let φ' tend to π. Then we show that there actually exists a unique closed operator B_α such that $F_\alpha(p,T) = S_R^{-1}(s, B_\alpha)$ and we define $T^\alpha := B_\alpha$. These results were again published in [21].

- Chapter 8 finally reveals a deep relation between complex and quaternionic operator theory and uses this relation in order to define the S-functional calculus for intrinsic slice hyperholomorphic functions on one-sided Banach spaces. Quaternionic operator theory is usually formulated on two-sided quaternionic Banach spaces. One does not only assume the existence of a multiplication of vectors with scalars from the right, but also the existence of a multiplication of vectors with scalars from the left. This seemed a requirement because both expressions for the left (or the right) S-resolvent, its closed form (1.7) and its series expansion in (1.6), contain the multiplication with non-real scalars. Such multiplication is supposed to act as $(aT)\mathbf{v} = a(T\mathbf{v})$ and $(Ta)\mathbf{v} = T(a\mathbf{v})$ and is hence only meaningful if a multiplication of vectors with quaternionic scalars from the left is defined.

  A right linear operator is via the linearity condition however only related with the right multiplication. Moreover, in important situations, such as on quaternionic Hilbert spaces, a multiplication with scalars from the left is not defined a priori. The spectral properties of a right linear operator should be independent from any left multiplication on the considered space.

  If V is a two-sided quaternionic Banach space, then we can choose $\mathsf{i} \in \mathbb{S}$ and consider V as a complex Banach space over the complex field $\mathbb{C}_\mathsf{i}$ by restricting the multiplication with quaternionic scalars from the right to $\mathbb{C}_\mathsf{i}$. We shall denote this $\mathbb{C}_\mathsf{i}$-complex Banach space by V_i. A closed quaternionic right linear operator T on V is then also a $\mathbb{C}_\mathsf{i}$-complex linear operator on V_i. We show that the S-spectrum $\sigma_S(T)$ of T as a quaternionic right linear operator and the classical (complex) spectrum $\sigma_{\mathbb{C}_\mathsf{i}}(T)$ of T as an operator on V_i satisfy the fundamental relation

  $$\sigma_{\mathbb{C}_\mathsf{i}}(T) = \sigma_S(T) \cap \mathbb{C}_\mathsf{i}$$

  and that the resolvent $R_z(T)$ of T as an operator on V_i, which is the inverse of the operator $(z\mathcal{I}_{V_\mathsf{i}} - T)\mathbf{v} = \mathbf{v}z - T\mathbf{v}$, is given by

  $$R_z(T)\mathbf{v} = \mathcal{Q}_z(T)^{-1}\mathbf{v}z - T\mathcal{Q}_z(T)^{-1}\mathbf{v}.$$

  Conversely, we have

  $$\mathcal{Q}_z(T)^{-1} = R_z(T)R_{\overline{z}}(T).$$

  Intrinsic slice hyperholomorphic functions play a special role in quaternionic operator theory because important results such as the product rule, the spectral mapping theorem or the composition rule only hold for these functions.

Using the symmetry $f(\overline{x}) = \overline{f(x)}$ that characterises this class of functions, we show that the formula of the S-functional calculus (1.8) can for any intrinsic slice hyperholomorphic function be rewritten as

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathsf{i}})} R_z(T)\mathbf{v}\, dz \frac{1}{2\pi\mathsf{i}}. \tag{1.15}$$

Hence, $f(T)$ coincides with the operator $f_{\mathsf{i}}(T)$ that we obtain if we interpret T as a $\mathbb{C}_{\mathsf{i}}$-linear operator on V_{i} and apply the classical Riesz-Dunford-functional calculus to the holomorphic function $f_{\mathsf{i}} := f|_{\mathbb{C}_{\mathsf{i}}}$. Since the formula (1.15) does not contain any multiplication of vectors with nonreal scalars from the left, the operator $f(T)$ does furthermore not depend on the left multiplication on the space V.

In analogy with this observation, we define the S-functional calculus of a closed right linear operator T on a quaternionic right Banach space V_R by choosing $\mathsf{i} \in \mathbb{S}$, considering T as a $\mathbb{C}_{\mathsf{i}}$-linear operator on V_{i} and applying the classical Riesz-Dunford-functional calculus to f_{i}. The fact that we restrict ourselves to intrinsic functions guarantees that this procedure actually yields a quaternionic theory: the operator $f(T)$ is again quaternionic right linear and the choice of the imaginary unit $\mathsf{i} \in \mathbb{S}$ is irrelevant. The function calculus we obtain has exactly the same properties as the S-functional calculus for operators on two-sided Banach spaces.

We conclude this chapter by addressing a possible concern of the reader, namely that this functional calculus only applies to the class of intrinsic functions. We show that this is not a problem, but actually a natural condition. The basic intuition of a functional calculus is that $f(T)$ should be defined by letting f act on the spectral values of T. In particular, if $T\mathbf{v} = \mathbf{v}s$, then one should have $f(T)\mathbf{v} = \mathbf{v}f(s)$. The fundamental symmetry (1.1) of the set of right eigenvalues of a quaternionic right linear operator however implies that any functional calculus that respects this condition is based on a class of intrinsic slice functions. Hence, the set of intrinsic slice hyperholomorphic functions is actually the natural domain of the slice hyperholomorphic functional calculus.

II) The second part of this thesis develops a theory of spectral integration and the theory of spectral operators in the quaternionic setting. Several authors, for instance [5, 51, 87], used spectral integrals over the quaternions, but they usually consider spectral measures defined on a complex halfplane

$$\mathbb{C}_{\mathsf{i}}^{\geq} = \{x_0 + \mathsf{i}x_1 : x_0 \in \mathbb{R}, x_1 \geq 0\}.$$

Since on a quaternionic Hilbert space a multiplication with scalars from the left is a priori not defined and the multiplication with scalars from the right is not linear, they then define a multiplication of vectors with scalars in $\mathbb{C}_{\mathsf{i}}$ (or all of $\mathbb{H}$) from the left. On one hand this has the disadvantage that is suggests a dependence of spectral integrals on the imaginary i, which does actually not exist. On the other hand, and this is more problematic, a normal operator on a quaternionic Hilbert space fully determines its spectral measure, but it determines the multiplication

with scalars from the left only partially. In order to define spectral integrals, one therefore has to extend this partially determined left multiplication randomly. Finally, this procedure cannot be applied in the Banach space setting, in which we want to develop the theory of spectral operators, because it relies on the spectral theorem for self-adjoint operators.

For these reasons we develop in Chapter 9 a new theory of quaternionic spectral integration, in which we specify ideas of [87]. We consider spectral measures E defined on the σ-algebra $\mathsf{B}_{\mathsf{S}}(\mathbb{H})$ of axially symmetric Borel sets on $\mathbb{H}$, which consists of Borel sets $\Delta \subset \mathbb{H}$ that satisfy $[s] \subset \Delta$ for any $s \in \Delta$ in accordance with the fundamental symmetry of the set of right eigenvalues resp. the S-spectrum of a quaternionic right linear operator. If $f(x) = f(x_0, x_1)$ is a real-valued $\mathsf{B}_{\mathsf{S}}(\mathbb{H})$-measurable function (that is, if it is measurable with respect to the classical Borel sets and constant on each of the equivalence classes $[x]$), then we can define the integral of f with respect to E as usual. We choose a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions

$$f_n(x) = \sum_{\ell=1}^{N_n} \alpha_{n,\ell}\chi_{\Delta_{n,\ell}}(x), \qquad \alpha_{n,\ell} \in \mathbb{R}, \Delta_{n,\ell} \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$$

that converges uniformly of f and set

$$\int_{\mathbb{H}} f(s)\,dE(s) := \lim_{n\to+\infty} \int_{\mathbb{H}} f_n(s)\,dE(s) := \lim_{n\to+\infty} \sum_{\ell=1}^{N_n} \alpha_{n,\ell} E(\Delta_{n,\ell}).$$

Since a multiplication of vectors with scalars from the left is not a priori not defined on a quaternionic Hilbert space $\mathcal{H}$, a multiplication of operators on $\mathcal{H}$ with nonreal scalars in note defined either. The above procedure is therefore only meaningful for real-valued functions. Other functions would require non-real coefficients $\alpha_{n,k}$ in the approximating sequence.

We therefore introduce the concept of a spectral system (E, J) consisting of a spectral measure E and an imaginary operator J. The spectral measure E is defined on the set $\mathsf{B}_{\mathsf{S}}(\mathbb{H})$ of axially symmetric Borel sets on $\mathbb{H}$ and its values are uniformly bounded projections on a quaternionic right Banach space V_R. The operator J satisfies $-J^2 = E(\mathbb{H} \setminus \mathbb{R})$ and it is essentially a multiplication with the sphere of imaginary units $\mathbb{S}$ from the right on the subspace that is $E(\mathbb{H}\setminus\mathbb{R})V_R$ that is associated via E with the set of quaternions with non-vanishing imaginary part.

Any measurable intrinsic slice function f is of the form $f(x) = \alpha(x_0, x_1) + \mathbf{i}_x\beta(x_0, x_1)$ with real-valued $\mathsf{B}_{\mathsf{S}}(\mathbb{H})$-measurable components α and β. For such functions we therefore define

$$\int_{\mathbb{H}} f(s)\,dE_J(s) := \int_{\mathbb{H}} \alpha(s_0, s_1)\,dE(s) + J\int_{\mathbb{H}} \beta(s_0, s_1)\,dE(s).$$

Spectral integration with respect to a spectral system has several advantages. First of all, a normal operator determines its spectral system completely and the proof of the spectral theorem for normal operators in [20] easily translates into the language of spectral system. This important result can hence be formulated without

randomly introducing any undetermined structure. Moreover, spectral integration with respect to a spectral system is consistent with the approaches using a left multiplication, but unlike them it has a clear interpretation in terms of the right linear structure of the space: while the spectral measure associates subspaces of V_R to equivalence classes of spectral values, the operator J determines how the different values in this equivalent class need to be multiplied onto the vectors in this subspaces. Finally, similar to the results in Chapter 8, we find that the quaternionic theory and the complex theory are consistent. If we restrict the multiplication with scalars to a complex plane $\mathbb{C}_{\mathsf{i}}$ and consider V_R as Hilbert space $V_{R,\mathsf{i}}$ over $\mathbb{C}_{\mathsf{i}}$, then any spectral system (E, J) determines a unique $\mathbb{C}_{\mathsf{i}}$-linear spectral measure E_{i} on $V_{R,\mathsf{i}}$ and integrating an intrinsic function f with respect to (E, J) is equivalent to integrating $f_{\mathsf{i}} := f|_{\mathbb{C}_{\mathsf{i}}}$ with respect to E_{i}.

In Chapter 10 we use these tools to develop a theory of bounded quaternionic spectral operators. A spectral operator on a quaternionic right Banach space V_R is a bounded right linear operator T that has a spectral decomposition, that is, there exists a spectral system (E, J) that commutes with T such that

(i) $\sigma_S(T_\Delta) \subset cl(\Delta)$ for any $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$, where $T_\Delta := T|_{\operatorname{ran} E(\Delta)}$ and $cl(\Delta)$ denotes the closure of the set Δ, and

(ii) $T - s_0\mathcal{I} + s_1 J$ has a bounded inverse on $\operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$ for any $s_0 \in \mathbb{R}$ and any $s_1 > 0$.

We show that this spectral decomposition is unique and we show how to construct it from the operator. Then we show that, similar to the complex case, any spectral operator has a canonical decomposition

$$T = S + N$$

into its scalar part S and its radical part N with

$$S = \int_{\mathbb{H}} s \, dE_J(s) \qquad \text{and} \qquad N = T - S,$$

where N is quasi-nilpotent.

We conclude by discussing the behavior of T under the intrinsic S-functional calculus. For any function f that is intrinsic slice hyperholomorphic on $\sigma_S(T)$, the operator $f(T)$ can be expressed as a power series in N that formally corresponds to the Taylor series expansion of f at S, namely

$$f(T) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!} \int_{\mathbb{H}} \left(\partial_S{}^n f\right)(s) \, dE_J(s).$$

The operator $f(T)$ is again a spectral operator and its spectral decomposition can be constructed from the one of T.

III) The third and final part of this thesis considers possible applications of quaternionic operator theory in the field of fractional evolution and discusses some of its consequences for the quaternionic formulation of quantum mechanics.

In order to explain the idea of fractional evolution equations, we recall the classical heat equation. This equation is derived from two principles, Fourier's law of thermal conductivity, an experimental law that states that the heat flow is proportional to the negative gradient of the temperature, and the law of conversation of energy. If $u(\mathbf{x}, t)$ is the temperature and $\mathbf{q}(\mathbf{x}, t)$ is the heat flow at the point $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ and at time t, then

$$\mathbf{q}(\mathbf{x}, t) = -\kappa \nabla u(\mathbf{x}, t) \qquad \text{(Fourier's law)} \tag{1.16}$$

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) + \operatorname{div} \mathbf{q}(\mathbf{x}, t) = 0 \qquad \text{(Conservation of Energy),} \tag{1.17}$$

where ∇ denotes the gradient of a function depending on $\mathbf{x}$, div denotes the divergence of a vector field depending on $\mathbf{x}$ and $\kappa > 0$ is the thermal conductivity. For mathematical treatment one usually sets $\kappa = 1$ and we shall do the same in this thesis. Combining (1.16) and (1.17) yields the heat equation

$$\frac{\partial}{\partial t} u(t, \mathbf{x}) - \Delta u(t, \mathbf{x}) = 0. \tag{1.18}$$

This equation has however the unphysical property that heat propagates with infinite speed and therefore mathematicians tried to find alternative models. One of there approaches was the definition of the fractional heat equation, in which the negative Laplacian in (1.18) is replaced by its fractional power so that

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) + (-\Delta)^{\alpha} u(\mathbf{x}, t) = 0, \tag{1.19}$$

usually with the parameter $\alpha \in (0, 1)$. This model takes long distance effects into account and has been studied extensively by several mathematicians. The same approach also applies to more general elliptic differential operators and an introduction to the theory can be found in [18, 85].

There exist several approaches for defining the fractional Laplacian. If a function u belongs to the Schwarz class of rapidly decreasing functions, then one can simply define $(-\Delta)^{\alpha} u$ via the Fourier transform so that the following identity holds

$$\widehat{(-\Delta)^{\alpha} f}(\xi) = |\xi|^{2\alpha} \hat{f}(\xi).$$

One can also define the fractional Laplacian via the heat semigroup. Starting from the formula

$$\lambda^{\alpha} = \frac{1}{\Gamma(-\alpha)} \int_0^{+\infty} \frac{1}{t^{1+\alpha}} \left(e^{-\lambda t} - 1 \right) dt,$$

which holds for $\alpha \in (0, 1)$ and $\lambda \geq 0$, one defines

$$(-\Delta)^{\alpha} f(\mathbf{x}) = \frac{1}{\Gamma(-s)} \int_0^{+\infty} \frac{1}{t^{1+\alpha}} \left(e^{\Delta t} f(\mathbf{x}) - f(\mathbf{x}) \right) dt,$$

where $e^{\Delta t} f(\mathbf{x})$ is the solution of the heat equation (1.18) with initial datum f. For $\alpha \in (0, 1)$, this yields the singular integral operator

$$(-\Delta)^{\alpha} f(\mathbf{x}) = c(n, \alpha) \int_{\mathbb{R}^n} \frac{f(\mathbf{x}) - f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n+2\alpha}} \, d\mathbf{y},$$

where the integral is intended in the sense of the principal value and $c(n, \alpha)$ is a constant depending on the dimension of the space and the parameter α. Finally, one can also define $(-\Delta)^{\alpha}$ using an abstract holomorphic functional calculus. All these approaches are equivalent.

Our idea was to go a different way. Instead of directly replacing $-\Delta$ by its fractional power in (1.18), we wanted to consider the gradient as a quaternionic linear operator and replace it in (1.16) by its fractional power using the theory develop in Chapter 7. This would yield the equation

$$\frac{\partial}{\partial t}u(t,\mathbf{x}) - \operatorname{div}\left(\nabla^{\alpha}u(\mathbf{x},t)\right) = 0. \tag{1.20}$$

This equation should have several advantages over (1.19). In particular, this procedure would allow a physical interpretation as a modification of the law for the flow, but it would preserve the law of conservation of energy. Since this equation is in divergence form, it moreover allows an immediate definition of weak solutions of this fractional evolution equation.

In Chapter 11 we therefore develop the spectral theory of the gradient, taking advantage also of the techniques developed in Chapter 8. We find that the nabla operator, the quaternionification the gradient, does not belong to the class of sectorial operators because $\sigma_S(T) = \mathbb{R}$ so that the theory developed in Chapter 7 is not directly applicable. Using a modification of Balakrishnan's formula for fractional powers of an operator that takes only positive spectral values into account, we hoped to define ∇^{α} at least on a subspace. Surprisingly this provided a way to deduce the original heat equation on the entire space so that we can propose a new method for deriving fractional evolution equations similar to the one in (1.20). In particular this method should be applicable for operators of form

$$T = a_1(\mathbf{x})\frac{\partial}{\partial x_1}e_1 + a_2(\mathbf{x})\frac{\partial}{\partial x_2}e_2 + a_3(\mathbf{x})\frac{\partial}{\partial x_3}e_3,$$

which correspond to Fourier laws with non-constant coefficients that could be used to model non-homogeneous materials.

Chapter 12 finally returns to the field that provided the original motivation for the development of quaternionic operator theory, namely quaternionic quantum mechanics. As pointed out in [1], researchers in this field found inconsistencies between this formulation and the classical complex version of quantum mechanics and therefore concluded that these two theories were not equivalent. In our last chapter, which contains results from [46], we conjecture that this idea is due to a logical mistake that was made in quaternionic quantum mechanics from its very beginning and that any quaternionic quantum system is actually equivalent to a complex quantum system. Precisely, motivated by certain techniques that are useful for treating normal operators on a quaternionic Hilbert space, we conjecture that for any quantum system on a quaternionic Hilbert space $\mathcal{H}$ there exists a unitary and anti-selfadjoint operator J that commutes with any observable and any time translation operator $U(t)$. If we choose $\mathsf{i} \in \mathbb{S}$, then the space

$$\mathcal{H}^{+}_{\mathsf{J},\mathsf{i}} := \{\mathbf{v} \in \mathcal{H} : \mathsf{J}\mathbf{v} = \mathbf{v}\mathsf{i}\},$$

turns out to be a complex Hilbert space over the complex field $\mathbb{C}_{\mathbf{i}}$ with the scalar product that it inherits from $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}^{+}_{\mathsf{J},\mathbf{i}} \oplus \mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}\mathbf{j}$ for any $\mathbf{j} \in \mathbb{S}$ with $\mathbf{j} \perp \mathbf{i}$. Any quaternionic right linear operator T on $\mathcal{H}$ that commutes with J is then the quaternionic linear extension of an operator $T_{\mathbb{C}_{\mathbf{i}}}$ on $\mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}$ that is obtained by componentwise application, that is

$$T(\mathbf{v}) = T_{\mathbb{C}_{\mathbf{i}}}(\mathbf{v}_1) + T_{\mathbb{C}_{\mathbf{i}}}(\mathbf{v}_2)\mathbf{j}$$

for any $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2\mathbf{j}$ in $\mathcal{H} = \mathcal{H}^{+}_{\mathsf{J},\mathbf{i}} \oplus \mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}\mathbf{j}$. The operator T and the operator $T_{\mathbb{C}_{\mathbf{i}}}$ have analogous properties: T is bounded, normal, (anti-)selfadjoint, unitary if and only if $T_{\mathbb{C}_{\mathbf{i}}}$ is bounded, normal, (anti-)selfadjoint, unitary. Since J commutes with any observable and any time translation operator, we find that the quaternionic quantum system on $\mathcal{H}$ is actually nothing else then the quaternionification of a complex quantum system on $\mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}$.

After making this conjecture, we show that it holds true for relativistic elementary systems (RES for short) in the sense of [70]. In this paper, the authors showed that any real RES admits a complex structure that turns it into a complex RES. We show that their arguments can also be applied in order to show that any quaternionic RES can be reduced to a complex RES on a space of the form $\mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}$. Moreover, we find that the operator J is the unitary part of the polar decomposition of the anti-selfadjoint Hamiltonian H. Precisely, we find $H = \mathsf{J}|H|$, where H is the infinitesimal generator of the group of time translations so that $U(t) = \exp(tH)$ for all $t \in \mathbb{R}$.

In a final section we show how the inconsistencies between complex and quaternionic quantum mechanics were caused by the random choice of a left multiplication on the quaternionic Hilbert space $\mathcal{H}$. The existence of such left multiplication was not implied by the logical structure of quantum mechanics developed in [15] and the arguments that seemed to determine and physically justify it were not correct.

CHAPTER 2

Preliminaries

The skew-field of quaternions consists of the real vector space

$$\mathbb{H} := \left\{ x = \xi_0 + \sum_{\ell=1}^{3} \xi_\ell e_\ell : \xi_\ell \in \mathbb{R} \right\},$$

which is endowed with an associative product with unity 1 such that $e_\ell^2 = -1$ and $e_\ell e_\kappa = -e_\kappa e_\ell$ for $\ell, \kappa \in \{1,2,3\}$ with $\ell \neq \kappa$. The real part of a quaternion $x = \xi_0 + \sum_{\ell=1}^{3} \xi_\ell e_\ell$ is defined as $\mathrm{Re}(x) := \xi_0$, its imaginary part as $\underline{x} := \sum_{\ell=1}^{3} \xi_\ell e_\ell$ and its conjugate as $\overline{x} := \mathrm{Re}(x) - \underline{x}$. The modulus of x is given by $|x|^2 = \overline{x}x = x\overline{x} = \sum_{\ell=0}^{3} \xi_\ell^2$ and the inverse of any non-zero quaternions x is hence $x^{-1} = \overline{x}|x|^{-2}$.

We denote the sphere of all normalised purely imaginary quaternions by

$$\mathbb{S} := \{x \in \mathbb{H} : \mathrm{Re}(x) = 0, |x| = 1\}.$$

If $\mathsf{i} \in \mathbb{S}$ then $\mathsf{i}^2 = -1$ and i is therefore called an imaginary unit. The subspace

$$\mathbb{C}_{\mathsf{i}} := \{x_0 + \mathsf{i}x_1 : x_0, x_1 \in \mathbb{R}\}$$

is then an isomorphic copy of the field of complex numbers. We furthermore introduce the notation

$$\begin{aligned}
\mathbb{C}_{\mathsf{i}}^{+} &:= \{x_0 + \mathsf{i}x_1 : x_0 \in \mathbb{R}, x_1 > 0\} \\
\mathbb{C}_{\mathsf{i}}^{-} &:= \{x_0 + \mathsf{i}x_1 : x_0 \in \mathbb{R}, x_1 < 0\} \\
\mathbb{C}_{\mathsf{i}}^{\geq} &:= \{x_0 + \mathsf{i}x_1 : x_0 \in \mathbb{R}, x_1 \geq 0\}
\end{aligned}$$

for the open upper half plane, the open lower half plane, and the closed upper halfplane in $\mathbb{C}_{\mathsf{i}}$.

If $\mathsf{i}, \mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$ and we set $\mathsf{k} = \mathsf{ij} = -\mathsf{ji}$, then 1, i, j and k form an orthonormal basis of $\mathbb{H}$ as a real vector space and 1 and j form a basis of $\mathbb{H}$ as a left or right vector space over the complex plane $\mathbb{C}_{\mathsf{i}}$, that is

$$\mathbb{H} = \mathbb{C}_{\mathsf{i}} + \mathbb{C}_{\mathsf{i}}\mathsf{j} \quad \text{and} \quad \mathbb{H} = \mathbb{C}_{\mathsf{i}} + \mathsf{j}\mathbb{C}_{\mathsf{i}}. \tag{2.1}$$

Any quaternion x belongs to such a complex plane $\mathbb{C}_{\mathsf{i}}$: if we set

$$\mathsf{i}_x := \begin{cases} \frac{1}{|\underline{x}|}\underline{x}, & \text{if } \underline{x} \neq 0 \\ \text{any } \mathsf{i} \in \mathbb{S}, & \text{if } \underline{x} = 0, \end{cases}$$

then we have

$$x = x_0 + \mathsf{i}_x x_1 \tag{2.2}$$

with $x_0 = \operatorname{Re}(x)$ and $x_1 = |\underline{x}|$. (Sometimes, when it is more convenient, we also set $\mathsf{i}_x = 0$ for $x \in \mathbb{R}$.) The set

$$[x] := \{x_0 + \mathsf{i}x_1 : \mathsf{i} \in \mathbb{S}\},$$

is a 2-sphere, that reduces to a single point if x is real. Quaternions that belong to the same sphere can be transformed into each other by multiplication with a third quaternion as the following well-known result shows [17].

Lemma 2.1. *Let $x \in \mathbb{H}$. A quaternion $y \in \mathbb{H}$ belongs to $[x]$ if and only if there exists $h \in \mathbb{H} \setminus \{0\}$ such that $x = h^{-1}yh$.*

Analogue to the complex case, any quaternion $x \in \mathbb{H} \setminus \{0\}$ can moreover be written as $x = |x|e^{\theta \mathsf{i}_x}$ with a unique angle $\theta \in [0, \pi]$. We define $\arg(x) := \theta$. Observe that $\arg(x)$ does not depend on the choice of i_x if x is real as $x = |x|e^{\mathsf{i}0}$ for any $\mathsf{i} \in \mathbb{S}$ if $x > 0$ and $x = |x|e^{\mathsf{i}\pi}$ for any $\mathsf{i} \in \mathbb{S}$ if $x < 0$.

These facts are well-known and they can be found in any standard textbook that considers quaternions, for instance in [73].

2.1 Slice Hyperholomorphic Functions

Complex operator theory is based on the theory of holomorphic functions. The function theory that underlies the theory of quaternionic linear operators is the theory of slice hyperholomorphic functions. Since this non-standard material is essential for developing our results, we recall the main properties of this class of functions. The corresponding proofs can be found in [36]. Note however that—for the reasons explained in Remark 2.16—the definition of slice hyperholomorphicity in [36] is different from the one we use and it is less restrictive unless the functions are defined on slice domains. The proofs of the presented results hold however also with the definition that we use in this thesis or they can be readapted by obvious modifications.

Definition 2.2. A set $U \subset \mathbb{H}$ is called

(i) axially symmetric if $[x] \subset U$ for any $x \in U$, and

(ii) a slice domain if U is open, $U \cap \mathbb{R} \neq 0$ and $U \cap \mathbb{C}_{\mathsf{i}}$ is a domain in $\mathbb{C}_{\mathsf{i}}$ for any $\mathsf{i} \in \mathbb{S}$.

In order to avoid confusion, we shall in this thesis denote the closure of a set U by $cl(U)$ and its conjugation by $\overline{U} = \{\overline{x} : x \in U\}$.

Definition 2.3. Let $U \subset \mathbb{H}$ be axially symmetric. A function $f : U \to \mathbb{H}$ is called left slice function, if it is of the form

$$f(x) = \alpha(x_0, x_1) + \mathbf{i}_x\beta(x_0, x_1) \qquad \forall x = x_0 + \mathbf{i}_x x_1 \in U, \tag{2.3}$$

where α and β are functions that take values in $\mathbb{H}$ and satisfy the compatibility conditions

$$\alpha(x_0, x_1) = \alpha(x_0, -x_1) \quad \text{and} \quad \beta(x_0, x_1) = -\beta(x_0, -x_1). \tag{2.4}$$

A function $f : U \to \mathbb{H}$ is called right slice function, if it is of the form

$$f(x) = \alpha(x_0, x_1) + \beta(x_0, x_1)\mathbf{i}_x \qquad \forall x = x_0 + \mathbf{i}_x x_1 \in U, \tag{2.5}$$

with functions α and β that satisfy (2.4). Finally, a left or right slice function $f = \alpha+\mathbf{i}\beta$ is called intrinsic if α and β take values in $\mathbb{R}$.

We denote the set of all left slice functions on U by $\mathcal{SF}_L(U)$, the set of all right slice functions on U by $\mathcal{SF}_R(U)$ and the set of all intrinsic slice functions on U by $\mathcal{SF}(U)$.

Remark 2.4. Any quaternion x can be represented using two different imaginary units, namely $x = x_0 + \mathbf{i}_x x_1 = x_0 + (-\mathbf{i}_x)(-x_1)$. If $x \in \mathbb{R}$, then we can even choose any imaginary unit in this representation. The compatibility conditions (2.4) assure that the choice of this imaginary unit is irrelevant. In particular it implies $\beta(x_0, x_1) = 0$ if $x_1 = 0$, that is if $x \in \mathbb{R}$.

Definition 2.5. Let $U \subset \mathbb{H}$ be an axially symmetric open set. A left slice function $f = \alpha + \mathbf{i}\beta : U \to \mathbb{H}$ is called left slice hyperholomorphic if α and β satisfy the Cauchy-Riemann-differential equations

$$\begin{aligned} \frac{\partial}{\partial x_0}\alpha(x_0, x_1) &= \frac{\partial}{\partial x_1}\beta(x_0, x_1) \\ \frac{\partial}{\partial x_1}\alpha(x_0, x_1) &= -\frac{\partial}{\partial x_0}\beta(x_0, x_1). \end{aligned} \tag{2.6}$$

Similarly, a right slice function $f = \alpha + \beta\mathbf{i} : U \to \mathbb{H}$ is called right slice hyperholomorphic if α and β satisfy (2.6).

For an arbitrary axially symmetric set U, we denote the set of all functions that are left slice hyperholomorphic on an open axially symmetric set U' with $U \subset U'$ by $\mathcal{SH}_L(U)$. Similarly, we denote the set of all functions that are right resp. intrinsic slice hyperholomorphic on an open axially symmetric set U' with $U \subset U'$ by $\mathcal{SH}_R(U)$ resp. $\mathcal{SH}(U)$.

Any intrinsic slice function is both a left and a right slice function because $\mathbf{i}$ and β commute. The converse is however not true: the constant function $f \equiv c \in \mathbb{H} \setminus \mathbb{R}$ is obviously a left and a right slice function, but not intrinsic. Intrinsic slice functions can be characterized in several ways.

Corollary 2.6. *If $f \in \mathcal{SF}_L(U)$ or $f \in \mathcal{SF}_R(U)$, then the following statements are equivalent.*

(i) The function f is an intrinsic slice function.

(ii) We have $f(\overline{x}) = \overline{f(x)}$ for any $x \in U$.

(iii) We have $f(U \cap \mathbb{C}_{\mathbf{i}}) \subset \mathbb{C}_{\mathbf{i}}$ for all $\mathbf{i} \in \mathbb{S}$.

The importance of this subclass is due to the fact that the pointwise multiplication and the composition with intrinsic slice functions preserve the slice structure. This is not true for arbitrary slice functions. Moreover, if the functions are even slice hyperholomorphic, then slice hyperholomorphicity is preserved, too.

Corollary 2.7. *Let $U \subset \mathbb{H}$ be axially symmetric.*

(i) If $f \in \mathcal{SF}(U)$ and $g \in \mathcal{SF}_L(U)$, then $fg \in \mathcal{SF}_L(U)$. If $f \in \mathcal{SF}_R(U)$ and $g \in \mathcal{SF}(U)$, then $fg \in \mathcal{SF}_R(U)$.

(ii) If $f \in \mathcal{SH}(U)$ and $g \in \mathcal{SH}_L(U)$, then $fg \in \mathcal{SH}_L(U)$. If $f \in \mathcal{SH}_R(U)$ and $g \in \mathcal{SH}(U)$, then $fg \in \mathcal{SH}_R(U)$.

(iii) If $g \in \mathcal{SF}(U)$ and $f \in \mathcal{SF}_L(g(U))$, then $f \circ g \in \mathcal{SF}_L(U)$. If $g \in \mathcal{SF}(U)$ and $f \in \mathcal{SF}_R(g(U))$, then $f \circ g \in \mathcal{SF}_R(U)$.

(iv) If $g \in \mathcal{SH}(U)$ and $f \in \mathcal{SH}_L(g(U))$, then $f \circ g \in \mathcal{SH}_L(U)$. If $g \in \mathcal{SH}(U)$ and $f \in \mathcal{SH}_R(g(U))$, then $f \circ g \in \mathcal{SH}_R(U)$.

The values of a slice function are uniquely determined by its values on a single arbitrary complex plane $\mathbb{C}_{\mathbf{i}}$.

Theorem 2.8. *Let U be an axially symmetric slice domain, let $f \in \mathcal{SH}_L(U)$ or let $f \in \mathcal{SH}_R(U)$ and set $\mathcal{Z} := \{x \in U : f(x) = 0\}$. If there exists $\mathbf{i} \in \mathbb{S}$ such that $\mathcal{Z}_{\mathbf{i}} := \mathcal{Z} \cap \mathbb{C}_{\mathbf{i}}$ has an accumulation point in $U_{\mathbf{i}} := U \cap \mathbb{C}_{\mathbf{i}}$, then $f \equiv 0$.*

Theorem 2.9 (Representation Formula)**.** *Let $U \subset \mathbb{H}$ be axially symmetric and $\mathbf{i} \in \mathbb{S}$. For any $x = x_0 + \mathbf{i}_x x_1 \in U$ set $x_{\mathbf{i}} := x_0 + \mathbf{i} x_1$. If $f \in \mathcal{SF}_L(U)$, then*

$$f(x) = \frac{1}{2}(1 - \mathbf{i}_x\mathbf{i})f(x_{\mathbf{i}}) + \frac{1}{2}(1 + \mathbf{i}_x\mathbf{i})f(\overline{x_{\mathbf{i}}}) \quad \text{for all } x \in U. \tag{2.7}$$

If $f \in \mathcal{SH}_R(U)$, then

$$f(x) = f(x_{\mathbf{i}})(1 - \mathbf{i}\mathbf{i}_x)\frac{1}{2} + f(\overline{x_{\mathbf{i}}})(1 + \mathbf{i}\mathbf{i}_x x)\frac{1}{2} \quad \text{for all } x \in U. \tag{2.8}$$

As a consequence, any quaternion-valued function that is defined on a suitable subset of a complex plane possesses a unique slice extension.

Corollary 2.10. *Let $\mathbf{i} \in \mathbb{S}$ and let $f : O \to \mathbb{H}$, where O is a set in $\mathbb{C}_{\mathbf{i}}$ that is symmetric with respect to the real axis. We define the axially symmetric hull of O as*

$$[O] := \bigcup_{z \in O} [z].$$

(i) There exists a unique function $f_L \in \mathcal{SF}_L([O])$ such that $f_L|_{O \cap \mathbb{C}_{\mathbf{i}}} = f$. Similarly, there exists a unique function $f_R \in \mathcal{SF}_R([O])$ such that $f_R|_{O \cap \mathbb{C}_{\mathbf{i}}} = f$.

(ii) If f satisfies $\frac{1}{2}\left(\frac{\partial}{\partial x_0}f(x) + \mathbf{i}\frac{\partial}{\partial x_1}f(x)\right) = 0$*, then f_L is left slice hyperholomorphic.*

(iii) If f satisfies $\frac{1}{2}\left(\frac{\partial}{\partial x_0}f(x) + \frac{\partial}{\partial x_1}f(x)\mathbf{i}\right) = 0$*, then f_R is right slice hyperholomorphic.*

Remark 2.11. If $O \subset \mathbb{C}_{\mathbf{i}}^{\geq}$ and $f : O \mapsto \mathbb{C}_{\mathbf{i}}$ is such that $f(O \cap \mathbb{R}) \subset \mathbb{R}$, then there exists a unique intrinsic slice extension $\tilde{f} \in \mathcal{SF}([O])$ of f to $[O]$. Indeed, we can use the relation (ii) in Corollary 2.6 to extend f to $O \cup \overline{O}$ by setting $f(\overline{z}) = \overline{f(z)}$. The set $O \cup \overline{O}$ is symmetric with respect to the real axis and hence Corollary 2.10 implies the existence of a left slice extension f_L of f. If we write $f(z) = \alpha(z_0, z_1) + \mathbf{i}\beta(z_0, z_1)$ with $\alpha(z_0, z_1), \beta(z_0, z_1) \in \mathbb{R}$ for $z \in O$, this slice extension is, because of (2.7), given by

$$\begin{aligned} f_L(x) =& \frac{1}{2}\left(f(x_{\mathbf{i}}) + \overline{f(x_{\mathbf{i}})}\right) + \mathbf{i}_x(-\mathbf{i})\frac{1}{2}\left(f(x_{\mathbf{i}}) - \overline{f(x_{\mathbf{i}})}\right) \\ =& \alpha(x_0, x_1) + \mathbf{i}_x\beta(x_0, x_1) \end{aligned}$$

As α and β take real values, f_L is intrinsic. Any intrinsic slice function is therefore already entirely determined by its values on one complex halfplane $\mathbb{C}_{\mathbf{i}}^{\geq}$.

Let us now turn our attention to the generalisation of results from classical function theory to the slice hyperholomorphic setting. Important examples of slice hyperholomorphic functions are power series with quaternionic coefficients. Any series of the form $\sum_{n=0}^{+\infty} x^n a_n$ with $a_n \in \mathbb{H}$ is left slice hyperholomorphic and any series of the form $\sum_{n=0}^{+\infty} a_n x^n$ is right slice hyperholomorphic on its domain of convergence. A power series is intrinsic if and only if its coefficients are real. Conversely, any slice hyperholomorphic function can be expanded into a power series at any real point in its domain.

Definition 2.12. Let $U \subset \mathbb{H}$ be an axially symmetric open set. For any $f \in \mathcal{SH}_L(U)$, the function

$$\partial_S f(x) = \lim_{\mathbb{C}_{\mathbf{i}} \ni s \to x} (s - x)^{-1}(f(s) - f(x)),$$

where $\lim_{\mathbb{C}_{\mathbf{i}_x} \ni s \to x} f(s)$ denotes the limit of $f(s)$ as s tends to x in $\mathbb{C}_{\mathbf{i}_x}$, is called the slice derivative of f. Similarly, if $f \in \mathcal{SH}_R(U)$, then the function

$$\partial_S f(x) = \lim_{\mathbb{C}_{\mathbf{i}_x} \ni s \to x} (f(s) - f(x))(s - x)^{-1}$$

is called the slice derivative of f.

Remark 2.13. The slice derivative of a left or right slice hyperholomorphic function is again left resp. right slice hyperholomorphic. Moreover, it coincides with the partial derivative $\frac{\partial}{\partial x_0}f(x)$ of f with respect to the real part x_0 of x. It is therefore also at points $x \in \mathbb{R}$ well defined and independent of the choice of $\mathbf{i}_x$.

Theorem 2.14. *If f is left slice hyperholomorphic on the ball $B_r(a)$ of radius $r > 0$ centered at $a \in \mathbb{R}$, then*

$$f(x) = \sum_{n=0}^{+\infty} (x - a)^n \frac{1}{n!} \left(\partial_S{}^n f\right)(a) \quad \textit{for } x \in B_r(a).$$

If f is right slice hyperholomorphic on $B_r(a)$, then

$$f(x) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\partial_S{}^n f\right)(a)(x-a)^n \quad \text{for } x \in B_r(a).$$

If we restrict a slice hyperholomorphic function to one slice $\mathbb{C}_{\mathsf{i}}$, then we obtain a vector-valued function that is holomorphic in the usual sense.

Lemma 2.15. *Let $U \subset \mathbb{H}$ be an axially symmetric open set. If $f \in \mathcal{SH}_L(U)$, then for any $\mathsf{i} \in \mathbb{S}$ the restriction $f_{\mathsf{i}} := f|_{U\cap\mathbb{C}_{\mathsf{i}}}$ is left holomorphic, that is*

$$\frac{1}{2}\left(\frac{\partial}{\partial x_0} f_{\mathsf{i}}(x) + \mathsf{i}\frac{\partial}{\partial x_1} f_{\mathsf{i}}(x)\right) = 0, \qquad \forall x = x_0 + \mathsf{i}x_1 \in U \cap \mathbb{C}_{\mathsf{i}}. \tag{2.9}$$

If $f \in \mathcal{SH}_R(U)$, then for any $\mathsf{i} \in \mathbb{S}$ the restriction $f_{\mathsf{i}} := f|_{U\cap\mathbb{C}_{\mathsf{i}}}$ is right holomorphic, that is

$$\frac{1}{2}\left(\frac{\partial}{\partial x_0} f_{\mathsf{i}}(x) + \frac{\partial}{\partial x_1} f_{\mathsf{i}}(x)\mathsf{i}\right) = 0, \qquad \forall x = x_0 + \mathsf{i}x_1 \in U \cap \mathbb{C}_{\mathsf{i}}. \tag{2.10}$$

Conversely, if $f \in \mathcal{SF}_L(U)$ satisfies (2.9) *for one (and hence for any) imaginary unit $\mathsf{i} \in \mathbb{S}$, then f is left slice hyperholomorphic. Similarly, if $f \in \mathcal{SF}_R(U)$ satisfies* (2.10) *for one (and hence for any) imaginary unit $\mathsf{i} \in \mathbb{S}$, then f is right slice hyperholomorphic.*

Remark 2.16. Slice hyperholomorphic functions were originally defined as functions that satisfied (2.9) resp. (2.10) on every complex plane $\mathbb{C}_{\mathsf{i}}$. One can show that, as a consequence of the identity principle, any function in this class that is defined on an axially symmetric slice domain satisfies the representation formula and is hence a slice function. On axially symmetric slice domains, this definition is therefore equivalent to the one given in Definition 2.3. On other sets, in particular on open sets that do not intersect the real axis, there exist functions that are not slice functions but satisfy (2.9) or (2.10). Definition 2.3 is therefore more restrictive than the original one.

However, the properties of slice hyperholomorphic functions that are essential in operator theory, such as the Cauchy formulas, depend on the slice structure of the function. For this reason, Definition 2.3 is more appropriate for applications to operator theory. Otherwise we are for instance in the S-functional calculus restricted to considering functions that are defined on axially symmetric slice domains which prevents the definition of Riesz-projectors via this calculus. (Another explanation why quaternionic operator theory must be built on slice functions will be given in Section 8.3.)

As pointed out above, the product of two slice hyperholomorphic functions is not slice hyperholomorphic unless the factor on the appropriate side is intrinsic. However, there exists a regularised product that preserves slice hyperholomorphicity.

Definition 2.17. For $f = \alpha + \mathsf{i}\beta, g = \gamma + \mathsf{i}\delta \in \mathcal{SH}_L(U)$, we define their left slice hyperholomorphic product as

$$f *_L g = (\alpha\gamma - \beta\delta) + \mathsf{i}(\alpha\delta + \beta\gamma).$$

For $f = \alpha + \beta\mathsf{i}, g = \gamma + \delta\mathsf{i} \in \mathcal{SH}_R(U)$, we define their right slice hyperholomorphic product as

$$f *_R g = (\alpha\gamma - \beta\delta) + (\alpha\delta + \beta\gamma)\mathsf{i}.$$

Remark 2.18. The slice hyperholomorphic product is associative and distributive, but in general not commutative. If however f is intrinsic, then $f *_L g$ coincides with the pointwise product fg and

$$f *_L g = fg = g *_L f. \tag{2.11}$$

Similarly, if g is intrinsic, then $f *_R g$ coincides with the pointwise product fg and

$$f *_R g = fg = g *_R f. \tag{2.12}$$

Example 2.19. If $f(x) = \sum_{n=0}^{+\infty} x^n a_n$ and $g(x) = \sum_{n=0}^{+\infty} x^n b_n$ are two left slice hyperholomorphic power series, then their slice hyperholomorphic product equals the usual product of formal power series with coefficients in a non-commutative ring

$$\left(\sum_{n=0}^{+\infty} x^n a_n\right) *_L \left(\sum_{n=0}^{+\infty} x^n b_n\right) = (f *_L g)(x) = \sum_{n=0}^{+\infty} x^n \sum_{k=0}^{n} a_k b_{n-k}. \tag{2.13}$$

Similarly, we have for right slice hyperholomorphic power series that

$$\left(\sum_{n=0}^{+\infty} a_n x^n\right) *_R \left(\sum_{n=0}^{+\infty} b_n x^n\right) = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n. \tag{2.14}$$

Definition 2.20. We define for $f = \alpha + \mathbf{i}\beta \in \mathcal{SH}_L(U)$ its slice hyperholomorphic conjugate as $f^c = \overline{\alpha} + \mathbf{i}\overline{\beta}$ and its symmetrisation as $f^s = f *_L f^c = f^c *_L f$. Similarly, we define for $f = \alpha + \beta\mathbf{i} \in \mathcal{SH}_R(U)$ its slice hyperholomorphic conjugate as $f^c = \overline{\alpha} + \overline{\beta}\mathbf{i}$ and its symmetrisation as $f^s = f *_R f^c = f^c *_R f$.

The symmetrisation of a left slice hyperholomorphic function $f = \alpha + \mathbf{i}\beta$ is explicitly given by

$$f^s = |\alpha|^2 - |\beta|^2 + \mathbf{i}2\mathrm{Re}\left(\alpha\overline{\beta}\right).$$

Hence, it is an intrinsic function. It is $f^s(x) = 0$ if and only if $f(\tilde{x}) = 0$ for some $\tilde{x} \in [x]$. Furthermore, one has

$$f^c(x) = \overline{\alpha(x_0, x_1)} + \mathbf{i}_x \overline{\beta(x_0, x_1)} = \overline{\alpha(x_0, x_1)} + \overline{\beta(x_0, x_1)(-\mathbf{i}_x)} = \overline{f(\overline{x})} \tag{2.15}$$

and an easy computation shows that

$$f *_L g(x) = f(x) g\left(f(x)^{-1} x f(x)\right) \quad \text{if } f(x) \neq 0 \tag{2.16}$$

and for $f(x) \neq 0$ it is

$$\begin{aligned} f^s(x) =& f(x) f^c\left(f(x)^{-1} x f(x)\right) \\ =& f(x) \overline{f\left(\overline{f(x)^{-1} x f(x)}\right)} = f(x) \overline{f\left(f(x)^{-1} \overline{x} f(x)\right)}. \end{aligned} \tag{2.17}$$

Similar computations hold true in the right slice hyperholomorphic case. Finally, if f is intrinsic, then $f^c(x) = f(x)$ and $f^s(x) = |f(x)|^2$.

Corollary 2.21. *The following statements hold true.*

(i) For $f \in \mathcal{SH}_L(U)$ with $f \not\equiv 0$, its slice hyperholomorphic inverse f^{-*_L}, which satisfies $f^{-*_L} *_L f = f *_L f^{-*_L} = 1$, is given by

$$f^{-*_L} = (f^s)^{-1} *_L f^c = (f^s)^{-1} f^c$$

and it is defined on $U \setminus [\mathcal{Z}_f]$, where $\mathcal{Z}_f = \{s \in U : f(s) = 0\}$.

(ii) For $f \in \mathcal{SH}_R(U)$ with $f \not\equiv 0$, its slice hyperholomorphic inverse f^{-*_R}, which satisfies $f^{-*_R} *_R f = f *_R f^{-*_R} = 1$, is given by

$$f^{-*_R} = f^c *_R (f^s)^{-1} = f^c (f^s)^{-1}$$

and it is defined on $U \setminus [\mathcal{Z}_f]$, where $\mathcal{Z}_f = \{s \in U : f(s) = 0\}$.

(iii) If $f \in \mathcal{SH}(U)$ with $f \not\equiv 0$, then $f^{-*_L} = f^{-*_R} = f^{-1}$.

We will need in Chapter 6 that $|f^{-*_L}|$ is in a certain sense comparable to $1/|f|$. Since f^s is intrinsic, we have $|f^s(x)| = |f^s(\tilde{x})|$ for any $\tilde{x} \in [x]$. Since $f(x)xf(x)^{-1} \in [x]$ by Lemma 2.1, we find for $f(x) \neq 0$ because of (2.17) that

$$\begin{aligned} |f^s(x)| &= \left|f^s\left(f(x)xf(x)^{-1}\right)\right| \\ &= \left|f\left(f(x)xf(x)^{-1}\right)\overline{f(\overline{x})}\right| = \left|f\left(f(x)xf(x)^{-1}\right)\right| |f(\overline{x})| . \end{aligned}$$

Therefore we have because of (2.15) that

$$\begin{aligned} \left|f^{-*_L}(x)\right| &= \left|f^s(x)^{-1}\right| |f^c(x)| \\ &= \frac{1}{|f(f(x)xf(x)^{-1})| |f(\overline{x})|} |f(\overline{x})| = \frac{1}{|f(f(x)\overline{x}f(x)^{-1})|} \end{aligned}$$

and so

$$\left|f^{-*_L}(x)\right| = \frac{1}{|f(\tilde{x})|} \quad \text{with } \tilde{x} = f(x)\overline{x}f(x)^{-1} \in [x]. \tag{2.18}$$

An analogous estimate holds for the slice hyperholomorphic inverse of a right slice hyperholomorphic function.

Finally, slice hyperholomorphic functions satisfy an adapted version of Cauchy's integral theorem and an integral formula of Cauchy-type with a modified kernel.

Definition 2.22. We define the left slice hyperholomorphic Cauchy kernel as

$$S_L^{-1}(s,x) = (x^2 - 2s_0x + |s|^2)^{-1}(\overline{s} - x) \quad \text{for } x \notin [s]$$

and the right slice hyperholomorphic Cauchy kernel as

$$S_R^{-1}(s,x) = (\overline{s} - x)(x^2 - 2s_0x + |s|^2)^{-1} \quad \text{for } x \notin [s].$$

Corollary 2.23. *The left slice hyperholomorphic Cauchy-kernel $S_L^{-1}(s,x)$ is left slice hyperholomorphic in the variable x and right slice hyperholomorphic in the variable s on its domain of definition. Moreover, we have $S_R^{-1}(s,x) = -S_L^{-1}(x,s)$.*

Remark 2.24. If x and s belong to the same complex plane, they commute and the slice hyperholomorphic Cauchy-kernels reduce to the classical one, i.e.

$$(s-x)^{-1} = S_L^{-1}(s,x) = S_R^{-1}(s,x).$$

The classical Cauchy kernel is the inverse of the function $x \mapsto s - x$. Similarly, the slice hyperholomorphic Cauchy kernels are the slice hyperholomorphic inverses of this function.

Corollary 2.25. *Let $s \in \mathbb{H}$ and consider the function $x \mapsto s - x$. Then*

$$S_L^{-1}(s,x) = (s-x)^{-*_L} \quad \text{and} \quad S_R^{-1}(s,x) = (s-x)^{-*_R}.$$

In analogy with the above relation, we introduce the following definition. Explicit formulas for these functions will be given in Section 4.6.

Definition 2.26. Let $s \in \mathbb{H}$ and let $n \in \mathbb{Z}$ be an arbitrary integer. We define

$$S_L^n(s,x) := (s-x)^{*_L n} \quad \text{and} \quad S_R^n(s,x) := (s-x)^{*_R n}.$$

We are now able to formulate the analogues of the Cauchy integral theorem and of Cauchy's integral formula for slice hyperholomorphic functions.

Theorem 2.27 (Cauchy's integral theorem)**.** *Let $U \subset \mathbb{H}$ be an axially symmetric open set, let $\mathsf{i} \in \mathbb{S}$ and let D_{i} be a bounded open subset of $U \cap \mathbb{C}_{\mathsf{i}}$ with $cl(D_{\mathsf{i}}) \subset U \cap \mathbb{C}_{\mathsf{i}}$ such that its boundary consists of a finite number of continuously differentiable Jordan curves. For any $f \in \mathcal{SH}_R(U)$ and $g \in \mathcal{SH}_L(U)$, it is*

$$\int_{\partial D_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, g(s) = 0,$$

where $ds_{\mathsf{i}} = -\mathsf{i}\, ds$.

Definition 2.28. An axially symmetric open set $U \subset \mathbb{H}$ is called a slice Cauchy domain if $U \cap \mathbb{C}_{\mathsf{i}}$ is a Cauchy domain for any $\mathsf{i} \in \mathbb{S}$, that is

(i) $U \cap \mathbb{C}_{\mathsf{i}}$ is open

(ii) $U \cap \mathbb{C}_{\mathsf{i}}$ has a finite number of components (i.e. maximal connected subsets), the closures of any two of which are disjoint

(iii) the boundary of $U \cap \mathbb{C}_{\mathsf{i}}$ consists of a finite positive number of closed piecewise continuously differentiable Jordan curves.

Remark 2.29. A slice Cauchy domain is either bounded or has exactly one unbounded component. If it is unbounded, then the unbounded component contains a neighborhood of infinity.

Theorem 2.30 (Cauchy's integral formula)**.** *Let $U \subset \mathbb{H}$ be a slice Cauchy domain, let $\mathsf{i} \in \mathbb{S}$ and set $ds_{\mathsf{i}} = -\mathsf{i}\, ds$. If f is left slice hyperholomorphic on an open set that contains $cl(U)$, then*

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,x)\, ds_{\mathsf{i}}\, f(s) \qquad \text{for all } x \in U.$$

If f is right slice hyperholomorphic on an open set that contains $cl(U)$, then

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathsf{i}})} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,x) \qquad \text{for all } x \in U.$$

Remark 2.31. The results presented in this section can be extended to functions with values in a two-sided quaternionic Banach space. Problems concerning vector-valued functions can be reduced to scalar problems by applying elements of the dual space, analogue to what is done in the complex setting [7].

2.2 Quaternionic Banach and Hilbert Spaces

The natural extension of the Riesz-Dunford-functional calculus for complex linear operators to quaternionic linear operators is the S-functional calculus. It is based on the theory of slice hyperholomorphic functions and follows the idea of the classical case: to formally replace the scalar variable x in the Cauchy formula by an operator. Unless stated differently, the proofs of the results in this section can again be found in [36].

Let us start with a precise definition of the various structures of quaternionic vector, Banach and Hilbert spaces.

Definition 2.32. A quaternionic right vector space is an additive group $(V, +)$ endowed with a quaternionic scalar multiplication from the right such that for all $\mathbf{u}, \mathbf{v} \in V$ and all $a, b \in \mathbb{H}$

$$(\mathbf{u}+\mathbf{v})a = \mathbf{u}a + \mathbf{v}a, \quad \mathbf{u}(a+b) = \mathbf{u}a + \mathbf{u}b, \quad \mathbf{v}(ab) = (\mathbf{v}a)b, \quad \mathbf{v}1 = \mathbf{v}. \tag{2.19}$$

A quaternionic left vector space is an additive group $(V, +)$ endowed with a quaternionic scalar multiplication from the left such that for all $\mathbf{u}, \mathbf{v} \in V$ and all $a, b \in \mathbb{H}$

$$a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}, \quad (a+b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}, \quad (ab)\mathbf{v} = a(b\mathbf{v}), \quad 1\mathbf{v} = \mathbf{v}. \tag{2.20}$$

Finally, a two sided quaternionic vector space is an additive group $(V, +)$ endowed with a quaternionic scalar multiplication from the right and a quaternionic scalar multiplication from the left that satisfy (2.19) resp. (2.20) such that in addition $a\mathbf{v} = \mathbf{v}a$ for all $\mathbf{v} \in V$ and all $a \in \mathbb{R}$.

Remark 2.33. Starting from a real vector space $V_{\mathbb{R}}$, we can easily construct a two-sided quaternionic vector space by setting

$$V_{\mathbb{R}} \otimes \mathbb{H} = \left\{ \sum_{\ell=0}^{3} \mathbf{v}_\ell \otimes e_\ell : \mathbf{v}_\ell \in V_{\mathbb{R}} \right\},$$

where we denote $e_0 = 1$ for neatness. Together with the componentwise addition $V_{\mathbb{R}} \otimes \mathbb{H}$ forms an additive group. It is a two-sided quaternionic vector space, if we endow it with the right and left scalar multiplications

$$a\mathbf{v} = \sum_{\ell,\kappa=0}^{3} (a_\ell \mathbf{v}_\kappa) \otimes (e_\ell e_\kappa) \qquad \text{and} \qquad \mathbf{v}a = \sum_{\ell,\kappa=0}^{3} (a_\ell \mathbf{v}_\kappa) \otimes (e_\kappa e_\ell)$$

for $a = \sum_{\ell=0}^{3} a_\ell e_\ell \in \mathbb{H}$ and $\mathbf{v} = \sum_{\kappa=0}^{3} \mathbf{v}_\kappa \otimes e_\kappa \in V_{\mathbb{R}} \otimes \mathbb{H}$. Usually one omits the symbol $\otimes$ and simply writes $\mathbf{v} = \sum_{\ell=0}^{3} \mathbf{v}_\ell e_\ell$.

It was shown in [71] that any two-sided quaternionic vector-space $\mathbf{v}$ is essentially of this form. Indeed, we can set

$$V_{\mathbb{R}} = \{\mathbf{v} \in V : a\mathbf{v} = \mathbf{v}a \ \forall a \in \mathbb{H}\} \tag{2.21}$$

and find that V is isomorphic to $V_{\mathbb{R}} \otimes \mathbb{H}$. If we set $\operatorname{Re}(\mathbf{v}) := \frac{1}{4}\sum_{\ell=0}^{3} \overline{e_\ell}\mathbf{v}e_\ell$, then $\operatorname{Re}(\mathbf{v}) \in V_{\mathbb{R}}$ and $\mathbf{v} = \sum_{\ell=0}^{3} \operatorname{Re}(\overline{e_\ell}\mathbf{v})e_\ell$.

Remark 2.34. A quaternionic right or left vector space also carries the structure of a real vector space: if we simply restrict the quaternionic scalar multiplication to $\mathbb{R}$, then we obtain a real vector space. Similarly, if we choose some $\mathsf{i} \in \mathbb{S}$ and identify $\mathbb{C}_{\mathsf{i}}$ with the field of complex numbers, then V also carries the structure of a complex vector space over $\mathbb{C}_{\mathsf{i}}$. Again we obtain this structure by restricting the quaternionic scalar multiplication to $\mathbb{C}_{\mathsf{i}}$.

If we consider a two-sided quaternionic vector space, then the left and the right scalar multiplication coincide for real numbers so that we can restrict them to $\mathbb{R}$ in order to obtain again a real vector space. This is however not true for the multiplication with scalars in one complex plane $\mathbb{C}_{\mathsf{i}}$. In general $z\mathbf{v} \neq \mathbf{v}z$ for $z \in \mathbb{C}_{\mathsf{i}}$ and $\mathbf{v} \in V$. Hence, we can only restrict either the left or the right multiplication to $\mathbb{C}_{\mathsf{i}}$ in order to consider V as a complex vector space over $\mathbb{C}_{\mathsf{i}}$, but not both simultaneously.

Definition 2.35. A norm on a right, left or two-sided quaternionic vector space V is a norm in the sense of real vector spaces (cf. Remark 2.34) that is compatible with the quaternionic right, left resp. two-sided scalar multiplication. Precisely, this means that$\|\mathbf{v}a\| = \|\mathbf{v}\|\,|a|$ (or $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$ resp. $\|a\mathbf{v}\| = |a|\|\mathbf{v}\| = \|\mathbf{v}x\|$) for all $a \in \mathbb{H}$ and all $\mathbf{v} \in V$. A quaternionic right, left or two-sided Banach space is a quaternionic right, left or two-sided vector space that is endowed with a norm $\|\cdot\|$ and complete with respect to the topology induced by this norm.

Remark 2.36. Throughout this thesis, we let V_R denote a right-sided and V denote a two-sided quaternionic Banach space.

Remark 2.37. Similar to Remark 2.34, we obtain a real Banach space if we restrict the left or right scalar multiplication on a quaternionic Banach space to $\mathbb{R}$ and we obtain a complex Banach space over $\mathbb{C}_{\mathsf{i}}$ if we restrict the left or right scalar multiplication to $\mathbb{C}_{\mathsf{i}}$ for some $\mathsf{i} \in \mathbb{S}$.

Definition 2.38. A quaternionic right Hilbert space $\mathcal{H}$ is a quaternionic right vector space equipped with a scalar product $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \to \mathbb{H}$ so that for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{H}$ and all scalars $a \in \mathbb{H}$

(i) $\langle\mathbf{u},\mathbf{u}\rangle \geq 0$

(ii) $\langle\mathbf{u},\mathbf{v}a + \mathbf{w}\rangle = \langle\mathbf{u},\mathbf{v}\rangle a + \langle\mathbf{u},\mathbf{w}\rangle$

(iii) $\langle\mathbf{u},\mathbf{v}\rangle = \overline{\langle\mathbf{v},\mathbf{u}\rangle}$

and so that $\mathcal{H}$ is complete with respect to the norm $\|\mathbf{v}\| = \sqrt{\langle\mathbf{v},\mathbf{v}\rangle}$.

Remark 2.39. In order to be consistent with our notation, we shall also assume the scalar product of a complex Hilbert space to be sesquilinear in the first and linear in the second variable.

Remark 2.40. Also a quaternionic Hilbert space carries natural real and complex Hilbert space structures. If we restrict the right scalar multiplication on $\mathcal{H}$ to $\mathbb{R}$ and define $\langle\mathbf{u},\mathbf{v}\rangle_{\mathbb{R}} := \mathrm{Re}\langle\mathbf{u},\mathbf{v}\rangle$, then $(\mathcal{H},\langle\cdot,\cdot\rangle_{\mathbb{R}})$ is a real Hilbert space.

If we choose $\mathsf{i},\mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$, then we can write any quaternion x as $x = z_1 + z_2\mathsf{j}$ with $z_1, z_2 \in \mathbb{C}_{\mathsf{i}}$ according to (2.1). The number $z_1 \in \mathbb{C}_{\mathsf{i}}$ is independent of j and hence we can define the $\mathbb{C}_{\mathsf{i}}$-part of x as $\{x\}_{\mathsf{i}} := z_1$. If we restrict the right scalar multiplication on $\mathcal{H}$ to $\mathbb{C}_{\mathsf{i}}$ and set $\langle\mathbf{u},\mathbf{v}\rangle_{\mathsf{i}} := \{\langle\mathbf{u},\mathbf{v}\rangle\}_{\mathsf{i}}$, then $(\mathcal{H},\langle.,.\rangle_{\mathsf{i}})$ is a complex Hilbert space over

$\mathbb{C}_{\mathsf{i}}$. For $\mathsf{j} \in \mathbb{S}$ with $\mathsf{j} \perp \mathsf{i}$ and any $\mathbf{v}$ in $\mathcal{H}$, the vectors $\mathbf{v}$ and $\mathbf{v}\mathsf{j}$ are orthogonal in this structure.

Remark 2.41. The fundamental concepts of complex Hilbert spaces such as orthogonality, orthonormal bases, the Riesz representation theorem etc. can be defined as in the complex case.

Notation 2.42. Since we are working with different number systems and vector space structures, we introduce for a set of vectors $\mathbf{B} := (\mathbf{b}_\ell)_{\ell\in\Lambda}$ the quaternionic right-linear span of $\mathbf{B}$

$$\operatorname{span}_{\mathbb{H}}\mathbf{B} := \left\{ \sum_{\ell\in I} \mathbf{b}_\ell x_\ell : x_\ell \in \mathbb{H}, I \subset \Lambda \text{ finite} \right\}$$

and the $\mathbb{C}_{\mathsf{i}}$-linear span of $\mathbf{B}$

$$\operatorname{span}_{\mathbb{C}_{\mathsf{i}}}\mathbf{B} := \left\{ \sum_{\ell\in I} \mathbf{b}_\ell z_\ell : z_\ell \in \mathbb{C}_{\mathsf{i}}, I \subset \Lambda \text{ finite} \right\}.$$

Definition 2.43. A mapping $T : V_R \to W_R$ between two quaternionic right Banach spaces V_R and W_R is called right linear if $T(\mathbf{u}a+\mathbf{v}) = T(\mathbf{u})a+T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V_R$ and all $a \in \mathbb{H}$. It is bounded if $\|T\| := \sup_{\|\mathbf{v}\|=1} \|T(\mathbf{v})\|$ is finite and closed if the graph of T is closed in $V_R \times W_R$.

We denote the set of all bounded right linear operators $T : V_R \to V_R$ by $\mathcal{B}(V_R)$ and the set of all closed operators $T : \operatorname{dom}(T) \subset V_R \to V_R$ by $\mathcal{K}(V_R)$.

Remark 2.44. We consider right linear operators by convention. One can also consider left linear operators, which leads to an equivalent theory.

Remark 2.45. The set $\mathcal{B}(V_R)$ of all bounded right linear operators on a quaternionic right Banach space V_R is a real Banach space with the pointwise addition $(T+U)(\mathbf{v}) := T(\mathbf{v})+U(\mathbf{v})$ and the multiplication $(Ta)(\mathbf{v}) := T(\mathbf{v}a)$ with scalars $a \in \mathbb{R}$. However, if we define $(Ta)(\mathbf{v}) := T(\mathbf{v}a)$ for $a \in \mathbb{H} \setminus \mathbb{R}$, then we do not obtain a quaternionic right linear operator as $(Ta)(\mathbf{v}b) = T(\mathbf{v}ab) \neq T(\mathbf{v}ba) = T(\mathbf{v}b)a = (Tb)(\mathbf{v})a$ if $a, b \in \mathbb{H}$ do not belong to the same complex plane. Hence, $\mathcal{B}(V_R)$ is not a quaternionic linear space.

The space $\mathcal{B}(V)$ of all bounded right linear operators on a two-sided quaternionic Banach space V is on the other hand again a two-sided quaternionic Banach space with the scalar multiplications

$$(aT)(\mathbf{v}) = a(T(\mathbf{v})) \quad \text{and} \quad (Ta)(\mathbf{v}) = T(a\mathbf{v}). \tag{2.22}$$

For this reason, the theory of quaternionic linear operators is usually developed on two-sided quaternionic Banach spaces. Also we will usually work on two-sided spaces. An exception is Chapter 8, where we discuss the minimal structure that is necessary to develop quaternionic operator theory and show that the essential results can also be obtained on one-sided spaces.

If T is a bounded operator on a two-sided quaternionic Banach space $V = V_{\mathbb{R}} \otimes \mathbb{H}$, then we can write T as

$$T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \tag{2.23}$$

with $\mathbb{R}$-linear components $T_\ell \in \mathcal{B}(V_\mathbb{R})$ for $\ell = 0, \ldots, 3$. If we set $e_0 = 1$ for neatness, then T acts as

$$T\mathbf{v} = \sum_{\ell,\kappa=0}^{3} T_\ell \mathbf{v}_\kappa e_\ell e_\kappa \quad \text{for} \quad \mathbf{v} = \sum_{\kappa=0}^{3} \mathbf{v}_\kappa e_\kappa \ \in V = V_\mathbb{R} \otimes \mathbb{H}.$$

We thus have

$$\mathcal{B}(V) = \mathcal{B}(V_\mathbb{R}) \otimes \mathbb{H}.$$

Definition 2.46. Let V be a two-sided quaternionic Banach space. We denote the set of all operators $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{B}(V)$ with commuting components by $\mathcal{BC}(V)$. Furthermore, we call a bounded operator T a scalar operator if it is of the form $T = T_0$, that is if $T_1 = T_2 = T_3 = 0$.

For an unbounded operator $T \in \mathcal{K}(V)$ a decomposition of the form (2.23) cannot always be obtained. This is only possible if $\operatorname{dom}(T)$ is a two-sided subspace of V, that is if it is of the form $\operatorname{dom}(T) = V_0 \otimes \mathbb{H}$ for some subspace V_0 of $V_\mathbb{R}$.

If on the other hand $T_0, \ldots, T_3$ are operators on $V_\mathbb{R}$, then we can define the operator

$$T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \quad \text{with} \quad \operatorname{dom}(T) = \left(\bigcap_{\ell=0}^{4} \operatorname{dom}(T_\ell) \right) \otimes \mathbb{H}.$$

Definition 2.47. Let V be a two-sided quaternionic Banach space. We define $\mathcal{KC}(V)$ as the set of all operators $T \in \mathcal{K}(V)$ that admit a decomposition of the form $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell$ with closed operators $T_\ell \in \mathcal{K}(V_\mathbb{R})$ such that

(i) $\operatorname{dom}(T^2) = \bigcap_{\ell,\kappa=0}^{3} \operatorname{dom}(T_\ell T_\kappa) = \bigcap_{\ell=0}^{3} \operatorname{dom}(T_\ell^2)$,

(ii) $\operatorname{dom}(T_\ell T_\kappa) = \operatorname{dom}(T_\kappa T_\ell)$ for $\ell, \kappa \in \{0, \ldots, 3\}$,

(iii) $T_\ell T_\kappa \mathbf{v} = T_\kappa T_\ell \mathbf{v}$ for all $\mathbf{v} \in \operatorname{dom}(T^2)$ for $\ell, \kappa \in \{0, \ldots, 3\}$.

Furthermore, we call a closed operator T a scalar operator if it is of the form $T = T_0$, that is if $T_1 = T_2 = T_3 = 0$ or equivalently if T is the extension of a closed operator on $V_\mathbb{R}$ to V.

Corollary 2.48. *A scalar operator $T \in \mathcal{K}(V)$ commutes with any quaternion $a \in \mathbb{H}$.*

It is essential for the theory that the Hahn-Banach theorem also holds on quaternionic spaces. It was first shown in [44], but a proof in English can be found in [36].

Theorem 2.49 (Hahn–Banach)**.** *Let V_R be a quaternionic right vector space, let V_0 be a right linear subspace of V_R and let $\rho : V_R \to [0, +\infty)$ satisfy $\rho(\mathbf{v}+\mathbf{u}) \leq \rho(\mathbf{v}) + \rho(\mathbf{u})$ and $\rho(\mathbf{v}a) = \rho(\mathbf{v})|a|$ for all $\mathbf{v}, \mathbf{u} \in V_R$ and all $a \in \mathbb{H}$. Moreover, let $\varphi : X_0 \to \mathbb{H}$ be a quaternionic right linear functional on V_0 such that $|\varphi(\mathbf{v})| \leq \rho(\mathbf{v})$ for all $\mathbf{v} \in V_0$. Then there exists a right linear functional $\Phi : V_R \to \mathbb{H}$ such that $\Phi(\mathbf{v}) = \varphi(\mathbf{v})$ for all $\mathbf{v} \in V_0$ and such that*

$$|\Phi(\mathbf{v})| \leq \rho(\mathbf{v}), \qquad \text{for all } \mathbf{v} \in V_R.$$

An analogous statement holds for left linear vector spaces.

As a corollary we obtain, as in the complex case, that the dual space of any quaternionic Banach space separates points.

Definition 2.50. Let V_R be a quaternionic right Banach space. Its topological dual space V_R^* is the quaternionic left Banach space of all bounded right linear mappings of V_R to $\mathbb{H}$.

Remark 2.51. For the reasons explained in Remark 2.45, the dual space V_R^* of a right Banach space V_R is only a left Banach space. The dual space of a two-sided quaternionic Banach space on the other hand is again a two-sided quaternionic Banach space.

Corollary 2.52. *The dual space of a quaternionic right Banach space separates points.*

We conclude this section by introducing the notions of self-adjoint, unitary and normal operators n a quaternionic Hilbert space, which are defined analogue to the complex case.

Definition 2.53. Let $\mathcal{H}$ be a quaternionic Hilbert space and let $T \in \mathcal{K}(\mathcal{H})$ with dense domain. The adjoint T^* of T is the unique operator with domain

$$\operatorname{dom}(T^*) = \{\mathbf{u} \in \mathcal{H} : \mathbf{v} \mapsto \langle \mathbf{u}, T\mathbf{v}\rangle \in \mathcal{H}^*\}$$

that is defined by the relation

$$\langle T^*\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, T\mathbf{v}\rangle. \tag{2.24}$$

The operator T is called

- self-adjoint if $T = T^*$,
- anti-selfadjoint if $T = -T^*$,
- normal if $T \in \mathcal{B}(\mathcal{H})$ and $TT^* = T^*T$,
- unitary if $T \in \mathcal{B}(\mathcal{H})$ and $T^* = T^{-1}$,
- partially unitary if $\mathcal{H}$ is the orthogonal sum $\mathcal{H} = \ker T \oplus \operatorname{ran} T$ and $T|_{\operatorname{ran} T}$ is unitary.

Remark 2.54. As in the complex case, it is the Riesz representation theorem that guarantees together with the density of $\operatorname{dom}(T)$ that (2.24) determines $T^*\mathbf{u}$ so that T^* is actually well defined.

2.3 The S-Spectrum and the S-Functional Calculus

For $T \in \mathcal{K}(V)$, we define

$$\mathcal{Q}_s(T) := T^2 - 2s_0T + |s|^2\mathcal{I}, \qquad \text{for } s \in \mathbb{H}.$$

Definition 2.55. Let $T \in \mathcal{K}(V)$. We define the S-resolvent set of T as

$$\rho_S(T) := \left\{ s \in \mathbb{H} : \mathcal{Q}_s(T)^{-1} \in \mathcal{B}(V) \right\}$$

and the S-spectrum of T as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

For $s \in \rho_S(T)$, the operator $\mathcal{Q}_s(T)^{-1}$ is called the pseudo-resolvent of T at s. Furthermore, we define the extended S-spectrum $\sigma_{SX}(T)$ as

$$\sigma_{SX}(T) := \begin{cases} \sigma_S(T) & \text{if } T \text{ is bounded,} \\ \sigma_S(T) \cup \{\infty\} & \text{if } T \text{ is unbounded.} \end{cases}$$

The S-spectrum generalizes the set of right eigenvalues of a quaternionic linear operator just as the spectrum of a complex linear operator generalizes the set of its eigenvalues. Moreover, it has properties that are analogous to those of the spectrum of a complex linear operator.

Theorem 2.56. *Let $T \in \mathcal{K}(V)$.*

(i) The S-spectrum $\sigma_S(T)$ of T is axially symmetric. It contains the set of right eigenvalues of T and if V_R has finite dimension, then it equals the set of right eigenvalues.

(ii) The S-spectrum $\sigma_S(T)$ is a closed subset of $\mathbb{H}$ and the extended S-spectrum $\sigma_{SX}(T)$ is a closed and hence compact subset of $\mathbb{H}_\infty := \mathbb{H} \cup \{\infty\}$.

(iii) If T is bounded, then $\sigma_S(T)$ is nonempty and bounded by the norm of T, that is $\sigma_S(T) \subset cl(B_{\|T\|}(0))$.

Remark 2.57. Analogue to the complex case, one can divide the S-spectrum of T into subsets with different properties. We shall distinguish the following subsets:

- The point S-spectrum $\sigma_{Sp}(T)$ of T is the set

  $$\sigma_{Sp}(T) := \{s \in \mathbb{H} : \ker \mathcal{Q}_s(T) \neq \{\mathbf{0}\}\}.$$

  It was shown in [26] that it coincides with the set of right eigenvalues of T.

- The continuous S-spectrum $\sigma_{Sc}(T)$ of T is the set

  $$\sigma_{Sc}(T) := \{s \in \mathbb{H} : \ker \mathcal{Q}_s(T) = \{\mathbf{0}\}, \operatorname{ran} \mathcal{Q}_s(T) \subsetneq V_R \text{ is dense}\}.$$

- The residual S-spectrum $\sigma_{Sr}(T)$ of T is the set

  $$\sigma_{Sc}(T) := \{s \in \mathbb{H} : \ker \mathcal{Q}_s(T) = \{\mathbf{0}\}, \operatorname{ran} \mathcal{Q}_s(T) \subsetneq V_R \text{ is not dense}\}.$$

Definition 2.58. Let $T \in \mathcal{K}(V)$. For $s \in \rho_S(T)$, the left S-resolvent operator is defined as

$$S_L^{-1}(s,T) := \mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1} \tag{2.25}$$

and the right S-resolvent operator is defined as

$$S_R^{-1}(s,T) := -(T - \mathcal{I}\overline{s})\mathcal{Q}_s(T)^{-1}. \tag{2.26}$$

Remark 2.59. One clearly obtains the right S-resolvent operator by formally replacing the variable x in the right slice hyperholomorphic Cauchy kernel by the operator T. The same procedure yields

$$S_L^{-1}(s,T)\mathbf{v} = -\mathcal{Q}_s(T)^{-1}(T - \overline{s}\mathcal{I})\mathbf{v}, \quad \text{for } \mathbf{v} \in \operatorname{dom}(T) \tag{2.27}$$

for the left S-resolvent operator. This operator is only defined on the domain $\operatorname{dom}(T)$ of T and not on the entire space V. However, $\mathcal{Q}_s(T)^{-1}T\mathbf{v} = T\mathcal{Q}_s(T)^{-1}\mathbf{v}$ for any $\mathbf{v} \in \operatorname{dom}(T)$ and commuting T and $Q_s(T)$ in (2.27) yields (2.25). For any $s \in \mathbb{H}$, the operator $\mathcal{Q}_s(T) = T^2 - 2s_0T + |s|^2\mathcal{I}$ maps $\operatorname{dom}(T^2)$ to V. Hence, the pseudo-resolvent $\mathcal{Q}_s(T)^{-1}$ maps V to $\operatorname{dom}(T^2) \subset \operatorname{dom}(T)$ if $s \in \rho_S(T)$. Since T is closed and $\mathcal{Q}_s(T)^{-1}$ is bounded, equation (2.25) then defines a continuous and therefore bounded right linear operator on the entire space V. Hence, the left resolvent $S_L^{-1}(s,T)$ is the closed extension of the operator (2.27) to V. In particular, if T is bounded, then $S_L^{-1}(s,T)$ can directly be defined by (2.27).

If one considers left linear operators, then one must modify the definition of the right S-resolvent operator for the same reasons.

Remark 2.60. The S-resolvent operators reduce to the classical resolvent if T and s commute, that is
$$S_L^{-1}(s,T) = S_R^{-1}(s,T) = (s\mathcal{I} - T)^{-1}.$$
This is in particular the case if s is real.

Lemma 2.61. *Let $T \in \mathcal{K}(V)$. The map $s \mapsto S_L^{-1}(s,T)$ is a $\mathcal{B}(V)$-valued right slice-hyperholomorphic function on $\rho_S(T)$ and the map $s \mapsto S_R^{-1}(s,T)$ is a $\mathcal{B}(V)$-valued left slice-hyperholomorphic function on $\rho_S(T)$.*

Remark 2.62. If T is bounded, the proof of the above crucial lemma consists of easy straightforward computations, which can be found for instance in [36]. The slice hyperholomorphicity of the S-resolvents of unbounded operators has also been assumed on several occasions in the theory although an explicit proof has never been given. It turned out that several additional technical difficulties have to be overcome in this case. The respective proof is hence presented in Chapter 3 as a part of this thesis, which fills an important gap in the existing theory.

The left and the right S-resolvent satisfy the following left resp. right S-resolvent equation.

Lemma 2.63 (Left and right S-resolvent equation)**.** *Let $T \in \mathcal{K}(V)$. For $s \in \rho_S(T)$, we have*
$$sS_R^{-1}(s,T)\mathbf{v} - S_R^{-1}(s,T)T\mathbf{v} = \mathbf{v} \qquad \forall \mathbf{v} \in \operatorname{dom}(T) \tag{2.28}$$
and
$$S_L^{-1}(s,T) - TS_L^{-1}(s,T) = \mathcal{I}. \tag{2.29}$$

The above equations provide a tool to compensate the fact that the S-resolvents do not commute with quaternionic scalars. They do however not generalise the classical resolvent equation, which allows to split the product of the resolvent at two points into a sum of the factors. Its role is played by the S-resolvent equation, which has been first introduced in [4] for bounded operators and then for unbounded operators in [21]. It is remarkable that the S-resolvent equation involves both the left and the right S-resolvent and that no generalisation of the classical resolvent that involves only one of them exists.

Theorem 2.64 (S-resolvent equation)**.** *Let $T \in \mathcal{K}(V)$. For $s, x \in \rho_S(T)$ with $s \notin [x]$, it is*
$$\begin{aligned} S_R^{-1}(s,T)S_L^{-1}(x,T) = &\big[[S_R^{-1}(s,T) - S_L^{-1}(x,T)]x \\ &- \overline{s}[S_R^{-1}(s,T) - S_L^{-1}(x,T)]\big](x^2 - 2s_0x + |s|^2)^{-1}. \end{aligned} \tag{2.30}$$

One can show by straight-forward computations that for any $x, s \in \mathbb{H}$ with $s \notin [x]$ and any $B \in \mathcal{B}(V)$, the identity

$$(s^2 - 2x_0 s + |x|^2)^{-1}(sB - B\overline{x}) = (\overline{s}B - Bx)(x^2 - 2s_0 x + |s|^2)^{-1} \tag{2.31}$$

holds. The S-resolvent equation can hence also be written as

$$\begin{aligned} S_R^{-1}(s,T)S_L^{-1}(x,T) &= (s^2 - 2x_0 s + |x|^2)^{-1} \\ &\quad \cdot \left[s[S_R^{-1}(s,T) - S_L^{-1}(x,T)] - [S_R^{-1}(s,T) - S_L^{-1}(x,T)]\overline{x} \right]. \end{aligned}$$

If we follow the idea of the Riesz-Dunford functional calculus and formally replace x by T in the slice hyperholomorphic Cauchy-formula, then we obtain its natural generalisation to the quaternionic setting [28].

Definition 2.65 (S-functional calculus for bounded operators)**.** Let $T \in \mathcal{B}(V)$, choose $\mathsf{i} \in \mathbb{S}$ and set $ds_{\mathsf{i}} = -\mathsf{i}\, ds$. For $f \in \mathcal{SH}_L(\sigma_S(T))$, we choose a bounded slice Cauchy domain U with $\sigma_S(T) \subset U$ and $cl(U) \subset \mathcal{D}(f)$ and define

$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s). \tag{2.32}$$

For $f \in \mathcal{SH}_R(\sigma_S(T))$, we choose again a bounded slice Cauchy domain U with $\sigma_S(T) \subset U$ and $cl(U) \subset \mathcal{D}(f)$ and define

$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathsf{i}})} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T). \tag{2.33}$$

These integrals are independent of the choices of the slice domain U and the imaginary unit $\mathsf{i} \in \mathbb{S}$.

The following properties of the S-functional calculus can be found in [4, 28, 36].

Lemma 2.66. *Let $T \in \mathcal{B}(V)$.*

(i) For any $f \in \mathcal{SH}_L(\sigma_S(T))$ and any $f \in \mathcal{SH}_R(\sigma_S(T))$, the operator $f(T)$ is bounded.

(ii) If $f, g \in \mathcal{SH}_L(\sigma_{SX}(T))$ and $a \in \mathbb{H}$, then $(fa + g)(T) = f(T)a + g(T)$. If $f, g \in \mathcal{SH}_R(\sigma_{SX}(T))$ and $a \in \mathbb{H}$, then $(af + g)(T) = af(T) + g(T)$.

(iii) If $f \in \mathcal{SH}(\sigma_S(T))$ and $g \in \mathcal{SH}_L(\sigma_S(T))$, then $(fg)(T) = f(T)g(T)$. Similarly, if $f \in \mathcal{SH}_R(\sigma_S(T))$ and $g \in \mathcal{SH}(\sigma_S(T))$, then also $(fg)(T) = f(T)g(T)$.

(iv) If $g \in \mathcal{SH}(\sigma_S(T))$, then $\sigma_S(g(T)) = g(\sigma_S(T))$ and $f(g(T)) = (f \circ g)(T)$ for any $f \in \mathcal{SH}_L(g(\sigma_S(T)))$ and any $f \in \mathcal{SH}_R(g(\sigma_S(T)))$.

(v) If σ is an open and closed subset of $\sigma_{SX}(T)$, let χ_σ be equal to 1 on an axially symmetric neighborhood of σ in $\mathbb{H}$ and equal to 0 on an axially symmetric neighborhood of $\sigma_{SX}(T) \setminus \sigma$ in $\mathbb{H}$. Then $\chi_\sigma \in \mathcal{SH}(\sigma_{SX}(T))$ and $\chi_\sigma(T)$ is a projection onto a right linear subspace of V that is invariant under T. Moreover, if we denote the restriction of T to the range of $\chi_\sigma(T)$ by T_σ, then $\sigma_{SX}(T_\sigma) = \sigma$.

Finally, using the above results, we can also introduce the S-functional calculus for unbounded operators.

Definition 2.67. We say that a function is left (or right) slice hyperholomorphic at infinity, if f is left (or right) slice hyperholomorphic on its domain $\mathcal{D}(f)$ and if there exists $r > 0$ such that $\mathbb{H} \setminus B_r(0)$ is contained in $\mathcal{D}(f)$ and the limit $f(\infty) := \lim_{x\to\infty} f(x)$ exists.

Let $T \in \mathcal{K}(V)$ and assume that there exists a real point $\alpha \in \rho_S(T)$. We define the function $\Phi_\alpha : \mathbb{H}_\infty \to \mathbb{H}_\infty$ by

$$\Phi_\alpha(x) := (\alpha - x)^{-1}, \qquad \Phi_\alpha(\infty) = 0, \quad \Phi_\alpha(\alpha) = 0$$

for $x \in \mathbb{H} \setminus \{\alpha\}$ and we set $\Phi_\alpha(T) := (\alpha\mathcal{I} - T)^{-1} = S_L^{-1}(\alpha, T) \in \mathcal{B}(V)$. The function Φ_α is intrinsic slice hyperholomorphic on $\mathbb{H} \setminus \{\alpha\}$ and at infinity and the mapping $f \mapsto f \circ \Phi_\alpha$ defines a bijection between $\mathcal{SH}_L(\sigma_{SX}(T))$ and $\mathcal{SH}_L(\sigma_S(\Phi_\alpha(T)))$ resp. between $\mathcal{SH}_R(\sigma_{SX}(T))$ and $\mathcal{SH}_R(\sigma_S(\Phi_\alpha(T)))$, where $\sigma_{SX}(T) = \sigma_S(T) \cup \{\infty\}$ because T was assumed to be unbounded.

Definition 2.68 (S-functional calculus for closed operators)**.** Let $T \in \mathcal{K}(V)$ be unbounded with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. We choose $\alpha \in \rho_S(T) \cap \mathbb{R}$ and define for any function $f \in \mathcal{SH}_L(\sigma_{SX}(T))$ or $f \in \mathcal{SH}_R(\sigma_{SX}(T))$ the operator $f(T)$ as

$$f(T) := \left(f \circ \Phi_\alpha^{-1}\right)(\Phi_\alpha(T)), \tag{2.34}$$

where this operator is intended in the sense of Definition 2.65.

Theorem 2.69. *Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. For any $f \in \mathcal{SH}_L(\sigma_S(T) \cup \{\infty\})$, any unbounded slice Cauchy domain U with $\sigma_S(T) \subset U$ and $cl(U) \subset \mathcal{D}(f)$ and any $\mathsf{i} \in \mathbb{S}$, we have*

$$f(T) := f(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_\mathsf{i})} S_L^{-1}(s, T)\, ds_\mathsf{i}\, f(s). \tag{2.35}$$

For any $f \in \mathcal{SH}_R(\sigma_S(T) \cup \{\infty\})$, any unbounded slice Cauchy domain U with $\sigma_S(T) \subset U$ and $cl(U) \subset \mathcal{D}(f)$ and any $\mathsf{i} \in \mathbb{S}$, we have

$$f(T) := f(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_\mathsf{i})} f(s)\, ds_\mathsf{i}\, S_R^{-1}(s, T). \tag{2.36}$$

These integrals are independent of the choices of the slice domain U and the imaginary unit $\mathsf{i} \in \mathbb{S}$ and $f(T)$ is in turn independent of the choice of $\alpha \in \rho_S(T) \cap \mathbb{R}$ used in (2.34)

The S-functional calculus allows to develop a theory of strongly continuous groups and semigroups of quaternionic linear operators that is analogue to the one for complex linear operators [32].

Definition 2.70. A family of bounded right-linear operators $(\mathcal{U}(t))_{t\in[0,+\infty)}$ on V is called a strongly continuous quaternionic semigroup if

$$\mathcal{U}(0) = \mathcal{I} \qquad \text{and} \qquad \mathcal{U}(t_1 + t_2) = \mathcal{U}(t_1)\mathcal{U}(t_2) \quad \forall t_1, t_2 \geq 0,$$

and if $t \mapsto \mathcal{U}(t)\mathbf{v}$ is a continuous function on $[0, +\infty)$ for any $\mathbf{v} \in V$.

Definition 2.71. Let $(\mathcal{U}(t))_{t\geq 0}$ be a strongly continuous quaternionic semigroup. Set

$$\operatorname{dom}(T) := \left\{ \mathbf{v} \in V : \lim_{h\to 0^+} \frac{1}{h}(\mathcal{U}(h)\mathbf{v} - \mathbf{v}) \text{ exists} \right\}$$

and

$$T\mathbf{v} = \lim_{h\to 0^+} \frac{1}{h}(\mathcal{U}(h)\mathbf{v} - \mathbf{v}), \quad \mathbf{v} \in \operatorname{dom}(T).$$

The operator T is called the quaternionic infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t\geq 0}$.We indicate that T is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ by writing $\mathcal{U}_T(t)$ instead of $\mathcal{U}(t)$.

The set $\operatorname{dom}(T)$ is a right subspace that is dense in V and $T : \operatorname{dom}(T) \to V$ is a right linear closed quaternionic operator.

Theorem 2.72. *Let $(\mathcal{U}_T(t))_{t\geq 0}$ be a strongly continuous quaternionic semigroup and let T be its quaternionic infinitesimal generator. Then*

$$\omega_0 := \lim_{t\to+\infty} \frac{1}{t} \ln \|\mathcal{U}_T(t)\| < +\infty.$$

If $s \in \mathbb{H}$ with $\operatorname{Re}(s) > \omega_0$ then s belongs $\rho_S(T)$ and

$$S_R^{-1}(s,T) = \int_0^{+\infty} e^{-ts}\mathcal{U}_T(t)\, dt.$$

The question whether a closed linear operator is the infinitesimal generator of a strongly continuous semigroup is answered by the Hille-Yosida-Phillips theorem.

Theorem 2.73. *Let T be a closed linear operator with dense domain. Then T is the infinitesimal generator of a strongly continuous semigroup if and only if there exist constants $\omega \in \mathbb{R}$ and $M > 0$ such that $\sigma_S(T) \subset \{s \in \mathbb{H} : \operatorname{Re}(s) \leq \omega\}$ and such that for any $s_0 \in \mathbb{R}$ with $s_0 > \omega$*

$$\left\|(S_R^{-1}(s_0,T))^n\right\| \leq \frac{M}{(s_0-\omega)^n} \qquad \textit{for } n \in \mathbb{N}.$$

We consider the problem of characterising when a strongly continuous semigroup of operators $(\mathcal{U}_T(t))_{t\geq 0}$ can be extended to a group $(\mathcal{U}_T(t))_{t\in\mathbb{R}}$ of operators. This extension is unique if it exists and if the family $\mathcal{U}_-(t) = \mathcal{U}_T(-t)$, $t \geq 0$, is a strongly continuous semigroup. Consider the identity

$$\frac{1}{h}[\mathcal{U}_-(h)\mathbf{v} - \mathbf{v}] = \frac{1}{-h}[-\mathcal{U}_T(-2)[\mathcal{U}_T(2-h)\mathbf{v} - \mathcal{U}_T(2)\mathbf{v}]], \quad \text{for } \ h \in (0,1).$$

By taking the limit for $h \to 0$ we have that the infinitesimal generator of $\mathcal{U}_-(t)$ is $-T$ and $\operatorname{dom}(-T) = \operatorname{dom}(T)$. In this case T is called the quaternionic infinitesimal generator of the group $(\mathcal{U}_T(t))_{t\in\mathbb{R}}$. The next theorem gives a necessary and sufficient condition such that a semigroup can be extended to a group [32, Theorem 5.1].

Theorem 2.74. *An operator $T \in \mathcal{K}(V)$ is the quaternionic infinitesimal generator of a strongly continuous group of bounded quaternionic linear operators if and only if there exist real numbers $M > 0$ and $\omega \geq 0$ such that*

$$\|(S_R^{-1}(s_0,T))^n\| \leq \frac{M}{(|s_0|-\omega)^n}, \quad \textit{for } \omega < |s_0|. \tag{2.37}$$

If T generates the group $(\mathcal{U}_T(t))_{t\in\mathbb{R}}$, then $\|\mathcal{U}_T(t)\| \leq Me^{\omega|t|}$.

2.4 The Spectral Theorem for Normal Operators

We conclude the preliminaries by recalling the spectral theorem for bounded normal operators on a quaternionic Hilbert space $\mathcal{H}$. This theorem was shown in [5] and later also in [51], which is largely a rewriting of [5, 86]. The two articles use however different strategies for proving this theorem and they consider two different approaches towards spectral integration in the quaternionic setting.

We recall that $\mathcal{H}$ is only a right Banach space. Therefore the space $\mathcal{B}(\mathcal{H})$ of all bounded right linear operators on $\mathcal{H}$ is only a real Banach space, cf. Remark 2.45.

Definition 2.75. A spectral measure E over $\mathbb{C}_{\mathsf{i}}^{\geq}$ on the quaternionic Hilbert space $\mathcal{H}$ is a set function defined on the Borel sets $\mathsf{B}\big(\mathbb{C}_{\mathsf{i}}^{\geq}\big)$ of $\mathbb{C}_{\mathsf{i}}^{\geq}$ the values of which are orthogonal projections on $\mathcal{H}$ such that

(i) $E\big(\mathbb{C}_{\mathsf{i}}^{\geq}\big) = \mathcal{I}$

(ii) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathsf{B}\big(\mathbb{C}_{\mathsf{i}}^{\geq}\big)$

(iii) $E(\cup_{n\in\mathbb{N}}\Delta_n)\mathbf{v} = \sum_{n\in\mathbb{N}} E(\Delta_n)\mathbf{v}$ for all $\mathbf{v} \in \mathcal{H}$ and any sequence $(\Delta_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathsf{B}\big(\mathbb{C}_{\mathsf{i}}^{\geq}\big)$.

In the complex setting, integrals with respect to a spectral measure E defined on the Borel sets $\mathsf{B}(\mathbb{C})$ of $\mathbb{C}$ are defined via approximation by simple functions. For a simple function $f(s) = \sum_{\ell=1}^{n} a_\ell \chi_{\Delta_\ell}(s)$, where χ_{Δ_ℓ} denotes the characteristic function of the set $\Delta_\ell \in \mathsf{B}(\mathbb{C})$, one defines

$$\int_{\mathbb{C}} f(z)\, dE(z) = \sum_{\ell=1}^{n} a_\ell E(\Delta_\ell), \tag{2.38}$$

and for arbitrary bounded and measurable f one defines

$$\int_{\mathbb{C}} f(z)\, dE(z) = \lim_{n\to+\infty} \int_{\mathbb{C}} f_n(z)\, dE(z), \tag{2.39}$$

where f_n is a suitable sequence of simple functions converging uniformly to f. In order to proceed similarly in the quaternionic setting, we need additional structure, a left multiplication, on $\mathcal{H}$. Otherwise only real-valued functions can be integrated because (2.38) is only defined for real coefficients a_ℓ.

Definition 2.76. A left multiplication on $\mathcal{H}$ is a homomorphism $\mathcal{L} : \mathbb{H} \to \mathcal{B}(\mathcal{H})$ of real algebras acting as $a \mapsto L_a$ such that $L_a = a\mathcal{I}$ for any $a \in \mathbb{R}$ and $(L_a)^* = L_{\overline{a}}$ for any $a \in \mathbb{H}$.

If it is clear which left-multiplication we consider, we will also write $a\mathbf{v}$ instead of $L_a\mathbf{v}$. Since $\mathcal{L}$ is a real algebra-homomorphism and $a\mathbf{v} = L_a\mathbf{v} = \mathbf{v}a$ for any $a \in \mathbb{R}$, such a left multiplication turns $\mathcal{H}$ into a two-sided quaternionic vector space. Since moreover $L_{\overline{a}} = (L_a)^*$, we have $\langle \mathbf{u}, a\mathbf{v}\rangle = \langle \overline{a}\mathbf{u}, \mathbf{v}\rangle$ and in turn

$$\|a\mathbf{v}\|^2 = \langle a\mathbf{v}, a\mathbf{v}\rangle = \langle \mathbf{v}, \overline{a}a\mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{v}\rangle |a|^2 = |a|^2 \|\mathbf{v}\|^2.$$

Hence, any left multiplication as in Definition 2.76 turns $\mathcal{H}$ into a two-sided quaternionic Banach space.

If $\mathbf{B} = (\mathbf{b}_\ell)_{\ell\in\Lambda}$ is an orthonormal basis of $\mathcal{H}$, then the left multiplication induced by $\mathbf{B}$ is the map $\mathcal{L}_{\mathbf{B}}$ given by $a \mapsto L_a := \sum_{\ell\in\Lambda} \mathbf{b}_\ell a\langle \mathbf{b}_\ell, \mathbf{v}\rangle$ for $a \in \mathbb{H}$. If on the other hand $\mathcal{H}$ is endowed with a left multiplication, then the real component space $\mathcal{H}_\mathbb{R}$ defined in (2.21) is, endowed with the scalar product on $\mathcal{H}$, a real Hilbert space. Indeed, since $a\mathbf{v} = \mathbf{v}a$ for $a \in \mathbb{H}$ and $\mathbf{v} \in \mathcal{H}_\mathbb{R}$, we have for $\mathbf{u}, \mathbf{v} \in \mathcal{H}_\mathbb{R}$ that

$$a\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}\overline{a}, \mathbf{v}\rangle = \langle \overline{a}\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, a\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}a\rangle = \langle \mathbf{u}, \mathbf{v}\rangle a \qquad \forall a \in \mathbb{H},$$

which implies that $\langle \mathbf{u}, \mathbf{v}\rangle$ is real. Any orthonormal basis $(\mathbf{b}_\ell)_{\ell\in\Lambda}$ of this space is also an orthonormal basis of the quaternionic Hilbert space $\mathcal{H}$ and induces the left multiplication given on $\mathcal{H}$ as $a\mathbf{v} = a\sum_{\ell\in\Lambda} \mathbf{b}_\ell\langle \mathbf{b}_\ell, \mathbf{v}\rangle = \sum_{\ell\in\Lambda} \mathbf{b}_\ell a\langle \mathbf{b}_\ell, \mathbf{v}\rangle$ for any $a \in \mathbb{H}$. (This fact was also shown in [51, Theorem 4.3] with different arguments).

For any imaginary unit $\mathsf{i} \in \mathbb{S}$, the mapping $\mathbf{v} \mapsto \mathsf{i}\mathbf{v} = L_\mathsf{i}\mathbf{v} = \sum_{\ell\in\Lambda} \mathbf{b}_\ell \mathsf{i}\langle \mathbf{b}_\ell, \mathbf{v}\rangle$ is a unitary anti-selfadjoint operator on $\mathcal{H}$. Conversely, if J is a unitary and anti-selfadjoint operator and $\mathsf{i} \in \mathbb{S}$ is any imaginary unit, then there exists an orthonormal basis $(\mathbf{b}_{\mathsf{i},\ell})_{\ell\in\Lambda}$ of $\mathcal{H}$ such that $\mathsf{J}\mathbf{v} = \sum_{\ell\in\Lambda} \mathbf{b}_{\mathsf{i},\ell}\mathsf{i}\langle \mathbf{b}_{\mathsf{i},\ell}, \mathbf{v}\rangle$. Hence, applying J can be considered as a multiplication with i on the left. Note however, that i is not determined by J. We can interpret applying J as a multiplication with any $\mathsf{i} \in \mathbb{S}$, but obviously we cannot make this identification for two different imaginary units simultaneously. If we choose a different imaginary unit $\mathsf{j} \in \mathbb{S}$, then we do moreover not have $\mathsf{J}\mathbf{v} = \sum_{\ell\in\Lambda} \mathbf{b}_{\mathsf{i},\ell}\mathsf{j}\langle \mathbf{b}_{\mathsf{i},\ell}, \mathbf{v}\rangle$. Instead, $\mathsf{J}\mathbf{v} = \sum_{\ell\in\Lambda} \mathbf{b}_{\mathsf{j},\ell}\mathsf{j}\langle \mathbf{b}_{\mathsf{j},\ell}, \mathbf{v}\rangle$, where $\mathbf{b}_{\mathsf{j},\ell} = \mathbf{b}_{\mathsf{i},\ell}h$ with $h \in \mathbb{H}$ such that $|h| = 1$ and $\mathsf{j} = h^{-1}\mathsf{i}h$, cf. Lemma 2.1.

Any left multiplication on $\mathcal{H}$ is fully determined by its multiplications $L_\mathsf{i}, L_\mathsf{j}$ with two different imaginary units $\mathsf{i}, \mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$. Indeed, setting $\mathsf{k} = \mathsf{ij}$, any quaternion $a \in \mathbb{H}$ can then be written as $a = a_0 + \mathsf{i}a_1 + \mathsf{j}a_2 + \mathsf{k}a_3$. We must then obviously have $a\mathbf{v} = \mathbf{v}a_0 + L_\mathsf{i}\mathbf{v}a_1 + L_\mathsf{j}\mathbf{v}a_2 + L_\mathsf{i}L_\mathsf{j}\mathbf{v}a_3$. Observe that L_i and L_j are two unitary anti-selfadjoint operators that anti-commute. Conversely, if I and J are two such operators, then we can choose $\mathsf{i}, \mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$ and define a left multiplication on $\mathcal{H}$ by setting $L_\mathsf{i} := \mathsf{I}$ and $L_\mathsf{j} := \mathsf{J}$. However, i and j are arbitrary. Hence, any couple consisting of two anti-commuting unitary and anti-selfadjoint operators can be used to generate an infinite number of distinct left multiplications on $\mathcal{H}$.

Assume now that we are given a spectral measure E on $\mathcal{H}$ over $\mathbb{C}_\mathsf{i}^{\geq}$ for some imaginary unit $\mathsf{i} \in \mathbb{S}$ and a unitary anti-selfadjoint operator J that commutes with E. We can interpret the application of J as a multiplication with the imaginary unit i from the left, i.e. we set $\mathsf{i}\mathbf{v} = \mathsf{J}\mathbf{v}$ for $\mathbf{v} \in \mathcal{H}$. Since E and J commute, the imaginary unit i does also commute with E so that (2.38) is meaningful not only for real coefficients a_ℓ, but even for coefficients in $\mathbb{C}_\mathsf{i}$. We can hence define spectral integrals for $\mathbb{C}_\mathsf{i}$-valued functions via the usual procedure in (2.38) and (2.39). Observe that if $f(z) = \alpha(z) + \mathsf{i}\beta(z)$ with $\alpha(z), \beta(z) \in \mathbb{R}$ and $f_n(z) = \sum_{\ell=1}^{N_n}(\alpha_{n,\ell} + \mathsf{i}\beta_{n,\ell})\chi_{\Delta_n}(z)$ with $\alpha_{n,\ell}, \beta_{n,\ell} \in \mathbb{R}$ for $n \in \mathbb{N}$

is a sequence of measurable simple functions tending uniformly to f, then

$$\begin{aligned}\int_{\mathbb{C}_{\mathsf{i}}^{\geq}} f(z)\, dE(z) &= \lim_{n\to+\infty} \sum_{\ell=1}^{N_n} (\alpha_{n,\ell} + \mathsf{i}\beta_{n,\ell}) E(\Delta_n) \\ &= \lim_{n\to+\infty} \sum_{\ell=1}^{N_n} \alpha_{n,\ell} E(\Delta_n) + \mathsf{J} \lim_{n\to+\infty} \sum_{\ell=1}^{N_n} \beta_{n,\ell} E(\Delta_n) \\ &= \int_{\mathbb{C}_{\mathsf{i}}^{\geq}} \alpha(z)\, dE(z) + \mathsf{J} \int_{\mathbb{C}_{\mathsf{i}}^{\geq}} \beta(z)\, dE(z).\end{aligned}$$

The characterization of this spectral integral given in the following lemma was shown in [5, Lemma 5.3].

Lemma 2.77. *Let E be a quaternionic spectral measure on $\mathcal{H}$ over $\mathbb{C}_{\mathsf{i}}^{\geq}$ and let J be a unitary and anti-selfadjoint operator J that commutes with E. If we interpret J as the multiplication with i from the left and f is a bounded measurable $\mathbb{C}_{\mathsf{i}}$-valued function on $\mathbb{C}_{\mathsf{i}}^{\geq}$ with $f(z) = \alpha(z) + \mathsf{i}\beta(z)$ where $\alpha(z), \beta(z) \in \mathbb{R}$, then*

$$\left\langle \mathbf{u}, \int_{\mathbb{C}_{\mathsf{i}}^{\geq}} f(z)\, dE(z)\mathbf{v} \right\rangle = \int_{\mathbb{C}_{\mathsf{i}}^{\geq}} \alpha(z)\, d\langle \mathbf{u}, E(z)\mathbf{v}\rangle + \int_{\mathbb{C}_{\mathsf{i}}^{\geq}} \beta(z)\, d\langle \mathbf{u}, E(z)\mathsf{J}\mathbf{v}\rangle$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{H}$, where the quaternion-valued measure $\langle \mathbf{u}, E\mathbf{v}\rangle$ on $\big(\mathbb{C}_{\mathsf{i}}^{\geq}, \mathsf{B}\big(\mathbb{C}_{\mathsf{i}}^{\geq}\big)\big)$ is given by $\Delta \mapsto \langle \mathbf{u}, E(\Delta)\mathbf{v}\rangle$.

If T is a bounded normal operator on $\mathcal{H}$ (i.e. a bounded operator that commutes with its adjoint T^*), then we have

$$T = A + \mathsf{J}_0 B = \frac{1}{2}(T + T^*) + \mathsf{J}_0 \frac{1}{2} |T - T^*|, \tag{2.40}$$

where $A := \frac{1}{2}(T + T^*)$ is self adjoint. The operator $C := \frac{1}{2}(T - T^*)$ is anti-selfadjoint. Hence the polar decomposition theorem [49, Theorem 2.20] implies the existence of an anti-selfadjoint partially unitary operator J_0 such that $C = \mathsf{J}_0|C|$, where $|C| = \sqrt{C^*C}$ is the unique positive square root of the positive operator C^*C. The operator J_0 is partially unitary with $\ker \mathsf{J}_0 = \ker C$, i.e the restriction of J_0 to $\operatorname{ran} J_0 = cl(\operatorname{ran} C) = \ker \mathsf{J}_0^{\perp}$ is a unitary operator. Moreover, A, B, and J_0 commute mutually and also with any operator that commutes with T and T^*. Finally, we have $T^* = A - \mathsf{J}_0 B$. Teichmüller already showed these facts in 1936 in [82], but proofs in English can also be found in [49]. As shown in [49], the operator J_0 can furthermore be extended to a unitary and anti-selfadjoint operator J on all of $\mathcal{H}$ that commutes with T and T^* such that

$$T = A + \mathsf{J}B. \tag{2.41}$$

We however stress that unlike J_0 the operator J is not unique.

Alpay, Colombo, and Kimsey used these facts to show the spectral theorem for normal quaternionic linear operators in [5]. We shall only recall the result for bounded operators [5, Theorem 4.7], the proof of which is based on the continuous functional calculus for quaternionic normal operators introduced in [49]. Let

$$p(z_0, z_1) = \sum_{0\leq|\ell|\leq n} a_\ell z_0^{\ell_1} z_1^{\ell_2}$$

with the multi-index $\ell = (\ell_1, \ell_2)$ be a polynomial in the variables z_0 and z_1 with real coefficients a_ℓ. The function $p(s) := p(s_0, s_1)$ for $s = s_0 + \mathbf{i}_s s_1 \in \mathbb{H}$ with $s_1 \geq 0$ is then a real-valued and hence intrinsic slice function. For a normal operator T decomposed as in (2.41), we can then define

$$p(T) := p(A, B) = \sum_{0 \leq |\ell| \leq n} a_\ell A^{\ell_1} B^{\ell_2}. \tag{2.42}$$

The operator J is essentially a multiplication with imaginary units $\mathbf{i} \in \mathbb{S}$. Hence, we can define

$$f(T) := p_1(A, B) + J p_2(A, B)$$

for any intrinsic slice function function $f(s) = p_1(s_0, s_1) + \mathbf{i} p_2(s_0, s_1)$ with real-valued polynomials p_1 and p_2. We have $\|f(T)\| = \sup_{z \in \sigma_S(T)} |f(z)|$. If f is any continuous slice function on $\sigma_S(T)$, then the Weierstrass approximation theorem implies the existence of a sequence of intrinsic slice functions $f_n = p_{n,1} + \mathbf{i} p_{n,2}$ with real-valued polynomials $p_{n,1}, p_{n,2}$ such that f_n tends to f uniformly on $\sigma_S(T)$. Hence we can define $f(T) = \lim_{n \to +\infty} f_n(T)$, where the sequence $f_n(T)$ converges in $\mathcal{B}(\mathcal{H})$. We then have $\sigma_S(f(T)) = f(\sigma_S(T))$.

Alpay, Colombo, and Kimsey follow in [5] a well-known strategy from the complex case in order to show the spectral theorem for normal quaternionic linear operators. We choose $\mathbf{i} \in \mathbb{S}$. By Remark 2.11, the mapping $f \mapsto f_\mathbf{i} = f|_{\mathbb{C}_\mathbf{i}^\geq}$ determines a bijective relation between the set $\mathcal{SC}(\sigma_S(T), \mathbb{R})$ of all real valued continuous slice functions on $\sigma_S(T)$ and the set $C(\Omega_\mathbf{i}, \mathbb{R})$ of all continuous functions on $\Omega_\mathbf{i} := \sigma_S(T) \cap \mathbb{C}_\mathbf{i}^\geq$ with values in $\mathbb{R}$. It also defines a bijective relation between the set $\mathcal{SC}(\sigma_S(T))$ of all continuous intrinsic slice functions on $\sigma_S(T)$ and the set $C_0(\Omega_\mathbf{i}, \mathbb{C}_\mathbf{i})$ of all continuous $\mathbb{C}_\mathbf{i}$-valued functions $f_\mathbf{i}$ on $\Omega_\mathbf{i}$ with $f_\mathbf{i}(\mathbb{R} \cap \Omega_\mathbf{i}) \subset \mathbb{R}$. For any $\mathbf{v} \in \mathcal{H}$, the mapping $f_\mathbf{i} \mapsto \langle \mathbf{v}, f(T)\mathbf{v} \rangle$ is a continuous and positive $\mathbb{R}$-linear functional on $C(\Omega_\mathbf{i}, \mathbb{R})$. The Riesz representation theorem hence implies the existence of a positive Borel measure $\mu_{\mathbf{v},\mathbf{v}}$ on $\Omega_\mathbf{i}$ such that

$$\langle \mathbf{v}, f(T)\mathbf{v} \rangle = \int_{\Omega_\mathbf{i}} f_\mathbf{i}(z)\, d\mu_{\mathbf{v},\mathbf{v}}(z) \qquad \forall f_\mathbf{i} \in C(\Omega_\mathbf{i}, \mathbb{R}).$$

Using the polarisation identity, the authors deduced for any $\mathbf{u}, \mathbf{v} \in \mathcal{H}$ the existence of a quaternion-valued Borel measure $\mu_{\mathbf{u},\mathbf{v}}$ such that

$$\langle \mathbf{u}, f(T)\mathbf{v} \rangle = \int_{\Omega_\mathbf{i}} f_\mathbf{i}(z)\, d\mu_{\mathbf{u},\mathbf{v}}(z) \qquad \forall f_\mathbf{i} \in C(\Omega_\mathbf{i}, \mathbb{R}).$$

For each $\Delta \in \mathsf{B}(\Omega_\mathbf{i})$ and each $\mathbf{u} \in \mathcal{H}$, the map $\mathbf{v} \mapsto \mu_{\mathbf{u},\mathbf{v}}(\Delta)$ is then a continuous quaternionic right linear functional on $\mathcal{H}$. Hence there exists a unique $\mathbf{w} \in \mathcal{H}$ such that $\mu_{\mathbf{u},\mathbf{v}}(\Delta) = \langle \mathbf{w}, \mathbf{v} \rangle$ for all $\mathbf{v} \in \mathcal{H}$. We define $E_\mathbf{i}(\Delta)\mathbf{u} := \mathbf{w}$ such that

$$\langle E_\mathbf{i}(\Delta)\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle = \mu_{\mathbf{u},\mathbf{v}}(\Delta).$$

The operator $E_\mathbf{i}(\Delta)$ is then an orthogonal projection on $\mathcal{H}$ and the mapping $\Delta \mapsto E_\mathbf{i}(\Delta)$ turns out to be a quaternionic spectral measure over $\mathbb{C}_\mathbf{i}^\geq$ on $\mathcal{H}$. Finally, one arrives at the spectral theorem for bounded normal operators on quaternionic Hilbert spaces [5, Theorem 4.7].

Theorem 2.78. *Let $T \in \mathcal{B}(\mathcal{H})$ be normal, let $T = A + \mathsf{J}B$ as in* (2.41)*, consider* J *as a multiplication with* $\mathsf{i} \in \mathbb{S}$ *and set* $\Omega_{\mathsf{i}} := \sigma_S(T) \cap \mathbb{C}_{\mathsf{i}}^{\geq}$ *. Then there exists a unique quaternionic spectral measure* E_{i} *over* $\mathbb{C}_{\mathsf{i}}^{\geq}$ *such that*

$$f(T) = \int_{\Omega_{\mathsf{i}}} f_{\mathsf{i}}(z)\, dE_{\mathsf{i}}(z) \qquad \forall f \in \mathcal{SC}(\sigma_S(T)), \tag{2.43}$$

where the spectral integral is intended as above, cf. also Lemma 2.77. Moreover, if $\mathsf{j} \in \mathbb{S}$ *and we define* $\varphi_{\mathsf{i},\mathsf{j}}(z_0 + \mathsf{i}z_1) = z_0 + \mathsf{j}z_1$*, then* $E_{\mathsf{i}}(\Delta) = E_{\mathsf{j}}(\varphi_{\mathsf{i},\mathsf{j}}(\Delta))$ *for any* $\Delta \in \mathsf{B}(\Omega_{\mathsf{i}})$.

Remark 2.79. We want to point out that the construction of the spectral measure E only used real-valued functions on Ω_{i}. These functions are restrictions of real-valued slice functions.

Moreover, the spectral measure was actually constructed using functions of A and B in (2.42). Hence, it depends only on A and B but not on J. The spectral measures of $T = A+\mathsf{J}B$ and $T^* = A-\mathsf{J}B$ therefore coincide. In the quaternionic setting, a normal operator is not fully determined by its spectral measure. Essential information is also contained in the operator J, which will force us to introduce the notion of a spectral system in Definition 9.19.

In [51] the authors go one step further in their theory of spectral integration and introduce *intertwining quaternionic projection-valued measures*, which allow them to define spectral integrals for $\mathbb{H}$-valued and not only for $\mathbb{C}_{\mathsf{i}}$-valued functions.

Definition 2.80. Let $\mathsf{i} \in \mathbb{S}$. An intertwining quaternionic projection-valued measure (for short iqPVM) over $\mathbb{C}_{\mathsf{i}}^{\geq}$ on $\mathcal{H}$ is a couple $\mathcal{E} = (E, \mathcal{L})$ consisting of a quaternionic spectral measure E over $\mathbb{C}_{\mathsf{i}}$ and a left multiplication $\mathcal{L} : \mathbb{H} \to \mathcal{B}(\mathcal{H}), a \mapsto L_a$ that commutes with E, that is $E(\Delta)L_a = L_aE(\Delta)$ for any $\Delta \in \mathsf{B}(\mathbb{C}_{\mathsf{i}}^{\geq})$ and any $a \in \mathbb{H}$.

With respect to an iqPVM $\mathcal{E}$ one can define spectral integrals of functions with arbitrary values in the quaternions since (2.38) is meaningful for simple functions with arbitrary quaternionic coefficients a_ℓ. Ghiloni, Moretti, and Perotti show the following result [51, Theorem 4.1]

Theorem 2.81. *Let $T \in \mathcal{B}(\mathcal{H})$ be normal and let $\mathsf{i} \in \mathbb{S}$. There exists an iqPVM $\mathcal{E} = (E, \mathcal{L})$ over $\mathbb{C}_{\mathsf{i}}^{\geq}$ on $\mathcal{H}$ such that*

$$T = \int_{\mathbb{C}_{\mathsf{i}}^{\geq}} z\, d\mathcal{E}(z). \tag{2.44}$$

The spectral measure E is uniquely determined by T and the left multiplication is uniquely determined for $a \in \mathbb{C}_{\mathsf{i}}$ on $\ker(T - T^*)^{\perp}$*. Precisely, we have for any other left multiplication $\mathcal{L}'$ such that $\mathcal{E}' = (E, \mathcal{L}')$ is an iqPVM satisfying* (2.44) *that $L_a\mathbf{v} = L'_a\mathbf{v}$ for any $a \in \mathbb{C}_{\mathsf{i}}$ and any* $\mathbf{v} \in \ker(T - T^*)^{\perp}$*. (Even more specifically, we have* $\mathsf{i}\mathbf{v} = \mathsf{J}_0\mathbf{v}$ *for any* $\mathbf{v} \in \ker(T - T^*)^{\perp} = \operatorname{ran} \mathsf{J}_0$*.)*

The proof of the above result in [51] follows a completely different strategy than the proof of the spectral theorem in [5]. Similar to [78, 87], it reduces the quaternionic problem to a complex problem in order to apply the well-known results from complex

operator theory. Instead of working with the symplectic image, the authors however apply the classical results to the restriction of T to a suitably chosen complex component space.

We decompose $T = A + \mathsf{J}B$ as in (2.41) and choose $\mathsf{i}, \mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$. Since J is unitary and anti-selfadjoint, there exists an orthonormal basis $(\mathbf{b}_\ell)_{\ell\in\Lambda}$ of $\mathcal{H}$ such that $\mathsf{J}\mathbf{b}_\ell = \mathbf{b}_\ell\mathsf{i}$. We can write any $\mathbf{v} = \sum_{\ell\in\Lambda}\mathbf{b}_\ell a_\ell \in \mathcal{H}$ with $a_\ell \in \mathbb{H}$ as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2\mathsf{j} = \sum_{\ell\in\Lambda}\mathbf{b}_\ell a_{\ell,1} + \sum_{\ell\in\Lambda}\mathbf{b}_\ell a_{\ell,2}\mathsf{j}$ with $a_{\ell,1}, a_{\ell,2} \in \mathbb{C}_\mathsf{i}$ such that $a_\ell = a_{\ell,1} + a_{\ell,2}\mathsf{j}$. Setting $\mathcal{H}^+_{\mathsf{J},\mathsf{i}} := cl(\operatorname{span}_{\mathbb{C}_\mathsf{i}}\{\mathbf{b}_\ell : \ell \in \Lambda\})$ and $\mathcal{H}^-_{\mathsf{J},\mathsf{i}} := \mathcal{H}^+_{\mathsf{J},\mathsf{i}}\mathsf{j}$, we find that $\mathcal{H} = \mathcal{H}^+_{\mathsf{J},\mathsf{i}} \oplus \mathcal{H}^-_{\mathsf{J},\mathsf{i}}$. The spaces $\mathcal{H}^+_{\mathsf{J},\mathsf{i}}$ and $\mathcal{H}^-_{\mathsf{J},\mathsf{i}}$ are $\mathbb{C}_\mathsf{i}$-complex Hilbert spaces if we endow them with the restriction of the right scalar multiplication on $\mathcal{H}$ to $\mathbb{C}_\mathsf{i}$ and with the scalar product on $\mathcal{H}$. Obviously $\mathcal{H}^+_{\mathsf{J},\mathsf{i}}$ consists of all eigenvectors of J associated with the eigenvalue i and since $\mathsf{ij} = \mathsf{j}(-\mathsf{i})$ the space $\mathcal{H}^-_{\mathsf{J},\mathsf{i}}$ consists of all eigenvectors of J associated with the eigenvalue $-\mathsf{i}$. For more detailed arguments we refer to [49], but we stress that these facts motivate Theorem 9.18 in this thesis.

For $\mathbf{v} \in \mathcal{H}^+_{\mathsf{J},\mathsf{i}}$, we have $\mathsf{J}(T\mathbf{v}) = T(\mathsf{J}\mathbf{v}) = (T\mathbf{v})\mathsf{i}$ and so $T\mathbf{v} \in \mathcal{H}^+_{\mathsf{J},\mathsf{i}}$. Hence, T leaves $\mathcal{H}^+_{\mathsf{J},\mathsf{i}}$ invariant. The restriction T_+ of T to $\mathcal{H}^+_{\mathsf{J},\mathsf{i}}$ defines a bounded normal operator on $\mathcal{H}^+_{\mathsf{J},\mathsf{i}}$ with $\sigma(T_+) = \sigma_S(T) \cap \mathbb{C}^{\geq}_\mathsf{i}$. Applying the spectral theorem for normal complex linear operators, one obtains a spectral measure E_+ on $\mathcal{H}^+_{\mathsf{j},\mathsf{i}}$, the support of which is $\sigma(T_+)$, such that $T_+ = \int_{\sigma(T_+)} z\,dE_+(z)$. The quaternionic linear extension E of E_+ to all of $\mathcal{H}$ is then a quaternionic spectral measure over $\mathbb{C}^{\geq}_\mathsf{i}$ on $\mathcal{H}$. This extension is obtained by writing $\mathbf{v} \in \mathcal{H}$ as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2\mathsf{j}$ with $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{H}^+_{\mathsf{J},\mathsf{i}}$ and setting $E(\Delta)\mathbf{v} = E_+(\Delta)\mathbf{v}_1 + (E(\Delta)\mathbf{v}_2)\mathsf{j}$ for $\Delta \in \mathsf{B}(\mathbb{C}^{\geq}_\mathsf{i})$. If we furthermore choose a suitable orthonormal basis $\mathbf{B} := (\mathbf{b}_\ell)_{\ell\in\Lambda}$ of the $\mathbb{C}_\mathsf{i}$-complex Hilbert space $\mathcal{H}^+_{\mathsf{J},\mathsf{i}}$, then we find that $(\mathbf{b}_\ell)_{\ell\in\Lambda}$ is also an orthonormal basis of the quaternionic Hilbert space $\mathcal{H}$ and that $\mathcal{E} = (E, \mathcal{L}_\mathbf{B})$, where $\mathcal{L}_\mathbf{B}$ is the left multiplication on $\mathcal{H}$ induced by $\mathbf{B}$, is an iqPVM such that (2.44) holds true. Again this is a rough summary of the strategy and we refer to [51] for the technical details.

Since $\mathbf{B}$ is an orthonormal basis of $\mathcal{H}^+_{\mathsf{J},\mathsf{i}}$, it consists of eigenvectors of J with respect to i. We hence find for $\mathbf{v} \in \mathcal{H}^+_{\mathsf{J},\mathsf{i}}$ that

$$\mathsf{i}\mathbf{v} = L_\mathsf{i}\mathbf{v} = \sum_{\ell\in\Lambda}\mathbf{b}_\ell\mathsf{i}\langle\mathbf{b}_\ell, \mathbf{v}\rangle = \sum_{\ell\in\Lambda}\mathsf{J}\mathbf{b}_\ell\langle\mathbf{b}_\ell, \mathbf{v}\rangle = \mathsf{J}\sum_{\ell\in\Lambda}\mathbf{b}_\ell\langle\mathbf{b}_\ell, \mathbf{v}\rangle = \mathsf{J}\mathbf{v}$$

and in turn also for arbitrary $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2\mathsf{j} \in \mathcal{H}$ with $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{H}^+_{\mathsf{J},\mathsf{i}}$ that

$$\mathsf{i}\mathbf{v} = \mathsf{i}\mathbf{v}_1 + (\mathsf{i}\mathbf{v}_2)\mathsf{j} = \mathsf{J}\mathbf{v}_1 + (\mathsf{J}\mathbf{v}_2)\mathsf{j} = \mathsf{J}\mathbf{v}.$$

Hence also in the approach using iqPVM the application of J is interpreted as multiplication with a randomly chosen imaginary unit i on the left. Moreover, this multiplication is only on $\ker(T^* - T)^\perp$ determined by T. Indeed, on this space J is coincides with J_0, whereas the extension of J_0 to a unitary anti-selfadjoint operator on all of $\mathcal{H}$ is arbitrary. The left-multiplication L_j for j with $\mathsf{j} \perp \mathsf{i}$, which together with $L_\mathsf{i} = \mathsf{J}$ fully determines $\mathcal{L}$ in $\mathcal{E}$, is completely arbitrary and not at all determined by T. We stress these facts because they shall be important in the discussion in Section 9.4.

Part I

Slice Hyperholomorphic Functional Calculi

CHAPTER 3

Properties of the Pseudo-Resolvent and the S-Resolvents

This chapter gives a precise proof of the slice hyperholomorphicity of the S-resolvents of an unbounded quaternionic linear operator. It furthermore shows that they do not have any nontrivial slice hyperholomorphic extension to a set that is larger than the S-resolvent set of this operator. Close to the spectrum, the S-resolvents explode, but in a sense that is somewhat different from the situation in the complex setting. These results are part of [21].

We start with a new series expansion for the pseudo-resolvent $\mathcal{Q}_s(T)^{-1}$. An heuristic approach for finding this expansion is to consider the immediate equation

$$\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_x(T)^{-1} = \mathcal{Q}_s(T)^{-1}(\mathcal{Q}_x(T) - \mathcal{Q}_s(T))\mathcal{Q}_x(T)^{-1} \tag{3.1}$$

and write it as

$$\mathcal{Q}_s(T)^{-1} = \mathcal{Q}_x(T)^{-1} + \mathcal{Q}_s(T)^{-1}(\mathcal{Q}_x(T) - \mathcal{Q}_s(T))\mathcal{Q}_x(T)^{-1}.$$

Recursive application of this equation then yields the series expansion proved in the following lemma. We recall that a series $\sum_{n=0}^{+\infty} T_n$ of operators $T_n \in \mathcal{B}(V)$ is called absolutely convergent if $\sum_{n=0}^{+\infty} \|T_n\| < +\infty$.

Lemma 3.1. *Let $T \in \mathcal{K}(V)$ and $x \in \rho_S(T)$ and let $s \in \mathbb{H}$. If the series*

$$\mathcal{J}(s) = \sum_{n=0}^{+\infty} (\mathcal{Q}_x(T) - \mathcal{Q}_s(T))^n \, \mathcal{Q}_x(T)^{-(n+1)} \tag{3.2}$$

converges absolutely in $\mathcal{B}(V)$, then $s \in \rho_S(T)$ and it equals the pseudo-resolvent $\mathcal{Q}_s(T)^{-1}$ of T at s.

The series converges in particular uniformly on any of the closed axially symmetric neighbourhoods

$$C_\varepsilon(x) = \{s \in \mathbb{H} : d_S(s,x) \leq \varepsilon\}$$

of x with

$$d_S(s,x) = \max\left\{2|s_0 - x_0|, \left||x|^2 - |s|^2\right|\right\}$$

and

$$\varepsilon < \frac{1}{\|T\mathcal{Q}_x(T)^{-1}\| + \|\mathcal{Q}_x(T)^{-1}\|}.$$

Proof. Let us first consider the question of convergence of the series. The sets $C_\varepsilon(x)$ are obviously axially symmetric: if s_i belongs to the sphere $[s]$ associated to s, then $s_0 = \mathrm{Re}(s) = \mathrm{Re}(s_\mathsf{i})$ and $|s|^2 = |s_\mathsf{i}|^2$. Thus $d_S(s_\mathsf{i}, x) = d_S(s,x)$ and in turn $s \in C_\varepsilon(x)$ if and only if $s_\mathsf{i} \in C_\varepsilon(x)$. Moreover, since the map $s \mapsto d_S(s,x)$ is continuous, the sets $U_\varepsilon(x) := \{s \in \mathbb{H} : d_S(s,x) < \varepsilon\}$ are open in $\mathbb{H}$. Since $U_\varepsilon(x) \subset C_\varepsilon(x)$, the sets C_ε are actually neighbourhoods of x.

In order to simplify the notation, we set

$$\Lambda(x,s) := \mathcal{Q}_x(T) - \mathcal{Q}_s(T) = 2(s_0 - x_0)T + (|x|^2 - |s|^2)\mathcal{I}.$$

Since $\mathcal{Q}_x(T)^{-1}$ maps V to $\mathrm{dom}(T^2)$ and $\Lambda(x,s)$ commutes with $\mathcal{Q}_x(T)^{-1}$ on $\mathrm{dom}(T^2)$, we have for any $s \in C_\varepsilon(x)$

$$\begin{aligned}
&\sum_{n=0}^{+\infty} \left\|\Lambda(x,s)^n \mathcal{Q}_x(T)^{-(n+1)}\right\| \\
=&\sum_{n=0}^{+\infty} \left\|\left(\Lambda(x,s)\mathcal{Q}_x(T)^{-1}\right)^n \mathcal{Q}_x(T)^{-1}\right\| \\
\leq&\sum_{n=0}^{+\infty} \left\|\Lambda(x,s)\mathcal{Q}_x(T)^{-1}\right\|^n \left\|\mathcal{Q}_x(T)^{-1}\right\|.
\end{aligned}$$

We further have

$$\begin{aligned}
\left\|\Lambda(x,s)\mathcal{Q}_x(T)^{-1}\right\| \leq& 2|s_0 - x_0| \left\|T\mathcal{Q}_x(T)^{-1}\right\| + \left||x|^2 - |s|^2\right| \left\|\mathcal{Q}_x(T)^{-1}\right\| \\
\leq& d_S(s,x)\left(\left\|T\mathcal{Q}_x(T)^{-1}\right\| + \left\|\mathcal{Q}_x(T)^{-1}\right\|\right) \\
\leq& \varepsilon\left(\left\|T\mathcal{Q}_x(T)^{-1}\right\| + \left\|\mathcal{Q}_x(T)^{-1}\right\|\right) =: \varrho.
\end{aligned}$$

If now $\varepsilon < 1/\left(\|T\mathcal{Q}_x(T)^{-1}\| + \|\mathcal{Q}_x(T)^{-1}\|\right)$, then $0 < \varrho < 1$ and thus

$$\sum_{n=0}^{+\infty} \left\|\Lambda(x,s)^n \mathcal{Q}_x(T)^{-(n+1)}\right\| \leq \left\|\mathcal{Q}_x(T)^{-1}\right\| \sum_{n=0}^{+\infty} \varrho^n < +\infty$$

and the series converges uniformly in $\mathcal{B}(V)$ on $C_\varepsilon(x)$.

Now assume that the series (3.2) converges and observe that $\mathcal{Q}_s(T)$, $\mathcal{Q}_x(T)$ and

$\mathcal{Q}_x(T)^{-1}$ commute on $\mathrm{dom}(T^2)$. Hence, we have for $\mathbf{v} \in \mathrm{dom}(T^2)$ that

$$\begin{aligned}\mathcal{J}(s)\mathcal{Q}_s(T)\mathbf{v} = &\sum_{n=0}^{+\infty} \Lambda(x,s)^n \mathcal{Q}_x(T)^{-(n+1)}\mathcal{Q}_s(T)\mathbf{v} \\ =&\sum_{n=0}^{+\infty} \Lambda(x,s)^n \mathcal{Q}_x(T)^{-(n+1)}\left[-\Lambda(x,s)+\mathcal{Q}_x(T)\right]\mathbf{v} \\ =&-\sum_{n=0}^{+\infty} \Lambda(x,s)^{n+1} \mathcal{Q}_x(T)^{-(n+1)}\mathbf{v} \\ &+\sum_{n=0}^{+\infty} \Lambda(x,s)^n \mathcal{Q}_x(T)^{-n}\mathbf{v} = \mathbf{v}.\end{aligned}$$

On the other hand

$$\mathbf{v}_N := \sum_{n=0}^{N} \Lambda(x,s)^n \mathcal{Q}_x(T)^{-(n+1)}\mathbf{v} = \mathcal{Q}_x(T)^{-1}\sum_{n=0}^{N} \Lambda(x,s)^n \mathcal{Q}_x(T)^{-n}\mathbf{v}$$

belongs to $\mathrm{dom}(T^2)$ for any $\mathbf{v} \in V$ and we have

$$\begin{aligned}\mathcal{Q}_s(T)\mathbf{v}_N =&(-\Lambda(x,s)+\mathcal{Q}_x(T))\sum_{n=0}^{N} \Lambda(x,s)^n \mathcal{Q}_x(T)^{-(n+1)}\mathbf{v} \\ =&-\sum_{n=0}^{N} \Lambda(x,s)^{n+1} \mathcal{Q}_x(T)^{-(n+1)}\mathbf{v} + \sum_{n=0}^{N} \Lambda(x,s)^n \mathcal{Q}_x(T)^{-n}\mathbf{v} \\ =&-\Lambda(x,s)^{N+1}\mathcal{Q}_p(T)^{-(n+1)}\mathbf{v} + \mathbf{v}.\end{aligned}$$

Now observe that $\Lambda(x,s) = 2(s_0 - p_0)T + (|p|^2 - |s|^2)\mathcal{I}$ is defined on $\mathrm{dom}(T)$ and maps $\mathrm{dom}(T^2)$ to $\mathrm{dom}(T)$. Hence $\Lambda(x,s)^2\mathcal{Q}_x(T)^{-1}$ belongs to $\mathcal{B}(V)$ and for $N \geq 1$

$$\begin{aligned}&\left\|-\Lambda(x,s)^{N+1}\mathcal{Q}_p(T)^{-(n+1)}\mathbf{v}\right\| \\ =&\left\|-\Lambda(x,s)^{N-1}\mathcal{Q}_p(T)^{-N}\Lambda(x,s)^2\mathcal{Q}_x(T)^{-1}\mathbf{v}\right\| \\ \leq&\left\|-\Lambda(x,s)^{N-1}\mathcal{Q}_p(T)^{-N}\right\|\left\|\Lambda(x,s)^2\mathcal{Q}_x(T)^{-1}\mathbf{v}\right\| \overset{N\to\infty}{\longrightarrow} 0\end{aligned}$$

because the series (3.2) converges in $\mathcal{B}(V)$ by assumption. Thus $\mathcal{Q}_s(T)\mathbf{v}_N \to \mathbf{v}$ and $\mathbf{v}_N \to \mathbf{v}_\infty := \mathcal{J}(s)\mathbf{v}$ as $N \to \infty$. Since $\mathcal{Q}_s(T)$ is closed, we obtain that

$$\mathcal{J}(s)\mathbf{v} \in \mathrm{dom}(\mathcal{Q}_s(T)) = \mathrm{dom}(T^2) \quad \text{and} \quad \mathcal{Q}_s(T)\mathcal{J}(s)\mathbf{v} = \mathbf{v}.$$

Hence, $\mathcal{J}(s) = \mathcal{Q}_s(T)^{-1}$ and in turn $s \in \rho_S(T)$.

$\square$

Lemma 3.2. *Let $T \in \mathcal{K}(V)$. The functions $s \to \mathcal{Q}_s(T)^{-1}$ and $s \to T\mathcal{Q}_s(T)^{-1}$, which are defined on $\rho_S(T)$ and take values in $\mathcal{B}(V)$, are continuous.*

Proof. Let $x \in \rho_S(T)$. Then $\mathcal{Q}_s(T)^{-1}$ can be represented by the series (3.2), which converges uniformly on a neighborhood of x. Hence, we have

$$\begin{aligned}\lim_{s\to x}\mathcal{Q}_s(T)^{-1} =& \sum_{n=0}^{+\infty}\lim_{s\to x}\left(2(s_0-x_0)T+\left(|x|^2-|s|^2\right)\mathcal{I}\right)^n\mathcal{Q}_x(T)^{-(n+1)}\\ =&\mathcal{Q}_x(T)^{-1},\end{aligned}$$

because each term in the sum is a polynomial in s_0 and s_1 with coefficients in $\mathcal{B}(V)$ and thus continuous. Indeed

$$\begin{aligned}&\left((s_0-x_0)T+\left(|x|^2-|s|^2\right)\mathcal{I}\right)^n\mathcal{Q}_x(T)^{-(n+1)}\\ =&\sum_{k=0}^{n}\binom{n}{k}(s_0-x_0)^k\left(|x|^2-|s|^2\right)^{n-k}T^k\mathcal{Q}_x(T)^{-(n+1)}\end{aligned}$$

and the coefficients $T^k\mathcal{Q}_x(T)^{-(n+1)}$ belongs to $\mathcal{B}(V)$ because $\mathcal{Q}_x(T)^{-(n+1)}$ maps V to $\mathrm{dom}(T^{2(n+1)})$ and $k<2(n+1)$. The function $s\mapsto T\mathcal{Q}_s(T)^{-1}$ is continuous because the identity (3.1) implies

$$\begin{aligned}&\lim_{h\to 0}\left\|T\mathcal{Q}_{s+h}(T)^{-1}-T\mathcal{Q}_s(T)^{-1}\right\|=\\ =&\lim_{h\to 0}\left\|T\mathcal{Q}_{s+h}(T)^{-1}(\mathcal{Q}_s(T)-\mathcal{Q}_{s+h}(T))\mathcal{Q}_s(T)^{-1}\right\|.\end{aligned}$$

The operator $\mathcal{Q}_s(T)^{-1}$ maps V to $\mathrm{dom}(T^2)$ and so

$$(\mathcal{Q}_s(T)-\mathcal{Q}_{s+h}(T))\mathcal{Q}_s(T)^{-1}=(2h_0T+(|s|^2-|s+h|^2)\mathcal{I})\mathcal{Q}_s(T)^{-1}$$

maps $V$to $\mathrm{dom}(T)$. Since T and $\mathcal{Q}_{s+h}(T)^{-1}$ commute on $\mathrm{dom}(T)$ we thus have

$$\begin{aligned}&\lim_{h\to 0}\left\|T\mathcal{Q}_{s+h}(T)^{-1}-T\mathcal{Q}_s(T)^{-1}\right\|\\ =&\lim_{h\to 0}\left\|\mathcal{Q}_{s+h}(T)^{-1}\left(2h_0T^2+\left(|s|^2-|s+h|^2\right)T\right)\mathcal{Q}_s(T)^{-1}\right\|\\ \leq&\lim_{h\to 0}\left\|\mathcal{Q}_{s+h}(T)^{-1}\right\|\lim_{h\to 0}2h_0\left\|T^2\mathcal{Q}_s(T)^{-1}\right\|\\ &+\lim_{h\to 0}\left\|\mathcal{Q}_{s+h}(T)^{-1}\right\|\lim_{h\to 0}\left(|s|^2-|s+h|^2\right)\left\|T\mathcal{Q}_s(T)^{-1}\right\|=0.\end{aligned}$$

□

Lemma 3.3. *Let $T\in\mathcal{K}(V)$ and $s\in\rho_S(T)$. The pseudo resolvent $\mathcal{Q}_s(T)^{-1}$ is continuously real differentiable with*

$$\frac{\partial}{\partial s_0}\mathcal{Q}_s(T)^{-1}=(2T-2s_0\mathcal{I})\mathcal{Q}_s(T)^{-2}\quad\text{and}\quad\frac{\partial}{\partial s_1}\mathcal{Q}_s(T)^{-1}=-2s_1\mathcal{Q}_s(T)^{-2}.$$

Proof. Let us first compute the partial derivative of $\mathcal{Q}_s(T)^{-1}$ with respect to the real part s_0. Applying equation (3.1), we have

$$\begin{aligned}\frac{\partial}{\partial s_0}\mathcal{Q}_s(T)^{-1}=&\lim_{\mathbb{R}\ni h\to 0}\frac{1}{h}\left(\mathcal{Q}_{s+h}(T)^{-1}-\mathcal{Q}_s(T)^{-1}\right)\\ =&\lim_{\mathbb{R}\ni h\to 0}\frac{1}{h}\mathcal{Q}_{s+h}(T)^{-1}\left(\mathcal{Q}_s(T)-\mathcal{Q}_{s+h}(T)\right)\mathcal{Q}_s(T)^{-1}\\ =&\lim_{\mathbb{R}\ni h\to 0}\mathcal{Q}_{s+h}(T)^{-1}\left(2T-2s_0\mathcal{I}-h\mathcal{I}\right)\mathcal{Q}_s(T)^{-1},\end{aligned}$$

where $\lim_{\mathbb{R}\ni h\to 0} f(h)$ denotes the limit of a function f as h tends to 0 in $\mathbb{R}$. Since the composition and the multiplication with scalars are continuous operations on $\mathcal{B}(V)$, we further have

$$\begin{aligned}\frac{\partial}{\partial s_0}\mathcal{Q}_s(T)^{-1} =& \lim_{\mathbb{R}\ni h\to 0}\mathcal{Q}_{s+h}(T)^{-1}\lim_{\mathbb{R}\ni h\to 0}\left((2T-2s_0\mathcal{I})\mathcal{Q}_s(T)^{-1}-h\mathcal{Q}_s(T)^{-1}\right)\\ =&\mathcal{Q}_s(T)^{-1}(2T-2s_0\mathcal{I})\mathcal{Q}_s(T)^{-1} = (2T-2s_0\mathcal{I})\mathcal{Q}_s(T)^{-2},\end{aligned}$$

where the last equation holds true because $\mathcal{Q}_s(T)^{-1}$ maps V to $\mathrm{dom}(T^2)\subset\mathrm{dom}(T)$ and T and $\mathcal{Q}_s(T)^{-1}$ commute on $\mathrm{dom}(T)$. Observe that $\frac{\partial}{\partial s_0}\mathcal{Q}_s(T)^{-1}$ is even continuous because it is the sum and product of continuous functions by Lemma 3.2.

If we write $s = s_0 + \mathsf{i}_s s_1$, then we can argue in a similar way to show that the derivative of $\mathcal{Q}_s(T)^{-1}$ with respect to s_1 is

$$\begin{aligned}\frac{\partial}{\partial s_1}\mathcal{Q}_s(T)^{-1} =& \lim_{\mathbb{R}\ni h\to 0}\frac{1}{h}\left(\mathcal{Q}_{s+h\mathsf{i}_s}(T)^{-1}-\mathcal{Q}_s(T)^{-1}\right)\\ =& \lim_{\mathbb{R}\ni h\to 0}\frac{1}{h}\mathcal{Q}_{s+h\mathsf{i}_s}(T)^{-1}\left(\mathcal{Q}_s(T)-\mathcal{Q}_{s+h\mathsf{i}_s}(T)\right)\mathcal{Q}_s(T)^{-1}\\ =& \lim_{\mathbb{R}\ni h\to 0}\mathcal{Q}_{s+h\mathsf{i}_s}(T)^{-1}\left(-2s_1-h\right)\mathcal{Q}_s(T)^{-1}\\ =& \lim_{\mathbb{R}\ni h\to 0}\mathcal{Q}_{s+h\mathsf{i}_s}(T)^{-1}\lim_{\mathbb{R}\ni h\to 0}\left(-2s_1\mathcal{Q}_s(T)^{-1}-h\mathcal{Q}_s(T)^{-1}\right)\\ =& -2s_1\mathcal{Q}_s(T)^{-2}.\end{aligned}$$

Again this derivative is continuous as the product of two continuous functions by Lemma 3.2.

Finally, we easily obtain that $\mathcal{Q}_s(T)^{-1}$ is continuously real differentiable from the fact that $\mathcal{Q}_s(T)^{-1}$ is continuously differentiable in the variables s_0 and s_1. If we write s in terms of its four real coordinates as $s = \xi_0 + \sum_{\ell=1}^{3}\xi_\ell e_\ell$, then the partial derivative with respect to ξ_0 corresponds to the partial derivative with respect to s_0 and thus exists and is continuous. The partial derivative with respect to ξ_ℓ for $1\leq\ell\leq 3$ on the other hand exists and is continuous for $s_1\neq 0$ because $\mathcal{Q}_s(T)^{-1}$ can be considered as the composition of the continuously differentiable functions $s\mapsto s_1 = \sqrt{\xi_1^2+\xi_2^2+\xi_3^2}$ and $s_1\to\mathcal{Q}_{s+\mathsf{i}s_1}(T)^{-1}$ with fixed $\mathsf{i}\in\mathbb{S}$ and find

$$\frac{\partial}{\partial\xi_\ell}\mathcal{Q}_s(T)^{-1} = -2s_1\mathcal{Q}_s(T)^{-2}\frac{\partial}{\partial\xi_\ell}s_0 = -2\xi_\ell\mathcal{Q}_s(T)^{-2}.$$

For $s_1 = 0$ (that is for $s\in\mathbb{R}$), we can simply choose $\mathsf{i} = e_\ell$ and then the partial derivative with respect to ξ_ℓ agrees with the partial derivative with respect to s_1. In particular, we see that also the partial derivatives with respect to the real coordinates $\xi_0,\ldots,\xi_3$ are continuous.

□

Lemma 3.4. *Let $T\in\mathcal{K}(V)$ and $s\in\rho_S(T)$. The function $s\mapsto T\mathcal{Q}_s(T)^{-1}$ is continuously real differentiable with*

$$\frac{\partial}{\partial s_0}T\mathcal{Q}_s(T)^{-1} = (2T^2-2s_0T)\mathcal{Q}_s(T)^{-2} \tag{3.3}$$

and

$$\frac{\partial}{\partial s_1}T\mathcal{Q}_s(T)^{-1} = -2s_1T\mathcal{Q}_s(T)^{-2}. \tag{3.4}$$

Proof. If $\lim_{\mathbb{R}\ni h\to 0} f(h)$ denotes again the limit of a function f as h tends to 0 in $\mathbb{R}$, then we obtain from (3.1) that

$$\begin{aligned}\frac{\partial}{\partial s_0} T\mathcal{Q}_s(T)^{-1} &= \lim_{\mathbb{R}\ni h\to 0} \frac{1}{h}\left(T\mathcal{Q}_{s+h}(T)^{-1} - T\mathcal{Q}_s(T)^{-1}\right)\\ &= \lim_{\mathbb{R}\ni h\to 0} \frac{1}{h} T\mathcal{Q}_{s+h}(T)^{-1}\left(\mathcal{Q}_s(T) - \mathcal{Q}_{s+h}(T)\right)\mathcal{Q}_s(T)^{-1}\\ &= \lim_{\mathbb{R}\ni h\to 0} \frac{1}{h} T\mathcal{Q}_{s+h}(T)^{-1}\left(2hT - 2hs_0\mathcal{I} - h^2\mathcal{I}\right)\mathcal{Q}_s(T)^{-1}\\ &= \lim_{\mathbb{R}\ni h\to 0} \mathcal{Q}_{s+h}(T)^{-1}\left(2T^2 - 2s_0T - hT\right)\mathcal{Q}_s(T)^{-1},\end{aligned}$$

because $(2hT - 2hs_0\mathcal{I} - h^2\mathcal{I})\,\mathcal{Q}_s(T)^{-1}$ maps V to $\operatorname{dom}(T)$ and T and $\mathcal{Q}_{s+h}(T)^{-1}$ commute on $\operatorname{dom}(T)$. Since the composition and the multiplication with scalars are continuous operations on the space $\mathcal{B}(V)$ and since the pseudo-resolvent is continuous by Lemma 3.2, we get

$$\begin{aligned}\frac{\partial}{\partial s_0} T\mathcal{Q}_s(T)^{-1} &= \lim_{\mathbb{R}\ni h\to 0} \mathcal{Q}_{s+h}(T)^{-1} \lim_{\mathbb{R}\ni h\to 0}\left(\left(2T^2 - 2s_0T\right)\mathcal{Q}_s(T)^{-1} - hT\mathcal{Q}_s(T)^{-1}\right)\\ &= \mathcal{Q}_s(T)^{-1}(2T^2 - 2s_0T)\mathcal{Q}_s(T)^{-1} = (2s_0T - 2T^2)\mathcal{Q}_s(T)^{-2}.\end{aligned}$$

This function is continuous because we can write it as the product of functions that are continuous by Lemma 3.2.

The derivative with respect to s_1 can be computed using similar arguments via

$$\begin{aligned}\frac{\partial}{\partial s_1} T\mathcal{Q}_s(T)^{-1} &= \lim_{\mathbb{R}\ni h\to 0} \frac{1}{h}\left(T\mathcal{Q}_{s+h\mathbf{i}_s}(T)^{-1} - T\mathcal{Q}_s(T)^{-1}\right)\\ &= \lim_{\mathbb{R}\ni h\to 0} \frac{1}{h} T\mathcal{Q}_{s+h\mathbf{i}_s}(T)^{-1}\left(\mathcal{Q}_s(T) - \mathcal{Q}_{s+h\mathbf{i}_s}(T)\right)\mathcal{Q}_s(T)^{-1}\\ &= \lim_{\mathbb{R}\ni h\to 0} \frac{1}{h} T\mathcal{Q}_{s+h\mathbf{i}_s}(T)^{-1}\left(-2hs_1 - h^2\right)\mathcal{Q}_s(T)^{-1}\\ &= \lim_{\mathbb{R}\ni h\to 0} \mathcal{Q}_{s+h\mathbf{i}_s}(T)^{-1} \lim_{\mathbb{R}\ni h\to 0}\left(-2s_1T\mathcal{Q}_s(T)^{-1} - hT\mathcal{Q}_s(T)^{-1}\right)\\ &= -\,2s_1T\mathcal{Q}_s(T)^{-2}.\end{aligned}$$

Also this derivative is continuous because $\frac{\partial}{\partial s_1}T\mathcal{Q}_s(T)^{-1} = -2s_1\left(T\mathcal{Q}_s(T)^{-1}\right)\mathcal{Q}_s(T)^{-1}$ is the product of functions that are continuous by Lemma 3.2.

Finally, we see as in the proof of Lemma 3.3 that $T\mathcal{Q}_s(T)^{-1}$ is continuously differentiable in the four real coordinates by considering it as the composition of the two continuously real differentiable functions $s \mapsto (s_0, s_1)$ and $(s_0, s_1) \mapsto T\mathcal{Q}_{s+\mathbf{i}s_1}(T)^{-1}$ resp. by choosing $\mathbf{i}_s$ appropriately if $s \in \mathbb{R}$.

□

Remark 3.5. The identities (3.3) and (3.4) seem to be immediate consequences of Lemma 3.3. However, since T is closed but not necessarily bounded, it is not obvious that $\frac{\partial}{\partial s_\ell}T\mathcal{Q}_s(T)^{-1} = T\frac{\partial}{\partial s_\ell}\mathcal{Q}_s(T)^{-1}$ so that we preferred to show them explicitly.

Corollary 3.6. *Let $T \in \mathcal{K}(V)$ and $s \in \rho_S(T)$. The left and the right S-resolvent are continuously real differentiable.*

Proof. The S-resolvents are sums of functions that are continuously real differentiable by Lemma 3.3 and Lemma 3.4 and hence continuously real differentiable themselves. □

Theorem 3.7. *Let $T \in \mathcal{K}(V)$. The left S-resolvent $S_L^{-1}(s,T)$ is right slice hyperholomorphic and the right S-resolvent $S_R^{-1}(s,T)$ is left slice hyperholomorphic in the variable s.*

Proof. We consider only the case of the left S-resolvent, the other one works with analogous arguments. We have

$$S_L^{-1}(s,T) = \alpha(s_0,s_1) + \mathbf{i}_s\beta(s_0,s_1)$$

with

$$\alpha(s_0,s_1) = \mathcal{Q}_s(T)^{-1}s_0 - T\mathcal{Q}_s(T) \quad \text{and} \quad \beta(s_0,s_1) = -\mathcal{Q}_s(T)^{-1}s_1.$$

Obviously α and β satisfy the compatibility conditions (2.4) and hence $S_L^{-1}(s,T)$ is a right slice function in s.

Applying Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned}\frac{\partial}{\partial s_0}S_L^{-1}(s,T) =&\frac{\partial}{\partial s_0}\mathcal{Q}_s(T)^{-1}\overline{s} - \frac{\partial}{\partial s_0}T\mathcal{Q}_s(T)^{-1}\\ =&(2T-2s_0\mathcal{I})\mathcal{Q}_s(T)^{-2}\overline{s} + \mathcal{Q}_s(T)^{-1} - \left(2T^2-2s_0T\right)\mathcal{Q}_s(T)^{-2}\\ =&(2T-2s_0\mathcal{I})\mathcal{Q}_s(T)^{-2}\overline{s} + \left(-T^2+|s|^2\mathcal{I}\right)\mathcal{Q}_s(T)^{-2}.\end{aligned}$$

Since s_0 and $|s|^2$ are real, they commute with $\mathcal{Q}_s(T)^{-2}$. If we apply the identities $2s_0 = s+\overline{s}$ and $|s|^2 = s\overline{s}$, we obtain

$$\frac{\partial}{\partial s_0}S_L^{-1}(s,T) = -T^2\mathcal{Q}_s(T)^{-2} + 2T\mathcal{Q}_s(T)^{-2}\overline{s} - \mathcal{Q}_s(T)^{-2}\overline{s}^2.$$

For the partial derivative with respect to s_1, we obtain

$$\begin{aligned}&\frac{\partial}{\partial s_1}S_L^{-1}(s,T) = \frac{\partial}{\partial s_1}\mathcal{Q}_s(T)^{-1}\overline{s} - \frac{\partial}{\partial s_1}T\mathcal{Q}_s(T)^{-1}\\ =&-2s_1\mathcal{Q}_s(T)^{-2}\overline{s} - \mathcal{Q}_s(T)^{-1}\mathbf{i}_s + 2s_1T\mathcal{Q}_s(T)^{-2}\\ =&-2s_1\mathcal{Q}_s(T)^{-2}\overline{s} - (T^2-2s_0T+|s|^2\mathcal{I})\mathcal{Q}_s(T)^{-2}\mathbf{i}_s + 2s_1T\mathcal{Q}_s(T)^{-2}.\end{aligned}$$

We can again commute $2s_0$, $2s_1$ and $|s|^2$ with $\mathcal{Q}_s(T)^{-1}$ because they are real. By exploiting the identities $2s_0 = s+\overline{s}$, $-2s_1 = (s-\overline{s})\mathbf{i}_s$ and $|s|^2 = s\overline{s}$, we obtain

$$\frac{\partial}{\partial s_1}S_L^{-1}(s,T) = \left(-T^2\mathcal{Q}_s(T)^{-2} + 2T\mathcal{Q}_s(T)^{-2}\overline{s} - \mathcal{Q}_s(T)^{-2}(T)\overline{s}^2\right)\mathbf{i}_s$$

By Lemma 2.15, the function $s \mapsto S_L^{-1}(s,T)$ is right slice hyperholomorphic as

$$\frac{1}{2}\left(\frac{\partial}{\partial s_0}S_L^{-1}(s,T) + \frac{\partial}{\partial s_1}S_L^{-1}(s,T)\mathbf{i}_s\right) = 0.$$

□

In Section 7.3 we need the fact that the S-resolvent set is the maximal domain of slice hyperholomorphicity of the S-resolvents such that they do not have a slice hyperholomorphic continuation. In the complex case this is guaranteed by the well-known estimate

$$\|R(z,A)\| \geq \frac{1}{\operatorname{dist}(z,\sigma(A))}, \tag{3.5}$$

where $R(z,A)$ denotes the resolvent and $\sigma(A)$ the spectrum of the complex linear operator A. This estimate assures that $\|R(z,A)\| \to +\infty$ as z approaches $\sigma(A)$ and in turn that the resolvent does not have any holomorphic continuation to a larger domain.

In the quaternionic setting, an estimate similar to (3.5) cannot hold true. We can for example consider the operator $T = \lambda\mathcal{I}$ on a two-sided Banach space V for some $\lambda = \lambda_0 + \mathsf{i}_\lambda\lambda_1$ with $\lambda_1 > 0$. Its S-spectrum $\sigma_S(T)$ coincides with the sphere $[\lambda]$ associated with λ and its left S-resolvent is

$$S_L^{-1}(s,T) = (\lambda^2 - 2s_0\lambda + |s|^2)^{-1}(\overline{s} - \lambda)\mathcal{I}.$$

If $s \in \mathbb{C}_{\mathsf{i}_\lambda}$, then λ and s commute so that the left S-resolvent reduces to $S_L^{-1}(s,T) = (s-\lambda)^{-1}\mathcal{I}$ with $\|S_L^{-1}(s,T)\| = 1/|s-\lambda|$. If s tends to $\overline{\lambda}$ in $\mathbb{C}_{\mathsf{i}_\lambda}$, then $\operatorname{dist}(s,\sigma_S(T)) \to 0$ because $\overline{\lambda} \in \sigma_S(T)$. But at the same time $\|S_L^{-1}(s,T)\| \to 1/|\lambda-\overline{\lambda}| = 1/(2\lambda_1) < +\infty$.

Nevertheless, although (3.5) does not have a pointwise counterpart in the quaternionic setting, we can show that the norms of the S-resolvents explode near the S-spectrum. As it happens often in quaternionic operator theory, this requires that we work with spectral spheres of associated quaternions instead of single spectral values.

Lemma 3.8. *Let $T \in \mathcal{K}(V)$ and $s \in \rho_S(T)$. Then*

$$\|\mathcal{Q}_s(T)^{-1}\| + \|T\mathcal{Q}_s(T)^{-1}\| \geq \frac{1}{d_S(s,\sigma_S(T))}, \tag{3.6}$$

where $d_S(s,\sigma_S(T)) = \inf_{x\in\sigma_S(T)} d_S(s,x)$ and $d_S(s,x)$ is defined as in the Lemma 3.1.

Proof. Set $C_s := \|\mathcal{Q}_s(T)^{-1}\| + \|T\mathcal{Q}_s(T)^{-1}\|$. If $d_S(s,x) < 1/C_s$, then $x \in \rho_S(T)$ by Lemma 3.1. Thus, $d_S(s,x) \geq 1/C_s$ for any $x \in \sigma_S(T)$. If we take the infimum over all $x \in \sigma_S(T)$, this inequality still holds true and we obtain $d_S(s,\sigma_S(T)) \geq 1/C_s$, which is equivalent to (3.6).

□

Lemma 3.9. *Let $T \in \mathcal{K}(V)$ and $s \in \rho_S(T)$. Then*

$$\sqrt{2\,\|\mathcal{Q}_s(T)^{-1}\|} \leq \left\|S_L^{-1}(s,T)\right\| + \left\|S_L^{-1}(\overline{s},T)\right\|$$

and in turn

$$\sqrt{\|\mathcal{Q}_s(T)^{-1}\|} \leq \sqrt{2}\sup_{s_{\mathsf{i}}\in[s]} \left\|S_L^{-1}(s_{\mathsf{i}},T)\right\|.$$

Analogous estimates hold for the right S-resolvent operator.

Proof. Observe that $\mathcal{Q}_s(T)^{-1} = \mathcal{Q}_{\overline{s}}(T)^{-1}$ for $s \in \rho_S(T)$. Because of $2s_0 = s + \overline{s}$, we

hence have

$$
\begin{aligned}
& S_L^{-1}(s,T)S_L^{-1}(s,T) + S_L^{-1}(s,T)S_L^{-1}(\overline{s},T) \\
={}& \left(\mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1}\right)\left(\mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1}\right) \\
& + \left(\mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1}\right)\left(\mathcal{Q}_s(T)^{-1}s - T\mathcal{Q}_s(T)^{-1}\right) \\
={}& \left(\mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1}\right) 2\left(s_0\mathcal{I} - T\right)\mathcal{Q}_s(T)^{-1}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
& S_L^{-1}(\overline{s},T)S_L^{-1}(s,T) + S_L^{-1}(\overline{s},T)S_L^{-1}(\overline{s},T) \\
={}& \left(\mathcal{Q}_s(T)^{-1}s - T\mathcal{Q}_s(T)^{-1}\right) 2\left(s_0\mathcal{I} - T\right)\mathcal{Q}_s(T)^{-1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& S_L^{-1}(s,T)S_L^{-1}(s,T) + S_L^{-1}(s,T)S_L^{-1}(\overline{s},T) \\
& + S_L^{-1}(\overline{s},T)S_L^{-1}(s,T) + S_L^{-1}(\overline{s},T)S_L^{-1}(\overline{s},T) \\
={}& \left(\mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1}\right) 2\left(s_0\mathcal{I} - T\right)\mathcal{Q}_s(T)^{-1} \\
& + \left(\mathcal{Q}_s(T)^{-1}s - T\mathcal{Q}_s(T)^{-1}\right) 2\left(s_0\mathcal{I} - T\right)\mathcal{Q}_s(T)^{-1} \\
={}& 2\left(s_0\mathcal{I} - T\right)\mathcal{Q}_s(T)^{-1} 2\left(s_0\mathcal{I} - T\right)\mathcal{Q}_s(T)^{-1} \\
={}& 4(T^2 - 2s_0T + s_0^2\mathcal{I})\mathcal{Q}_s(T)^{-2} = 4\mathcal{Q}_s(T)^{-1} - 4s_1^2\mathcal{Q}_s(T)^{-2},
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
4\mathcal{Q}_s(T)^{-1} ={}& S_L^{-1}(s,T)S_L^{-1}(s,T) + S_L^{-1}(s,T)S_L^{-1}(\overline{s},T) \\
& + S_L^{-1}(\overline{s},T)S_L^{-1}(s,T) + S_L^{-1}(\overline{s},T)S_L^{-1}(\overline{s},T) + 4s_1^2\mathcal{Q}_s(T)^{-2}.
\end{aligned}
$$

Thus, we can estimate

$$
\begin{aligned}
& 4\left\|\mathcal{Q}_s(T)^{-1}\right\| = \\
={}& \left\|S_L^{-1}(s,T)\right\|\left\|S_L^{-1}(s,T)\right\| + \left\|S_L^{-1}(s,T)\right\|\left\|S_L^{-1}(\overline{s},T)\right\| \\
& + \left\|S_L^{-1}(\overline{s},T)\right\|\left\|S_L^{-1}(s,T)\right\| + \left\|S_L^{-1}(\overline{s},T)\right\|\left\|S_L^{-1}(\overline{s},T)\right\| \\
& + 4\left\|s_1^2\mathcal{Q}_s(T)^{-2}\right\| \\
={}& \left(\left\|S_L^{-1}(s,T)\right\| + \left\|S_L^{-1}(\overline{s},T)\right\|\right)^2 + \left\|2s_1\mathcal{Q}_s(T)^{-1}\right\|\left\|2s_1\mathcal{Q}_s(T)^{-1}\right\|. \qquad (3.7)
\end{aligned}
$$

Finally, we observe that

$$
\begin{aligned}
2\mathcal{Q}_s(T)^{-1}s_1\mathbf{i}_s ={}& T\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T)^{-1}(s_0 - \mathbf{i}_s s_1) \\
& - \left(T\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T)^{-1}(s_0 + \mathbf{i}_s s_1)\right) = S_L^{-1}(s,T) - S_L^{-1}(\overline{s},T)
\end{aligned}
$$

and therefore

$$
\left\|2s_1\mathcal{Q}_s(T)^{-1}\right\| = \left\|2\mathcal{Q}_s(T)^{-1}s_1\mathbf{i}_s\right\| \leq \left\|S_L^{-1}(s,T)\right\| + \left\|S_L^{-1}(\overline{s},T)\right\|.
$$

Combining this estimate with (3.7), we finally obtain

$$2\left\|\mathcal{Q}_s(T)^{-1}\right\| \leq \left(\left\|S_L^{-1}(s,T)\right\| + \left\|S_L^{-1}(\overline{s},T)\right\|\right)^2$$

and hence the statement for the left S-resolvent operator. The estimates for the right S-resolvent operator can be shown with similar computations.

□

Lemma 3.10. *Let $T \in \mathcal{K}(V)$. If $(s_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\rho_S(T)$ with*

$$\lim_{n\to\infty} \operatorname{dist}(s_n, \sigma_S(T)) = 0,$$

then

$$\lim_{n\to\infty} \sup_{s\in[s_n]} \left\|S_L^{-1}(s,T)\right\| = +\infty \qquad \textit{and} \qquad \lim_{n\to\infty} \sup_{s\in[s_n]} \left\|S_R^{-1}(s,T)\right\| = +\infty.$$

Proof. First of all observe that $\operatorname{dist}(s_n, \sigma_S(T)) \to 0$ if and only if $d_S(s_n, \sigma_S(T)) \to 0$ because $\sigma_S(T)$ is axially symmetric. Indeed, for any $n \in \mathbb{N}$ there exits $x_n \in \sigma_S(T)$ such that

$$|s_n - x_n| < \operatorname{dist}(s_n, \sigma_S(T)) + 1/n.$$

If $\operatorname{dist}(s_n, \sigma_S(T)) \to 0$, then $|s_n - x_n| \to 0$ and hence $|s_{n,0} - x_{n,0}| \to 0$. Since the sequence s_n is bounded, the sequence x_n is bounded too and we also have

$$\left||s_n|^2 - |x_n|^2\right| \leq |s_n||\overline{s_n} - \overline{x_n}| + |s_n - x_n||\overline{x_n}| \to 0$$

and in turn

$$0 < d_S(s_n, \sigma_S(T)) \leq d_S(s_n, x_n) = \max\left\{|s_{n,0} - x_{n,0}|, \left||s_n|^2 - |x_n|^2\right|\right\} \longrightarrow 0.$$

If on the other hand $d_S(s_n, \sigma_S(T))$ tends to zero, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\sigma_S(T)$ such that

$$d_S(s_n, x_n) < d_S(s_n, \sigma_S(T)) + 1/n$$

and in turn $d_S(s_n, x_n) \to 0$. Since $\sigma_S(T)$ is axially symmetric and $d(s_n, x_{n,\mathbf{i}}) = d(s_n, x_n)$ for any $x_{n,\mathbf{i}} \in [x_n]$, we can moreover assume that $\mathbf{i}_{x_n} = \mathbf{i}_{s_n}$. Then

$$0 \leq |s_{n,0} - x_{n,0}| \leq d_S(s_n, x_n) \to 0.$$

Since s_n and in turn also x_n are bounded, this implies $|s_{n,0}^2 - x_{n,0}^2| \to 0$, from which we deduce that also $|s_{n,1}^2 - x_{n,1}^2| \to 0$ because

$$0 \leq \left|s_{n,0}^2 - x_{n,0}^2 + s_{n,1}^2 - x_{n,1}^2\right| = \left||s_n|^2 - |x_n|^2\right| \leq d_S(s_n, x_n) \to 0.$$

Since $s_{n,1} \geq 0$ and $x_{n,1} \geq 0$, we conclude $s_{n,1} - x_{n,1} \to 0$ and, since $\mathbf{i}_s = \mathbf{i}_x$, also

$$0 < \operatorname{dist}(s_n, \sigma_S(T)) \leq |s_n - x_n| = \sqrt{(s_{n,0} - x_{n,0})^2 + (s_{n,1} - x_{n,1})^2} \to 0.$$

Now assume that $s_n \in \rho_S(T)$ with $\operatorname{dist}(s_n, \sigma_S(T)) \to 0$. By the above considerations and (3.6), we have

$$\|\mathcal{Q}_{s_n}(T)^{-1}\| + \|T\mathcal{Q}_{s_n}(T)^{-1}\| \to +\infty. \tag{3.8}$$

We show now that every subsequence $(s_{n_k})_{k\in\mathbb{N}}$ has a subsequence $(s_{n_{k_j}})_{j\in\mathbb{N}}$ such that

$$\lim_{j\to+\infty} \sup_{s\in[s_{n_{k_j}}]} \|S_L^{-1}(s,T)\| = +\infty, \tag{3.9}$$

which implies $\lim_{n\to+\infty}\sup_{s\in[s_n]}\|S_L^{-1}(s,T)\| = +\infty$. We consider an arbitrary subsequence $(s_{n_k})_{k\in\mathbb{N}}$ of $(s_n)_{n\in\mathbb{N}}$. If this subsequence has a subsequence $(s_{n_{k_j}})_{j\in\mathbb{N}}$ such that $\|\mathcal{Q}_{s_{n_{k_j}}}(T)^{-1}\| \to +\infty$, then Lemma 3.9 implies (3.9). Otherwise $\|\mathcal{Q}_{s_{n_j}}(T)^{-1}\| \leq C$ for some constant $C > 0$ and we deduce from (3.8) that $\|T\mathcal{Q}_{s_{n_j}}(T)^{-1}\| \to +\infty$. Observe that

$$T\mathcal{Q}_{s_{n_k}}(T)^{-1} = -\frac{1}{2}S_L^{-1}(s_{n_k},T) - \frac{1}{2}S_L^{-1}(\overline{s_{n_k}},T) + s_{n_k,0}\mathcal{Q}_{s_{n_k}}(T)^{-1},$$

from which we obtain the estimate

$$\begin{aligned}\left\|T\mathcal{Q}_{s_{n_k}}(T)^{-1}\right\| &\leq \sup_{s\in[s_{n_k}]}\left\|S_L^{-1}(s_{n_k},T)\right\| + |s_{n_k,0}|\left\|\mathcal{Q}_{s_{n_k}}(T)^{-1}\right\| \\ &\leq \sup_{s\in[s_{n_k}]}\left\|S_L^{-1}(s_{n_k},T)\right\| + CM\end{aligned}$$

with $M = \sup_{n\in\mathbb{N}}|s_n| < +\infty$. Since the left-hand side tends to infinity as $k\to+\infty$, we obtain that also $\sup_{s\in[s_{n_k}]}\left\|S_L^{-1}(s_{n_k},T)\right\| \to +\infty$ and thus the statement holds true. The case of the right S-resolvent can be shown with analogous arguments.

□

Definition 3.11. Let f be a left (or right) slice hyperholomorphic function defined on an axially symmetric open set U. A left (or right) slice hyperholomorphic function g defined on an axially symmetric open set U' with $U \subsetneq U'$ is called a slice hyperholomorphic continuation of f if $f(s) = g(s)$ for all $s \in U$. It is called nontrivial if $V = U' \setminus U$ cannot be separated from U, i.e. if $U' \neq U \cup V$ for some open set V with $\overline{V} \cap \overline{U} = \emptyset$.

Theorem 3.12. *Let $T \in \mathcal{K}(V)$. There does not exist any nontrivial slice hyperholomorphic continuation of the left or of the right S-resolvent operator.*

Proof. Assume that there exists a nontrivial extension f of $S_L^{-1}(s,T)$ to an axially symmetric open set U with $\rho_S(T) \subsetneq U$. Then there exists a point $s \in U \cap \partial(\rho_S(T))$ and a sequence $s_n \in \rho_S(T)$ with $\lim_{n\to+\infty} s_n = s$ such that

$$\lim_{n\to+\infty}\left\|S_L^{-1}(s_n,T)\right\| = \lim_{n\to+\infty}\|f(s_n)\| = \|f(s)\| < +\infty.$$

Moreover, also $\overline{s_n} \to \overline{s}$ as $n \to +\infty$ and in turn

$$\lim_{n\to+\infty}\left\|S_L^{-1}(\overline{s_n},T)\right\| = \lim_{n\to+\infty}\|f(\overline{s_n})\| = \|f(\overline{s})\| < +\infty.$$

From the representation formula (2.8) we then deduce

$$\lim_{n\to+\infty}\sup_{s\in[s_n]}\left\|S_L^{-1}(s,T)\right\| \leq \lim_{n\to+\infty}\left\|S_L^{-1}(s_n,T)\right\| + \left\|S_L^{-1}(\overline{s_n},T)\right\| < +\infty.$$

On the other hand the sequence s_n is bounded and

$$\operatorname{dist}(s_n, \sigma_S(T)) \leq |s_n - s| \to 0.$$

Lemma 3.10 therefore implies $\lim_{n\to+\infty} \sup_{s\in[s_n]} \left\| S_L^{-1}(s,T) \right\| = +\infty$, which is a contradiction. Thus, the analytic continuation (f, U) cannot exist.

For the right S-resolvent, we argue analogously.

□

Remark 3.13. We suspected that it might be possible to improve the above results by finding an estimate of the form (3.5) for the pseudo-resolvent $\mathcal{Q}_s(T)^{-1}$ instead of the S-resolvents. In this case Lemma 3.9 would yield an estimate of the form (3.5) for the norm of the S-resolvents on an entire sphere instead of a single point. This is however not possible as the following example shows: consider for $p \in [1, +\infty)$ the space $\ell^p(\mathbb{N})$ of p-summable sequences with quaternionic entries. Any sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \in \mathbb{H}$ does obviously define a right linear, densely defined and closed operator on $\ell^p(\mathbb{N})$ via $T(\mathbf{v}) = (\lambda_n v_n)_{n\in\mathbb{N}}$ for $\mathbf{v} = (v_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$. If $(\lambda_n)_{n\in\mathbb{N}}$ is unbounded, then T is unbounded. Otherwise $\|T\| = \sup_{n\in\mathbb{N}} |\lambda_n| = \|(\lambda_n)_{n\in\mathbb{N}}\|_\infty$. Indeed,

$$\|T(a)\|_p = \sqrt[p]{\sum_{n\in\mathbb{N}} |\lambda_n a_n|^p} \leq \|(\lambda_n)_{n\in\mathbb{N}}\|_\infty \sqrt[p]{\sum_{n\in\mathbb{N}} |a_n|^p} = \|(\lambda_n)_{n\in\mathbb{N}}\|_\infty \|a\|_p$$

such that $\|T\| \leq \|(\lambda_n)_{n\in\mathbb{N}}\|_\infty$ and, with $e_m = (\delta_{m,n})_{n\in\mathbb{N}}$ where $\delta_{m,n}$ is the Kronecker delta, on the other hand

$$\|\lambda_m\| = \sqrt[p]{\sum_{n\in\mathbb{N}} |\lambda_n \delta_{m,n}\|^p} = \|T(e_m)\| \leq \|T\|$$

for any $m \in \mathbb{N}$ such that also $\|(\lambda_n)_{n\in\mathbb{N}}\|_\infty \leq \|T\|$. The S-spectrum of T is

$$\sigma_S(T) = \overline{\bigcup_{n\in\mathbb{N}} [\lambda_n]} \tag{3.10}$$

as one can see easily: any λ_n is a right eigenvalue since for instance $T(e_n) = e_n\lambda_n$ and hence the relation $\supset$ in (3.10) holds true by Theorem 2.56. If on the other hand s does not belong to the right hand side of (3.10), then $\delta_s = \inf_{n\in\mathbb{N}} \operatorname{dist}(s, [\lambda_n]) = \inf_{n\in\mathbb{N}} |s_{\mathbf{i}_{\lambda_n}} - \lambda_n| > 0$, where $s_{\mathbf{i}_{\lambda_n}} = s_0 + \mathbf{i}_{\lambda_n} s_1$. As

$$\mathcal{Q}_s(T)\mathbf{v} = \left((\lambda_n - s_{\mathbf{i}_{\lambda_n}})(\lambda_n - \overline{s_{\mathbf{i}_{\lambda_n}}})v_n\right)_{n\in\mathbb{N}}$$

and in turn

$$\mathcal{Q}_s(T)^{-1}\mathbf{v} = \left((\lambda_n - s_{\mathbf{i}_{\lambda_n}})^{-1}(\lambda_n - \overline{s_{\mathbf{i}_{\lambda_n}}})^{-1}v_n\right)_{n\in\mathbb{N}},$$

we have $\|\mathcal{Q}_s(T)^{-1}\| \leq 1/\delta_s^2 < +\infty$ such that $s \in \rho_S(T)$. Thus, the relation $\subset$ in (3.10) also holds true.

Now choose a sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lambda_{n,1} \to +\infty$ as $n \to +\infty$ and consider the respective operator T on $\ell^p(\mathbb{N})$. For simplicity, consider for instance $\lambda_n = \mathbf{i}n$ with $\mathbf{i} \in \mathbb{S}$. By the above considerations, the sequence $s_N = \mathbf{i}(N + 1/N)$ with $N = 2, 3, \ldots$ does then satisfy $\operatorname{dist}(s_N, \sigma_S(T)) \to 0$ as $N \to +\infty$ and

$$\left\|\mathcal{Q}_{s_n}(T)^{-1}\right\| = \sup_{n\in\mathbb{N}} \frac{1}{|\lambda_n - s_N||\lambda_n - \overline{s_N}|} = \frac{1}{|\lambda_N - s_N||\lambda_N - \overline{s_N}|} = \frac{1}{2 + \frac{1}{N^2}}. \tag{3.11}$$

Indeed, if $n < N$, then some simple computations show that the inequality

$$\begin{aligned}\frac{1}{|\lambda_n - s_N||\lambda_n - \overline{s_N}|} &= \frac{1}{N + \frac{1}{N} - n}\frac{1}{n + N + \frac{1}{N}} \\ &< \frac{1}{2 + \frac{1}{N^2}} = \frac{1}{|\lambda_N - s_N||\lambda_N - \overline{s_N}|}\end{aligned}$$

is equivalent to $0 < N^2 - n^2$, which is obviously true. Similarly, in the case $n > N$, the inequality

$$\begin{aligned}\frac{1}{|\lambda_n - s_N||\lambda_n - \overline{s_N}|} &= \frac{1}{n - N - \frac{1}{N}}\frac{1}{n + N + \frac{1}{N}} \\ &< \frac{1}{2 + \frac{1}{N^2}} = \frac{1}{|\lambda_N - s_N||\lambda_N - \overline{s_N}|}\end{aligned}$$

is equivalent to $4 + 1/N^2 < n^2 - N^2$, which holds true since $2 \leq N < n$.

From (3.11), we see that $\|\mathcal{Q}_{s_n}(T)^{-1}\| \leq 2$ although $\operatorname{dist}(s_N, \sigma_S(T)) \to 0$. Consequently, the pseudo-resolvent cannot satisfy an estimate that is analogue to (3.5).

Also controlling the norm of $T\mathcal{Q}_s(T)^{-1}$ by the norm of $\mathcal{Q}_s(T)^{-1}$ in order to improve (3.6) is not possible: if we consider the operator $T\mathcal{Q}_{s_n}(T)^{-1}$ in the above example, then

$$T\mathcal{Q}_{s_n}(T)^{-1}\mathbf{v} = \left(\frac{n}{n - N - \frac{1}{N}}\frac{1}{\mathbf{i}\left(n + N + \frac{1}{N}\right)}v_n\right)_{n\in\mathbb{N}}$$

and

$$\|T\mathcal{Q}_{s_n}(T)^{-1}\| \leq \|T\mathcal{Q}_{s_n}(T)^{-1}(e_N)\| = \frac{N^2}{2N + \frac{1}{N}} \to +\infty$$

shows that $\|T\mathcal{Q}_{s_n}(T)^{-1}\|$ tends to infinity although $\|\mathcal{Q}_{s_n}(T)^{-1}\|$ stays bounded.

CHAPTER 4

A Direct Approach to the S-Functional Calculus for Closed Operators

The S-functional calculus for unbounded operators in Definition 2.68 was introduced by transforming the unbounded operator to a bounded one and then applying the S-functional calculus for bounded operators as it was done in [36]. The technique of reducing the functional calculus for unbounded operators to the one for bounded operators is very useful in the classical complex setting. It is also applicable in the quaternionic setting, but not to every operator because it requires the S-resolvent set of the operator to contain a real point. Otherwise, for nonreal s, the map $x \mapsto (s-x)^{-1}$ does not correspond to the S-resolvent operators at s. In fact, this map is then not even slice hyperholomorphic. The natural candidates for replacing this function are the left and the right slice hyperholomorphic Cauchy kernels. They are slice hyperholomorphic, but they are not intrinsic slice hyperholomorphic. The essential principles, on which this technique is based, are the spectral mapping theorem and the compatibility of the functional calculus with the composition of functions. In the case of the S-functional calculus in the quaternionic setting, they do however only hold for intrinsic functions. Moreover, the composition of two slice hyperholomorphic functions is, in general, only slice hyperholomorphic if the inner function is intrinsic. The left and right Cauchy kernels can hence neither be used for reducing the problem of defining the S-functional calculus for unbounded operators to the bounded case.

In this chapter we introduce the S-functional calculus for unbounded operators therefore directly via the slice hyperholomorphic Cauchy integrals (2.35) and (2.36), similar to the approach that Taylor chose in [81] for the complex setting. This allows us drop the assumption $\rho_S(T) \cap \mathbb{R} \neq \emptyset$ and to extend the S-functional calculus introduced in Definition 2.68 to arbitrary operators in $\mathcal{K}(V)$ with non-empty S-resolvent set. We then study the properties of this functional calculus in considerable detail and obtain

results that go beyond the known properties of the S-functional calculus for unbounded operators.

We point out that one of the most important quaternionic linear operator is the nabla operator, which is the quaternionification of both the gradient and the divergence operator. When we study its spectral properties in Chapter 11, we will see that this operator does not satisfy $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. Hence, although one does not seem to gain a lot by removing the restriction $\rho_S(T) \cap \mathbb{R} \neq \emptyset$ at the first glance, because one would assume that most quaternionic linear operators satisfy this condition, it is a worthwhile effort to remove it.

All results in this chapter were published in [45], except for those in Section 4.6, which were shown in [22].

4.1 Some Remarks on Slice Cauchy Domains

The following theorem is well known in the complex case. Implicitly, it has also been assumed to hold true in our settings but, to the best of the author's knowledge, it has never been stated explicitly, which we shall do for the sake of completeness.

Theorem 4.1. *Let C be a closed and let O be an open axially symmetric subsets of $\mathbb{H}$ such that $C \subset O$ and such that ∂O is nonempty and bounded. Then there exists a slice Cauchy domain U such that $C \subset U$ and $cl(U) \subset O$ and such that U is unbounded if O is unbounded.*

Proof. Let $\mathsf{i} \in \mathbb{S}$ and set $C_{\mathsf{i}} = C \cap \mathbb{C}_{\mathsf{i}}$ and $O_{\mathsf{i}} = O \cap \mathbb{C}_{\mathsf{i}}$. We cover the plane $\mathbb{C}_{\mathsf{i}}$ by a honeycomb network of non-overlapping congruent hexagons of side $\delta/4$ with

$$0 < \delta < \operatorname{dist}(C_{\mathsf{i}}, O_{\mathsf{i}}^{\mathsf{c}}) := \inf\{|z - z'| : z \in C_{\mathsf{i}}, z' \in O_{\mathsf{i}}^{\mathsf{c}}\},$$

where $O_{\mathsf{i}}^{\mathsf{c}}$ denotes the complement of O_{i} in $\mathbb{C}_{\mathsf{i}}$ and we choose this network symmetric with respect to the real axis. We call the closure of such a hexagon a cell and denote the set of all cells in our network by $\mathfrak{S}$. Set

$$S := \bigcup\{\Delta \in \mathfrak{S} : \Delta \cap O_{\mathsf{i}}^{\mathsf{c}} \neq \emptyset\}.$$

By standard arguments, we deduce that $U_{\mathsf{i}} := S^{\mathsf{c}}$ is a Cauchy domain in $\mathbb{C}_{\mathsf{i}}$ such that $C_{\mathsf{i}} \subset U_{\mathsf{i}}$ and $cl(U_{\mathsf{i}}) \subset O_{\mathsf{i}}$, which is unbounded if O_{i} is unbounded. We refer to the proof of [81, Theorem 3.3] for the technical details. Since both the network of hexagons and the set $O_{\mathsf{i}}^{\mathsf{c}}$ are symmetric with respect to the real axis, the set S and in turn also U_{i} are symmetric with respect to the real axis.

Now set $U := [U_{\mathsf{i}}]$, where $[U_{\mathsf{i}}]$ is the axially symmetric hull of U_{i}. Since U_{i} is symmetric with respect to the real axis, we have $U_{\mathsf{i}} = U \cap \mathbb{C}_{\mathsf{i}}$. Moreover, as $C_{\mathsf{i}} \subset U_{\mathsf{i}}$ and $cl(U_{\mathsf{i}}) \subset O_{\mathsf{i}}$, we find

$$C = [C_{\mathsf{i}}] \subset [U_{\mathsf{i}}] = U \quad \text{and} \quad cl(U) = [cl(U_{\mathsf{i}})] \subset [O_{\mathsf{i}}] = O.$$

If O is unbounded then O_{i} and U_{i} are unbounded. Thus, U is unbounded too.

It remains to show that U is actually a slice Cauchy domain. Let $\mathsf{j} \in \mathbb{S}$ and observe that $U \cap \mathbb{C}_{\mathsf{j}} = \{z_0 + \mathsf{j} z_1 : z_0 + \mathsf{i} z_1 \in U_{\mathsf{i}}\}$ because U is axially symmetric and $U_{\mathsf{i}} = U \cap \mathbb{C}_{\mathsf{i}}$. Since the mapping $\Phi : z_0 + \mathsf{i} z_1 \mapsto z_0 + \mathsf{j} z_1$ is a homeomorphism from $\mathbb{C}_{\mathsf{i}}$ to $\mathbb{C}_{\mathsf{j}}$ and the

set U_{i} is a Cauchy domain in $\mathbb{C}_{\mathsf{i}}$, we conclude that $U \cap \mathbb{C}_{\mathsf{j}} = \Phi(U_{\mathsf{i}})$ is a Cauchy domain in $\mathbb{C}_{\mathsf{j}}$.

□

The boundary of a slice Cauchy domain in a complex plane $\mathbb{C}_{\mathsf{i}}$ is of course symmetric with respect to the real axis. Hence, it can be fully described by the part that lies in the upper half plane $\mathbb{C}_{\mathsf{i}}^{\geq} := \{z_0 + \mathsf{i}z_1 : z_0 \in \mathbb{R}, z_1 \geq 0\}$. We specify this idea in the following statements.

Definition 4.2. For a path $\gamma : [0,1] \to \mathbb{C}_{\mathsf{i}}$, we define the paths $(-\gamma)(t) := \gamma(1-t)$ and $\overline{\gamma}(t) := \overline{\gamma(t)}$.

Lemma 4.3. *Let γ be a Jordan curve in $\mathbb{C}_{\mathsf{i}}$ whose image is symmetric with respect to the real axis. Then $\gamma_+ := \gamma \cap \mathbb{C}_{\mathsf{i}}^{\geq}$ consists of a single curve and $\gamma = \gamma_+ \cup \gamma_-$ with $\gamma_- := -\overline{\gamma_+}$.*

Proof. Since its image is symmetric with respect to the real axis, γ must take values in the upper and in the lower complex halfplane. Hence, as it is closed and continuous, it intersects the real line at least twice: once passing from the lower to the upper halfplane and once passing from the upper to the lower halfplane. Consider now a parametrisation $\gamma(t)$, $t \in [0,1]$ of γ with constant speed such that $\gamma(0) \in \mathbb{R}$ and such that $\gamma(t) \in \mathbb{C}_{\mathsf{i}}^{\geq}$ for t small enough. Then $\overline{\gamma}(t) := \overline{\gamma(t)}$ defines a parametrisation of γ with inverse orientation and constant speed because the image of γ is symmetric with respect to the real axis. On the other hand $(-\gamma)(t) := \gamma(1-t)$ is also a parametrisation of γ with inverse orientation and the same speed and starting point. We deduce $-\gamma = \overline{\gamma}$ and in turn $\gamma = -\overline{\gamma}$. Thus, $\gamma(1/2) = (-\overline{\gamma})(1/2) = \overline{\gamma(1/2)}$ and hence $\gamma(1/2) \in \mathbb{R}$. Moreover, there are no other points of γ that lie on the real line: if $\gamma(\tau) \in \mathbb{R}$ for some $\tau \notin \{0, 1/2\}$, then $\gamma(\tau) = \overline{\gamma(\tau)} = \gamma(1-\tau)$, which yields a contradictions as γ does not intersect itself.

Therefore, $\gamma_+(t) := \gamma(t/2)$, $t \in [0,1]$ takes values in $\mathbb{C}_{\mathsf{i}}^{\geq}$. Otherwise, by continuity, this path would have to intersect the real line when passing from the upper to the lower halfplane, which is impossible by the above argumentation. Moreover, the image of γ_+ coincides with $\gamma \cap \mathbb{C}_{\mathsf{i}}^{\geq}$ because $\gamma = -\overline{\gamma}$ and hence

$$\gamma \setminus \gamma_+ = \left\{\gamma(t) : \frac{1}{2} < t < 1\right\} = \left\{\overline{\gamma(t)} : 0 < t < \frac{1}{2}\right\} = \left\{\overline{\gamma_+(t)} : 0 < t < 1\right\},$$

which is a subset of $\mathbb{C}_{\mathsf{i}} \setminus : \mathbb{C}_{\mathsf{i}}^{\geq}$ as $\gamma_+(t) \in \mathbb{C}_{\mathsf{i}}^{+} = \{z_0 + \mathsf{i}z_1 : z_0 \in \mathbb{R}, z_1 > 0\}$ for $t \in (0,1)$.

Finally, $\gamma(t) = \gamma_+(2t)$ if $t \in [0,1/2]$ and $\gamma(t) = \overline{\gamma_+}(2-2t)$ if $t \in [1/2,1]$ and hence $\gamma = \gamma_+ \cup \gamma_-$.

□

Let now U be a slice Cauchy domain and consider any $\mathsf{i} \in \mathbb{S}$. The boundary $\partial(U \cap \mathbb{C}_{\mathsf{i}})$ of U in $\mathbb{C}_{\mathsf{i}}$ consists of a finite union of piecewise continuously differentiable Jordan curves and is symmetric with respect to the real axis. Hence, whenever a curve γ belongs to $\partial(U \cap \mathbb{C}_{\mathsf{i}})$, the curve $-\overline{\gamma}$ belongs to $\partial(U \cap \mathbb{C}_{\mathsf{i}})$ too. We can therefore decompose $\partial(U \cap \mathbb{C}_{\mathsf{i}})$ as follows:

- First define $\gamma_{+,1}, \ldots, \gamma_{+,\kappa}$ as those Jordan curves that belong to $\partial(U \cap \mathbb{C}_{\mathsf{i}})$ and lie entirely in the open upper complex halfplane $\mathbb{C}_{\mathsf{i}}^{+}$. The curves $-\overline{\gamma_{+,1}}, \ldots, -\overline{\gamma_{+,\kappa}}$ are then exactly those Jordan curves that belong to $\partial(U \cap \mathbb{C}_{\mathsf{i}})$ and lie entirely in the lower complex halfplane $\mathbb{C}_{\mathsf{i}}^{-}$
- In a second step consider the curves $\gamma_{\kappa+1}, \ldots, \gamma_N$ that belong to $\partial(U \cap \mathbb{C}_{\mathsf{i}})$ and take values both in $\mathbb{C}_{\mathsf{i}}^{+}$ and $\mathbb{C}_{\mathsf{i}}^{-}$. Define $\gamma_{+,\ell}$ for $\ell = \kappa + 1, \ldots, N$ as the part of γ_ℓ that lies in $\mathbb{C}_{\mathsf{i}}^{+}$ and $\gamma_{-,\ell} = -\overline{\gamma_{+,\ell}}$ as the part of γ_ℓ that lies in $\mathbb{C}_{\mathsf{i}}^{-}$, cf. Lemma 4.3.

Overall, we obtain the following decomposition of $\partial(U \cap \mathbb{C}_{\mathsf{i}})$:

$$\partial(U \cap \mathbb{C}_{\mathsf{i}}) = \bigcup_{1 \leq \ell \leq N} \gamma_{+,\ell} \cup -\overline{\gamma_{+,\ell}}.$$

Definition 4.4. We call the set $\{\gamma_{1,+}, \ldots, \gamma_{N,+}\}$ the part of $\partial(U \cap \mathbb{C}_{\mathsf{i}})$ that lies in $\mathbb{C}_{\mathsf{i}}^{\geq}$.

4.2 Definition and Algebraic Properties of the S-Functional Calculus

We want to define the S-functional calculus for an arbitrary operator in $\mathcal{K}(V)$ with nonempty S-resolvent set via the slice hyperholomorphic Cauchy integral in (2.35) and (2.36). The domain of integration is thereby the boundary of a suitable slice Cauchy domain U in one of the complex planes $\mathbb{C}_{\mathsf{i}}$. In order for the S-functional calculus to be well-defined, we have to show that these integrals are independent of the choice of the slice Cauchy domain U and the complex plane $\mathbb{C}_{\mathsf{i}}$. For the bounded case, this was first shown in [27]. We follow the strategy known from the bounded case, which can also be found in the monograph [36].

Theorem 4.5. *Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \neq \emptyset$. If $f \in \mathcal{SH}_L(\sigma_S(T) \cup \{\infty\})$, then there exists an unbounded slice Cauchy domain U with $\sigma_S(T) \subset U$ and $cl(U) \subset \mathcal{D}(f)$. The integral*

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathsf{i}})} S_L^{-1}(s, T)\, ds_{\mathsf{i}}\, f(s) \tag{4.1}$$

defines an operator in $\in \mathcal{B}(V)$ and this operator is the same for any choice of the imaginary unit $\mathsf{i} \in \mathbb{S}$ and for any choice of the slice Cauchy domain U that satisfies the above conditions.

Similarly, if $f \in \mathcal{SH}_R(\sigma_S(T) \cup \{\infty\})$, then there exists an unbounded slice Cauchy domain U such that $\sigma_S(T) \subset U$ and $cl(U) \subset \mathcal{D}(f)$. Again, the integral

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathsf{i}})} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s, T)$$

defines an operator in $\mathcal{B}(V)$ and this operator is the same for any choice of the imaginary unit $\mathsf{i} \in \mathbb{S}$ and for any choice of the slice Cauchy domain U that satisfies the above conditions.

Proof. Let $f \in \mathcal{SH}_L(\sigma_S(T) \cup \{\infty\})$ and $x \in \rho_S(T)$. Since $\rho_S(T)$ is open, there exists a closed ball $cl(B_\varepsilon(x)) \subset \rho_S(T)$ and since $\rho_S(T)$ is axially symmetric we have

$$[cl(B_\varepsilon(x))] = \{s = s_0 + \mathsf{i}_s s_1 \in \mathbb{H} : (s_0 - x_0)^2 + (s_1 - x_1)^2 \leq \varepsilon\} \subset \rho_S(T).$$

The existence of the slice Cauchy domain U follows now from Theorem 4.1 applied with $C = \sigma_S(T)$ and $O = \mathcal{D}(F) \cap (\mathbb{H} \setminus cl(B_\varepsilon(x)))$.

The boundary of U in $\mathbb{C}_{\mathsf{i}}$ consists of a finite set of closed piecewise differentiable Jordan curves and so it is compact. Hence, (4.1) is the integral of a bounded integrand over a compact domain. Thus it converges in $\mathcal{B}(V)$ and defines an operator in $\mathcal{B}(V)$.

We now show the independence of the slice Cauchy domain. Consider first the case of another unbounded slice Cauchy domain U' such that $\sigma_S(T) \subset U'$ and $cl(U') \subset \mathcal{D}(f)$. Let us for the moment furthermore assume that $cl(U') \subset U$. Then the set $W = U \setminus cl(U')$ is a bounded slice Cauchy domain and

$$\partial(W \cap \mathbb{C}_{\mathsf{i}}) = \partial(U \cap \mathbb{C}_{\mathsf{i}}) \cup -\partial(U' \cap \mathbb{C}_{\mathsf{i}}),$$

where $-\partial(U' \cap \mathbb{C}_{\mathsf{i}})$ denotes the inversely orientated boundary of U' in $\mathbb{C}_{\mathsf{i}}$. Moreover, the function $s \mapsto S_L^{-1}(s,T)$ is right and the function $s \mapsto f(s)$ is left slice hyperholomorphic on $cl(W)$. Thus, Theorem 2.27 implies

$$\begin{aligned} 0 &= \frac{1}{2\pi} \int_{\partial(W\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s) \\ &= \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s) - \frac{1}{2\pi} \int_{\partial(U'\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s). \end{aligned}$$

If $cl(U')$ is not contained in U, then $U \cap U'$ is an axially symmetric open set that contains $\sigma_S(T)$ such that $\partial(U \cap U')$ is nonempty and bounded. Theorem 4.1 implies the existence of a third slice Cauchy domain W such that $\sigma_S(T) \subset W$ and $cl(W) \subset U \cap U'$. By the above arguments, the choice of any of them yields the same operator in (4.1).

Finally, we consider another imaginary unit $\mathsf{j} \in \mathbb{S}$ and choose another unbounded slice Cauchy domain W with $\sigma_S(T) \subset W$ and $cl(W) \subset U$. By the above arguments and Theorem 2.30, we have

$$\begin{aligned} &\frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s) = \frac{1}{2\pi} \int_{\partial(W\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s) \\ =&\frac{1}{(2\pi)^2} \int_{\partial(W\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}} \left(f(\infty) + \int_{\partial(U\cap\mathbb{C}_{\mathsf{j}})} S_L^{-1}(x,s)\, dx_{\mathsf{j}}\, f(p) \right) \\ =&\frac{1}{(2\pi)^2} \int_{\partial(W\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(\infty) \\ &- \frac{1}{(2\pi)^2} \int_{\partial(U\cap\mathbb{C}_{\mathsf{j}})} \int_{\partial(W^{\mathsf{c}}\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, S_L^{-1}(x,s)\, dx_{\mathsf{j}}\, f(x), \end{aligned}$$

where Fubini's theorem allows us to exchange the order of integration in the last equation because we integrate a bounded function over a finite domain. The set W^{c} is a bounded slice Cauchy domain and the left S-resolvent is right slice hyperholomorphic in s on $cl(W^{\mathsf{c}})$. Theorem 2.27 implies

$$\frac{1}{(2\pi)^2} \int_{\partial(W\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(\infty) = -\frac{1}{(2\pi)^2} \int_{\partial(W^{\mathsf{c}}\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(\infty) = 0.$$

Since any $p \in \partial(U \cap \mathbb{C}_{\mathsf{i}})$ belongs to W^{c} by our choices of U and W and since

$S_L^{-1}(p,s) = -S_R^{-1}(s,p)$, we deduce from Theorem 2.30

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,f(s) = \\
&= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{j}})}\left(\frac{1}{2\pi}\int_{\partial(W^c\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,S_R^{-1}(s,p)\right)dp_{\mathsf{j}}\,f(p) \\
&= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{j}})} S_L^{-1}(p,T)\,dp_{\mathsf{j}}\,f(p).
\end{aligned}$$

□

Definition 4.6. Let $T\in\mathcal{K}(V)$ with $\rho_S(T)\neq\emptyset$. For any $f\in\mathcal{SH}_L(\sigma_S(T)\cup\{\infty\})$, we define

$$f(T) := f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,f(s), \tag{4.2}$$

and for $f\in\mathcal{SH}_R(\sigma_S(T)\cup\{\infty\})$, we define

$$f(T) := f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} f(s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T), \tag{4.3}$$

where $\mathsf{i}\in\mathbb{S}$ is arbitrary and U is any slice Cauchy domain as in Theorem 4.5.

Remark 4.7. If $\rho_S(T)\cap\mathbb{R}\neq\emptyset$, then our approach is because of Theorem 2.69 obviously consistent with the one used in [36]. Moreover, it also includes the case of bounded operators: if $f\in\mathcal{SH}_L(\sigma_S(T))$ for a bounded operator T, then we can choose $r>0$ such that $cl(U)$ is contained in the ball $B_r(0)$ because the slice domain U in Definition 2.65 is bounded. Since we do not require connectedness of $\mathcal{D}(f)$ in Definition 4.6, we might then extend f to a function in $\mathcal{SH}_L(\sigma_S(T)\cup\{\infty\})$, for instance by setting $f(s)=c$ with $c\in\mathbb{H}$ on $\mathbb{H}\setminus B_r(0)$, and use the unbounded slice Cauchy domain $(\mathbb{H}\setminus B_r(0))\cup U$ in (4.2). Since the left S-resolvent is then right slice hyperholomorphic on $\mathbb{H}\setminus B_r(0)$ and $f(s)$ is left slice hyperholomorphic on this set, we obtain

$$\begin{aligned}
f(T) =& f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{-\partial(B_r(0)\cap\mathbb{C}_{\mathsf{i}})} f(s)\,ds_{\mathsf{i}}\,S_L^{-1}(s,T) \\
&+ \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} f(s)\,ds_{\mathsf{i}}\,S_L^{-1}(s,T) \\
=& \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} f(s)\,ds_{\mathsf{i}}\,S_L^{-1}(s,T)
\end{aligned}$$

because Theorem 2.27 implies that the sum of $f(\infty)\mathcal{I}$ and the integral over the boundary of $B_r(0)$ vanishes.

Example 4.8. Let $T\in\mathcal{K}(V)$ with $\rho_S(T)\neq\emptyset$. Consider the left slice hyperholomorphic function $f(s)=a$ for some $a\in\mathbb{H}$ and choose an arbitrary unbounded slice Cauchy domain U with $\sigma_S(T)\subset U$ and an imaginary unit $\mathsf{i}\in\mathbb{S}$. Then

$$f(T) = f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,f(s) = a\mathcal{I}, \tag{4.4}$$

because $f(\infty) = a$ and the integral vanishes by Theorem 2.27 as the left S-resolvent is right slice hyperholomorphic in s on a superset of $cl(\mathbb{H} \setminus U)$ and vanishes at infinity. An analogue argument shows that also $f(T) = \mathcal{I}a = a\mathcal{I}$ if f is considered right slice hyperholomorphic.

The following algebraic properties of the S-functional calculus follow immediately from the left and right linearity of the integral.

Corollary 4.9. *Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \neq \emptyset$.*

(i) If $f, g \in \mathcal{SH}_L(\sigma_S(T) \cup \{\infty\})$ and $a \in \mathbb{H}$, then

$$(f+g)(T) = f(T) + g(T) \quad \text{and} \quad (fa)(T) = f(T)a.$$

(ii) If $f, g \in \mathcal{SH}_R(\sigma_S(T) \cup \{\infty\})$ and $a \in \mathbb{H}$, then

$$(f+g)(T) = f(T) + g(T) \quad \text{and} \quad (af)(T) = af(T).$$

Remark 4.10. Theorem 4.5 ensures that these functional calculi are well-defined in the sense that they are independent of the choices of the imaginary unit $\mathbf{i} \in \mathbb{S}$ and the slice Cauchy domain U. However, they are not consistent unless one restricts to functions that are defined on axially symmetric slice domains. As we shall see in the following, there exist functions that are left and right slice hyperholomorphic such that (4.2) and (4.3) do not give the same operator, cf. Remark 4.17 and Example 4.34. However, at least for intrinsic functions (4.2) and (4.3) are two representations for the same operator as the next theorem shows. An heuristic explanation for this inconsistency between the left and right slice hyperholomorphic version of the S-functional calculus is given in Section 8.3.

Lemma 4.11. *Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \neq \emptyset$ and let $f \in \mathcal{SH}(\sigma_S(T) \cup \{\infty\})$. Furthermore consider a slice Cauchy domain U such that $\sigma_S(T) \subset U$ and $cl(U) \subset \mathcal{D}(f)$ and some imaginary unit $\mathbf{i} \in \mathbb{S}$. If $\gamma_1, \ldots, \gamma_N$ is the part of $\partial(U \cap \mathbb{C}_{\mathbf{i}})$ that lies in $\mathbb{C}_{\mathbf{i}}^+$ as in Definition 4.4, then*

$$\begin{aligned}
&\int_{\partial(U\cap\mathbb{C}_{\mathbf{i}})} f(s)\, ds_{\mathbf{i}}\, S_R^{-1}(s,T) \\
&= \sum_{\ell=1}^{N} \int_0^1 2\mathrm{Re}\left(f(\gamma_\ell(t))(-\mathbf{i})\gamma_\ell'(t)\overline{\gamma_\ell(t)} \right) \mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\, dt \\
&\quad - \sum_{\ell=1}^{N} \int_0^1 2\mathrm{Re}\Big(f(\gamma_\ell(t))(-\mathbf{i})\gamma_\ell'(t) \Big) T \mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\, dt.
\end{aligned} \tag{4.5}$$

Proof. We have

$$\begin{aligned}
&\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} f(s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)\\
&=\sum_{\ell=1}^{N}\int_{\gamma_\ell} f(s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)+\sum_{\ell=1}^{N}\int_{-\overline{\gamma_\ell}} f(s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)\\
&=\sum_{\ell=1}^{N}\int_0^1 f(\gamma_\ell(t))(-\mathsf{i})\gamma_\ell'(t)\left(\overline{\gamma_\ell(t)}-T\right)\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\,dt\\
&\quad+\sum_{\ell=1}^{N}\int_0^1 f\left(\overline{\gamma_\ell(1-t)}\right)\mathsf{i}\overline{\gamma_\ell'(1-t)}(\gamma_\ell(1-t)-T)\mathcal{Q}_{\overline{\gamma_\ell(1-t)}}(T)^{-1}\,dt.
\end{aligned}$$

Since $f(\overline{x}) = \overline{f(x)}$ as f is intrinsic and $\mathcal{Q}_{\overline{s}}(T)^{-1} = \mathcal{Q}_s(T)^{-1}$ for $s\in\rho_S(T)$, we get after a change of variables in the integrals of the second sum

$$\begin{aligned}
&\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} f(s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)\\
&=\sum_{\ell=1}^{N}\int_0^1 f(\gamma_\ell(t))(-\mathsf{i})\gamma_\ell'(t)\left(\overline{\gamma_\ell(t)}-T\right)\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\,dt\\
&\quad+\sum_{\ell=1}^{N}\int_0^1 \overline{f(\gamma_\ell(t))(-\mathsf{i})\gamma_\ell'(t)}(\gamma_\ell(t)-T)\mathcal{Q}_{\gamma(t)}(T)^{-1}\,dt\\
&=\sum_{\ell=1}^{N}\int_0^1 2\mathrm{Re}\left(f(\gamma_\ell(t))(-\mathsf{i})\gamma_\ell'(t)\overline{\gamma_\ell(t)}\right)\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\,dt\\
&\quad-\sum_{\ell=1}^{N}\int_0^1 2\mathrm{Re}\Big(f(\gamma_\ell(t))(-\mathsf{i})\gamma_\ell'(t)\Big)T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\,dt.
\end{aligned}$$

□

Theorem 4.12. *Let $T\in\mathcal{K}(V)$ with $\rho_S(T)\neq\emptyset$. If $f\in\mathcal{SH}(\sigma_S(T)\cup\{\infty\})$, then*

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,f(s)=\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} f(s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)$$

for any $\mathsf{i}\in\mathbb{S}$ and any slice Cauchy domain as in Theorem 4.5.

Proof. Fix U and $\mathsf{i}\in\mathbb{S}$, let $\gamma_1,\dots\gamma_N$ be the part of $\partial(U\cap\mathbb{C}_{\mathsf{i}})$ that lies in $\mathbb{C}_{\mathsf{i}}^+$ and write the integral involving the right S-resolvent as an integral over these paths as in (4.5). Any operator commutes with real numbers and $f(\gamma_\ell(t))$, $\gamma_\ell'(t)$ and $\overline{\gamma_\ell(t)}$ commute

mutually since they all belong to the same complex plane $\mathbb{C}_{\mathsf{i}}$. Hence

$$
\begin{aligned}
&\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)\\
=&\sum_{\ell=1}^{N}\int_0^1 \mathcal{Q}_{\gamma(t)}(T)^{-1} 2\mathrm{Re}\left(\overline{\gamma_\ell(t)}\gamma_\ell'(t)(-\mathsf{i})f(\gamma_\ell(t))\right)\, dt\\
&-\sum_{\ell=1}^{N}\int_0^1 T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}2\mathrm{Re}\Big(\gamma_\ell'(t)(-\mathsf{i})f(\gamma_\ell(t))\Big)\, dt\\
=&\sum_{\ell=1}^{N}\int_0^1 \left(T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1} - \mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\overline{\gamma_\ell(t)}\right)\gamma_\ell'(t)(-\mathsf{i})f(\gamma_\ell(t))\, dt\\
&+\sum_{\ell=1}^{N}\int_0^1 \left(T\mathcal{Q}_{\overline{\gamma_\ell(t)}}(T)^{-1} - \mathcal{Q}_{\overline{\gamma_\ell(t)}}(T)^{-1}\gamma_\ell(t)\right)\overline{\gamma_\ell'(t)}\mathsf{i}\overline{f(\gamma_\ell(t))}\, dt\\
=&\sum_{\ell=1}^{N}\int_{\gamma_\ell} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s) + \sum_{\ell=1}^{N}\int_{-\overline{\gamma_\ell}} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s)\\
=&\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s).
\end{aligned}
$$

□

Corollary 4.13. *Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \neq \emptyset$. The S-functional calculus for left slice hyperholomorphic functions and the S-functional calculus for right slice hyperholomorphic functions agree for intrinsic functions: if $f \in \mathcal{SH}(\sigma_S(T)\cup\{\infty\})$, then* (4.2) *and* (4.3) *give the same operator.*

Remark 4.14. For intrinsic functions, slice hyperholomorphic Cauchy integrals of the form (4.2) and (4.3) are always equivalent. We have shown this only for the S-functional calculus, but with the same technique one can show this equivalence for instance also for the H^∞-functional calculus or for fractional powers of quaternionic linear operators. Since the technique for showing this equivalence is the same in any situation, we will use it without proving it explicitly at every occurrence.

Recall that a function f on U is called locally constant if every point $x \in U$ has a neighborhood $B_x \subset U$ such that f is constant on U. A locally constant function f is constant on every connected subset of the its domain. Thus, since every sphere $[x]$ is connected, the function f is constant on every sphere if its domain U is axially symmetric, i.e. it is of the form $f(x) = c(x_0, x_1)$, where c is locally constant on an appropriate subset of $\mathbb{R}^2$. Therefore f can be considered a left and a right slice function and it is even left and right slice hyperholomorphic because the partial derivatives of a locally constant function vanish.

Lemma 4.15. *A function f is left and right slice hyperholomorphic if and only if $f = c + \tilde{f}$, where c is a locally constant slice function and $\tilde{f}$ is intrinsic.*

Proof. Obviously any function that admits a decomposition of this type is both left and right slice hyperholomorphic. Assume on the other hand that f is left and right slice

hyperholomorphic such that

$$f(x) = \alpha(x_0, x_1) + \mathbf{i}_x \beta(x_0, x_1)$$

and

$$f(x) = \hat{\alpha}(x_0, x_1) + \hat{\beta}(x_0, x_1)\mathbf{i}_x.$$

The compatibility conditions (2.4) imply

$$\alpha(x_0, x_1) = \frac{1}{2}\left(f(x) + f(\overline{x})\right) = \hat{\alpha}(x_0, x_1),$$

from which we deduce $\mathbf{i}\beta(x_0, x_1) = f(x_{\mathbf{i}}) - \alpha(x_0, x_1) = \hat{\beta}(x_0, x_1)\mathbf{i}$ for any $\mathbf{i} \in \mathbb{S}$, where $x_{\mathbf{i}} = x_0 + \mathbf{i}x_1$. Hence we have

$$\mathbf{i}\beta(x_0, x_1)\mathbf{i}^{-1} = \hat{\beta}(x_0, x_1).$$

If we choose $\mathbf{i} = \mathbf{i}_{\beta(x_0,x_1)}$, then $\mathbf{i}$ and $\beta(x_0, x_1)$ commute and we obtain $\beta(x_0, x_1) = \hat{\beta}(x_0, x_1)$. Moreover, $\beta(x_0, x_1)$ commutes with every $\mathbf{i} \in \mathbb{S}$ because $\mathbf{i}\beta(x_0, x_1) = \hat{\beta}(x_0, x_1)\mathbf{i} = \beta(x_0, x_1)\mathbf{i}$, which implies that $\beta(x_0, x_1)$ is real.

Since β takes real values, its partial derivatives $\frac{\partial}{\partial x_0}\beta(x_0, x_1)$ and $\frac{\partial}{\partial x_1}\beta(x_0, x_1)$ are real-valued too. Thus, since α and β satisfy the Cauchy-Riemann-equations (2.6), the partial derivatives of α also take real-values.

Now define $\tilde{\alpha}(x_0, x_1) = \mathrm{Re}(\alpha(x_0, x_1))$ and $\tilde{\beta}(x_0, x_1) = \beta(x_0, x_1)$ and set $\tilde{f}(x) = \tilde{\alpha}(x_0, x_1) + \mathbf{i}_x\beta(x_0, x_1)$ and $c(x) = f(x) - \tilde{f}(x) = \mathrm{Im}(\alpha(x_0, x_1))$. Obviously, $\tilde{\alpha}$ and $\tilde{\beta}$ satisfy the compatibility conditions (2.4). They also satisfy the Cauchy-Riemann-equations (2.6) because α and β do and

$$\frac{\partial}{\partial x_\ell}\tilde{\alpha}(x_0, x_1) = \frac{\partial}{\partial x_\ell}\mathrm{Re}(\alpha(x_0, x_1)) = \mathrm{Re}\left(\frac{\partial}{\partial x_\ell}\alpha(x_0, x_1)\right) = \frac{\partial}{\partial x_\ell}\alpha(x_0, x_1)$$

for $\ell = 1, 2$. Therefore $\tilde{f}$ is a left slice hyperholomorphic function with real-valued components, thus intrinsic.

It remains to show that c is locally constant. Since $c(x) = \mathrm{Im}(\alpha(x_0, x_1))$ depends only on x_0 and x_1 but not on the imaginary unit $\mathbf{i}_x$, it is constant on every sphere $[x] \subset U$. Moreover, as the sum of two left and right slice hyperholomorphic functions, it is left (and right) slice hyperholomorphic and thus its restriction $c_{\mathbf{i}}$ to any complex plane $\mathbb{C}_{\mathbf{i}}$ is an $\mathbb{H}$-valued left holomorphic function. But

$$c_{\mathbf{i}}'(x) = \frac{\partial}{\partial x_0}c_{\mathbf{i}}(x) = \frac{\partial}{\partial x_0}f(x) - \frac{\partial}{\partial x_0}\tilde{f}(x) = 0 \quad x \in U \cap \mathbb{C}_{\mathbf{i}}$$

and hence c is locally constant on $U \cap \mathbb{C}_{\mathbf{i}}$. If $x \in U$, we can therefore find a neighborhood $B_{\mathbf{i}_x}$ of x in $U \cap \mathbb{C}_{\mathbf{i}_x}$ such that $c_{\mathbf{i}_x}$ is constant on $B_{\mathbf{i}_x}$. Since c is constant on any sphere, it is even constant on the axially symmetric hull $B = [B_{\mathbf{i}_x}]$ of $B_{\mathbf{i}_x}$, which is a neighborhood of x in U.

□

Corollary 4.16. *Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \neq \emptyset$ and let f be both left and right slice hyperholomorphic on $\sigma_S(T)$ and at infinity. If $\mathcal{D}(f)$ is connected, then* (4.2) *and* (4.3) *define the same operator.*

Proof. By applying Lemma 4.15 we obtain a decomposition $f = c + \tilde{f}$ of f into the sum of a locally constant function c and an intrinsic function $\tilde{f}$. Since $\mathrm{dom}(f)$ is connected, c is even constant. Thus, Corollary 4.13 and Example 4.8 imply

$$
\begin{aligned}
& f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \\
=& c\left(\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\, S_R^{-1}(s,T)\right) + \tilde{f}(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} \tilde{f}(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \\
=& c\mathcal{I} + \tilde{f}(T) = \mathcal{I}c + \tilde{f}(T) \\
=& \left(\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\right) c + \tilde{f}(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, \tilde{f}(s) \\
=& f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s),
\end{aligned}
$$

where U and $\mathsf{i} \in \mathbb{S}$ are chosen as in Definition 4.6.

□

Remark 4.17. We point out that Corollary 4.16 does not hold true in general. The S-functional calculus has usually been considered for functions that are defined on connected sets, namely on axially symmetric slice domains. Hence, the calculi for left and right slice hyperholomorphic functions were consistent as we have seen in Corollary 4.16.

However, this restriction occurred only due to the reasons explained in Remark 2.16. Since it excludes the class of slice hyperholomorphic functions whose domain is not connected, which in particular contains those functions that generate spectral projections as they are studied in Section 4.5, it is worthwhile to remove it. The price one has to pay in this case is that the two functional calculi become inconsistent. Indeed, in Corollary 4.16 the function c is constant since $\mathcal{D}(f)$ is connected and hence, by Example 4.8, the functional calculi for left and right slice hyperholomorphic functions yield $c(T) = c\mathcal{I}$ and $c(T) = \mathcal{I}c$, respectively. Since the identity operator commutes with every constant c, these operators coincide.

If on the contrary $\mathcal{D}(f)$ is not connected, then c is only locally constant, i.e. it will in general be of the form $c(x) = \sum \chi_{\Delta_\ell}(x)c_\ell$ with $c_\ell \in \mathbb{H}$, where the Δ_ℓ are disjoint axially symmetric sets and χ_{Δ_ℓ} denotes the characteristic function of Δ_ℓ, which is obviously intrinsic. The functional calculi for left and right slice hyperholomorphic functions yield then $c(T) = \sum \chi_{\Delta_\ell}(T)c_\ell$ and $c(T) = \sum c_\ell\chi_{\Delta_\ell}(T)$, respectively. These two operators coincide only if the operators $\chi_{\Delta_\ell}(T)$ commute with the scalars c_ℓ. As we will see in Section 4.5, the operators $\chi_{\Delta_\ell}(T)$ are spectral projections onto invariant subspaces of the operator T. Since the operator T is right linear, its invariant subspaces are right subspaces of V. But if a projection $\chi_{\Delta_\ell}(T)$ commutes with with any scalar, then $a\mathbf{v} = a\chi_{\Delta_\ell}(T)\mathbf{v} = \chi_{\Delta}(T)a\mathbf{v} \in \chi_{\Delta_\ell}(T)V$ for any $\mathbf{v} \in \chi_{\Delta_\ell}(T)V$ and any $a \in \mathbb{H}$. Thus $\chi_{\Delta_\ell}(T)V$ is also a left-sided and therefore even a two-sided subspace of V. In general, this is not true: the invariant subspaces obtained from spectral projections are only right sided. Hence, the projections $\chi_{\Delta_\ell}(T)$ do not necessarily commute with any scalar and it might be that $\sum \chi_{\Delta_\ell}(T)c_\ell \neq \sum c_\ell\chi_{\Delta_\ell}(T)$, i.e. the two functional calculi

give different operators for the same function. An explicit example for this situation is given in Example 4.34.

4.3 The Product Rule and Polynomials in T

In order to prove the product rule for the S-functional calculus, we recall Lemma 3.23 in [4]. For the reasons explained in Remark 2.16, the lemma was originally stated assuming that U is a slice domain. However, the same proof works also in the case that U is a only a bounded slice Cauchy domain.

Lemma 4.18. *Let $B \in \mathcal{B}(V)$, let U be a bounded slice Cauchy domain and assume that $f \in \mathcal{SH}(cl(U))$. For $p \in U$ and any $\mathsf{i} \in \mathbb{S}$, we have*

$$Bf(p) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} f(s)\, ds_{\mathsf{i}}\, (\overline{s}B - Bp)(p^2 - 2s_0 p + |s|^2)^{-1}.$$

Theorem 4.19. *Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \neq \emptyset$. If $f \in \mathcal{SH}(\sigma_S(T) \cup \{\infty\})$ and $g \in \mathcal{SH}_L(\sigma_S(T) \cup \{\infty\})$, then*

$$(fg)(T) = f(T)g(T). \tag{4.6}$$

Similarly, if $f \in \mathcal{SH}_R(\sigma_S(T) \cup \{\infty\})$ and $g \in \mathcal{SH}(\sigma_S(T) \cup \{\infty\})$, then the product rule (4.6) *also holds true.*

Proof. Let $f \in \mathcal{SH}(\sigma_S(\sigma_S(T) \cup \{\infty\})$ and let $g \in \mathcal{SH}_L(\sigma_S(T) \cup \{\infty\})$. By Theorem 4.5, there exist unbounded slice Cauchy domains U_p and U_s such that $\sigma_S(T) \subset U_p$ and $cl(U_p) \subset U_s$ and $cl(U_s) \subset \mathcal{D}(f) \cap \mathcal{D}(g)$. The subscripts s and p indicate the respective variable of integration in the following computation. Moreover, we use the notation $[\partial O]_{\mathsf{i}} := \partial(O \cap \mathbb{C}_{\mathsf{i}})$ for an axially symmetric set O in order to obtain more compact formulas.

Recall that the operator $f(T)$ can by Theorem 4.12 also be represented using the right S-resolvent operator and hence

$$\begin{aligned} f(T)g(T) = \left(f(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{[\partial U_s]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \right) \cdot \\ \cdot \left(g(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{[\partial U_p]_{\mathsf{i}}} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, g(p) \right). \end{aligned}$$

For the product of the integrals, the S-resolvent equation (2.30) gives us that

$$\begin{aligned} &\int_{[\partial U_s]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \int_{[\partial U_p]_{\mathsf{i}}} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, g(p) \\ &= \int_{[\partial U_s]_{\mathsf{i}}} \int_{[\partial U_p]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, g(p) \end{aligned}$$

$$
\begin{aligned}
&= \int_{[\partial U_s]_{\mathsf{i}}} \int_{[\partial U_p]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) p (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p) \\
&\quad - \int_{[\partial U_s]_{\mathsf{i}}} \int_{[\partial U_p]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_L^{-1}(p,T) p (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p) \\
&\quad - \int_{[\partial U_s]_{\mathsf{i}}} \int_{[\partial U_p]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, \overline{s} S_R^{-1}(s,T) (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p) \\
&\quad + \int_{[\partial U_s]_{\mathsf{i}}} \int_{[\partial U_p]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, \overline{s} S_L^{-1}(p,T) (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p).
\end{aligned}
$$

For the sake of readability, let us denote these last four integrals by $I_1, \dots I_4$.

If $r > 0$ is large enough, then $\mathbb{H} \setminus U_s$ is entirely contained in $B_r(0)$. In particular, $W := B_r(0) \cap U_p$ is then a bounded slice Cauchy domain with boundary $\partial(W \cap \mathbb{C}_{\mathsf{i}}) = \partial(U_p \cap \mathbb{C}_{\mathsf{i}}) \cup \partial(B_r(0) \cap \mathbb{C}_{\mathsf{i}})$. From Lemma 4.18, we deduce

$$
\begin{aligned}
I_1 &= \int_{[\partial U_s]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \int_{[\partial U_p]_{\mathsf{i}}} p (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p) \\
&= \int_{[\partial U_s]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \int_{[\partial W]_{\mathsf{i}}} p (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p) \\
&\quad - \int_{[\partial U_s]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \int_{[\partial B_r(0)]_{\mathsf{i}}} p (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p) \\
&= - \int_{[\partial U_s]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \int_{[\partial B_r(0)]_{\mathsf{i}}} p (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p),
\end{aligned}
$$

where the last equality follows from the Cauchy integral theorem since the function $p \mapsto p(p^2 - 2s_0 p + |s|^2)^{-1}$ is left slice hyperholomorphic and the function $p \mapsto g(p)$ is right slice hyperholomorphic on $cl(W)$ by our choice of U_s and U_p. If we let r tend to $+\infty$ and apply Lebesgue's theorem in order to exchange limit and integration, the inner integral tends to $2\pi g(\infty)$ and hence

$$
I_1 = -2\pi \left(\int_{[\partial U_s]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \right) g(\infty).
$$

We also have

$$
\begin{aligned}
-I_2 + I_4 = \int_{[\partial U_s]_{\mathsf{i}}} \int_{[\partial U_p]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, &\left(\overline{s} S_L^{-1}(p,T) - p S_L^{-1}(p,T) \right) \cdot \\
&\cdot (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p).
\end{aligned}
$$

and applying Fubini's theorem allows us to change the order of integration. If we now set $W = B_r(0) \cap U_s$ with r sufficiently large we obtain as before a bounded slice Cauchy domain with $\partial(W \cap \mathbb{C}_{\mathsf{i}}) = \partial(U_s \cap \mathbb{C}_{\mathsf{i}}) \cup \partial(B_r(0) \cap \mathbb{C}_{\mathsf{i}})$. Applying Lemma 4.18

with $B = S_L^{-1}(p,T)$, we find

$$\begin{aligned}
-I_2 + I_4 =& \int_{[\partial U_p]_{\mathsf{i}}} \int_{[\partial W]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, \left(\overline{s} S_L^{-1}(p,T) - p S_L^{-1}(p,T)\right) \cdot \\
&\qquad\qquad\qquad\qquad \cdot (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p) \\
&- \int_{[\partial U_p]_{\mathsf{i}}} \int_{[\partial B_r(0)]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, \left(\overline{s} S_L^{-1}(p,T) - p S_L^{-1}(p,T)\right) \cdot \\
&\qquad\qquad\qquad\qquad \cdot (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p) \\
=& 2\pi \int_{[\partial U_p]_{\mathsf{i}}} S_L^{-1}(p,T) f(p)\, dp_{\mathsf{i}}\, g(p) \\
&- \int_{[\partial U_p]_{\mathsf{i}}} \int_{[\partial B_r(0)]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, \overline{s} S_L^{-1}(p,T)(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p) \\
&- \int_{[\partial U_p]_{\mathsf{i}}} \int_{[\partial B_r(0)]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, p S_L^{-1}(p,T)(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p).
\end{aligned}$$

Observe that the third integral tends to zero as $r \to +\infty$. For the second one, we obtain by applying Lebesgue's theorem

$$\begin{aligned}
&\int_{[\partial U_p]_{\mathsf{i}}} \int_{[\partial B_r(0)]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, \overline{s} S_L^{-1}(p,T)(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p) \\
=& \int_{[\partial U_p]_{\mathsf{i}}} \left(\int_0^{2\pi} f(re^{i\phi}) r^2 S_L^{-1}(p,T)(p^2 - 2r\cos(\phi) p + r^2)^{-1}\, d\phi \right) dp_{\mathsf{i}}\, g(p) \\
&\qquad \overset{r \to +\infty}{\longrightarrow} 2\pi f(\infty) \int_{[\partial U_p]_{\mathsf{i}}} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, g(p).
\end{aligned}$$

Since f is intrinsic, $f(p)$ commutes with dp_{i}, and hence

$$\begin{aligned}
-I_2 + I_4 =& 2\pi \int_{[\partial U_p]_{\mathsf{i}}} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, f(p) g(p) \\
&- 2\pi f(\infty) \int_{[\partial U_p]_{\mathsf{i}}} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, g(p).
\end{aligned}$$

Finally, we consider the integral I_3. If we set again $W = B_r(0) \cap U_p$ with r sufficiently large, then

$$\begin{aligned}
-I_3 =& - \int_{[\partial U_s]_{\mathsf{i}}} \int_{[\partial W]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, \overline{s} S_R^{-1}(s,T)(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p) \\
&+ \int_{[\partial U_s]_{\mathsf{i}}} \int_{[\partial B_r(0)]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, \overline{s} S_R^{-1}(s,T)(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p).
\end{aligned}$$

By our choice of U_s and U_p, the functions $p \mapsto (p^2 - 2s_0 p + |s|^2)^{-1}$ and $p \mapsto g(p)$ are left resp. right slice hyperholomorphic on $cl(W)$. Hence, Cauchy's integral theorem implies that the first integral equals zero. Letting r tend to infinity, we can apply

Lebesgue's theorem in order to exchange limit and integration and we see that

$$-I_3 = \int_{[\partial U_s]_{\mathsf{i}}} \int_{[\partial B_r(0)]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, \overline{s} S_R^{-1}(s,T)(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, g(p) \to 0.$$

Altogether, we obtain

$$\begin{aligned}
&\frac{1}{(2\pi)^2} \int_{[\partial U_s]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \int_{[\partial U_p]_{\mathsf{i}}} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, g(p) \\
&= -\frac{1}{2\pi}\left(\int_{[\partial U_s]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)\right) g(\infty) + \frac{1}{2\pi}\int_{[\partial U_p]_{\mathsf{i}}} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, f(p)g(p) \\
&\quad - f(\infty)\frac{1}{2\pi}\int_{[\partial U_p]_{\mathsf{i}}} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, g(p).
\end{aligned}$$

We thus have

$$\begin{aligned}
f(T)g(T) =& f(\infty)g(\infty)\mathcal{I} + f(\infty)\frac{1}{2\pi}\int_{[\partial U_p]_{\mathsf{i}}} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, g(p) \\
&+ \left(\frac{1}{2\pi}\int_{[\partial U_s]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)\right) g(\infty) \\
&+ \frac{1}{(2\pi)^2}\int_{[\partial U_s]_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \int_{[\partial U_p]_{\mathsf{i}}} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, g(p) \\
=& f(\infty)g(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{[\partial U_p]_{\mathsf{i}}} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, f(p)g(p) = (fg)(T).
\end{aligned}$$

□

If the operator T is bounded, then slice hyperholomorphic polynomials of T belong to the class of functions that are admissible for the S-functional calculus. In the unbounded cases, this is not true, but the S-functional calculus is in some sense still compatible, at least with intrinsic polynomials. For such polynomial $P(s) = \sum_{k=0}^{n} a_k s^k$ with $a_k \in \mathbb{R}$, the operator $P(T)$ is as usual defined as the operator

$$P(T)\mathbf{v} := \sum_{k=0}^{n} a_k T^k \mathbf{v} \qquad \mathbf{v} \in \operatorname{dom}(T^n).$$

Lemma 4.20. *Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \neq \emptyset$, let $f \in \mathcal{SH}(\sigma_S(T) \cup \{\infty\})$ and let P be an intrinsic polynomial of degree $n \in \mathbb{N}_0$. If $\mathbf{v} \in \operatorname{dom}(T^n)$, then $f(T)\mathbf{v} \in \operatorname{dom}(T^n)$ and $f(T)P(T)\mathbf{v} = P(T)f(T)\mathbf{v}$.*

Proof. We consider first the special case $P(s) = s$. Let U be an unbounded slice Cauchy domain with $\sigma_S(T) \subset U$, let $\mathsf{i} \in \mathbb{S}$ and let $\{\gamma_1, \ldots, \gamma_n\}$ be the part of the boundary $\partial(U \cap \mathbb{C}_{\mathsf{i}})$ of U in $\mathbb{C}_{\mathsf{i}}$ that lies in $\mathbb{C}_{\mathsf{i}}^{\geq}$, cf. Definition 4.4. We consider $\mathbf{v} \in \operatorname{dom}(T)$ and apply the operatorial equality (4.5) to $T\mathbf{v}$ taking into account that

$\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}T\mathbf{v} = T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\mathbf{v}$. Since T commutes with real numbers, we find

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathbf{i}})} f(s)\,ds_{\mathbf{i}}\,S_R^{-1}(s,T)T\mathbf{v}\\
=&\sum_{\ell=1}^{n} T\frac{1}{2\pi}\int_0^1 2\mathrm{Re}\left(f(\gamma_\ell(t))(-\mathbf{i})\gamma_\ell'(t)\overline{\gamma_\ell(t)}\right)\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\mathbf{v}\\
&-\sum_{\ell=1}^{n} T\frac{1}{2\pi}\int_0^1 2\mathrm{Re}\Big(f(\gamma_\ell(t))(-\mathbf{i})\gamma_\ell'(t)\Big)T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\mathbf{v}\\
=&T\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathbf{i}})} f(s)\,ds_{\mathbf{i}}\,S_R^{-1}(s,T)\mathbf{v},
\end{aligned}$$

where Hille's theorem for the Bochner integral, Theorem 20 in [38, Chapter III.6], allowed us to move T in front of the integral and the last equation follows again from (4.5). Finally, observe that $f(\infty) = \lim_{s\to\infty} f(s)$ is real since $f(s)\in\mathbb{R}$ for any $s\in\mathbb{R}$ because f is intrinsic. Hence, we find

$$\begin{aligned}
f(T)T\mathbf{v} &= f(\infty)T\mathbf{v} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathbf{i}})} f(s)\,ds_{\mathbf{i}}\,S_R^{-1}(s,T)T\mathbf{v}\\
&= Tf(\infty)\mathbf{v} + T\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathbf{i}})} f(s)\,ds_{\mathbf{i}}\,S_R^{-1}(s,T)\mathbf{v} = Tf(T)\mathbf{v}.
\end{aligned}$$

In particular, this implies $f(T)\mathbf{v}\in\mathrm{dom}(T)$.

We show the general statement by induction with respect to the degree n of the polynomial. If $n=0$ then the statement follows immediately from Example 4.8. Now assume that it is true for $n-1$ and consider $P(s) = a_k s^n + P_{n-1}(s)$, where $a_n\in\mathbb{R}$ and $P_{n-1}(s)$ is an intrinsic polynomial of degree lower or equal to $n-1$. For $\mathbf{v}\in\mathrm{dom}(T^n)$ the above argumentation implies then $f(T)T^{n-1}\mathbf{v}\in\mathrm{dom}(T)$ and

$$\begin{aligned}
f(T)P(T)\mathbf{v} &= f(T)a_nT^n\mathbf{v} + f(T)P_{n-1}(T)\mathbf{v}\\
&= a_nTf(T)T^{n-1}\mathbf{v} + f(T)P_{n-1}(T)\mathbf{v}.
\end{aligned}$$

From the induction hypothesis, we further deduce $f(T)T^{n-1}\mathbf{v} = T^{n-1}f(T)\mathbf{v}$ and $f(T)P_{n-1}(T)\mathbf{v} = P_{n-1}(T)f(T)\mathbf{v}$ and hence

$$f(T)P(T)\mathbf{v} = a_nT^nf(T)\mathbf{v} + P_{n-1}(T)f(T)\mathbf{v} = P(T)f(T)\mathbf{v}.$$

In particular, we see that $f(T)\mathbf{v}$ belongs to $\mathrm{dom}(T^n)$ and we obtain that the statement is true.

$\square$

As in the complex case, we say that f has a zero of order n at ∞ if the first $n-1$-coefficients in the Taylor series expansion of $s\mapsto f(s^{-1})$ at 0 vanish and the n-th coefficient does not. Equivalently, f has a zero of order n if $\lim_{s\to\infty} f(s)s^n$ is bounded and nonzero. We say that f has a zero of infinite order at infinity, if it vanishes on a neighborhood of ∞.

Lemma 4.21. *Let $T\in\mathcal{K}(V)$ with $\rho_S(T)\neq\emptyset$ and assume that $f\in\mathcal{SH}(\sigma_S(T)\cup\{\infty\})$ has a zero of order $n\in\mathbb{N}_0\cup\{+\infty\}$ at infinity.*

(i) For any intrinsic polynomial P of degree lower than or equal to n, we have $P(T)f(T) = (Pf)(T)$.

(ii) If $\mathbf{v} \in \operatorname{dom}(T^m)$ for some $m \in \mathbb{N}_0 \cup \{\infty\}$, then $f(T)\mathbf{v} \in \operatorname{dom}(T^{m+n})$.

Proof. Assume first that f has a zero of order greater than or equal to one at infinity and consider $P(s) = s$. Then $Pf \in \mathcal{SH}(\sigma_S(T) \cup \{\infty\})$ and for $\mathbf{v} \in V$

$$(Pf)(T)\mathbf{v} = \lim_{s\to\infty} sf(s)\mathbf{v} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,sf(s)\mathbf{v},$$

with an appropriate slice Cauchy domain U and any imaginary unit $\mathsf{i} \in \mathbb{S}$. Since s and ds_{i} commute, we deduce from the left S-resolvent equation (2.29) that

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,sf(s)\mathbf{v}\\ =&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} TS_L^{-1}(s,T)\,ds_{\mathsf{i}}\,f(s)\mathbf{v} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\,f(s)\mathbf{v}\end{aligned}$$

Any sufficiently large ball $B_r(0)$ contains ∂U. The function $f(s)\mathbf{v}$ is then right slice hyperholomorphic on $cl(B_r(0)\cap U)$ and Cauchy's integral theorem implies

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\,f(s)\mathbf{v} &= \lim_{r\to+\infty} -\frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\,f(s)\mathbf{v}\\ &= \lim_{r\to+\infty} -\frac{1}{2\pi}\int_0^{2\pi} re^{\mathsf{i}\varphi}f(re^{\mathsf{i}\varphi})\mathbf{v}\,d\varphi = -\lim_{s\to+\infty} sf(s)\mathbf{v}.\end{aligned}$$

Thus, after applying Hille's theorem for the Bochner integral, Theorem 20 in [38, Chapter III.6], in order to write the operator T in front of the integral, we obtain

$$(Pf)(T)\mathbf{v} = T\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,f(s)\mathbf{v} = P(T)f(T)\mathbf{v}.$$

In particular, we see that $f(T)\mathbf{v} \in \operatorname{dom}(T)$.

We show (i) for monomials by induction and assume that it is true for $P(s) = s^{n-1}$ if f has a zero of order greater than or equal to $n-1$ at infinity. If the order of f at infinity is even greater than or equal to n, then $g(s) = s^{n-1}f(s)$ has a zero of order at least 1 at infinity and, from the above argumentation and the induction hypothesis, we conclude for $P(s) = s^n$

$$(Pf)(T)\mathbf{v} = Tg(T)\mathbf{v} = TT^{n-1}f(T)\mathbf{v} = T^nf(T)\mathbf{v},$$

which implies also $f(T)\mathbf{v} \in \operatorname{dom}(T^n)$. For arbitrary intrinsic polynomials the statement finally follows from the linearity of the S-functional calculus.

In order to show (ii) assume first $\mathbf{v} \in \operatorname{dom}(T^m)$ for $m \in \mathbb{N}$. If f has a zero of order $n \in \mathbb{N}$ at infinity, then (i) with $P(s) = s^n$ and Lemma 4.20 imply

$$(Pf)(T)T^m\mathbf{v} = T^nf(T)T^m\mathbf{v} = T^nT^mf(T)\mathbf{v} = T^{m+n}f(T)\mathbf{v}$$

and hence $f(T)\mathbf{v} \in \operatorname{dom}(T^{m+n})$. Finally, if $m = +\infty$ then $\mathbf{v} \in \operatorname{dom}(T^k)$ and hence $f(T)\mathbf{v} \in \operatorname{dom}(T^{k+n})$ for any $k \in \mathbb{N}$. Thus, $\mathbf{v} \in \operatorname{dom}(T^\infty)$.

□

Corollary 4.22. *Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \neq \emptyset$. For any intrinsic polynomial P, the operator $P(T)$ is closed.*

Proof. We choose $s \in \rho_S(T)$ and $n \in \mathbb{N}$ such that $m \leq 2n$, where m is the degree of P. Then $f(p) = P(p)\mathcal{Q}_s(p)^{-n}$ belongs to $\mathcal{SH}(\sigma_S(T) \cup \{\infty\})$ and has a zero of order $2n - m$ at infinity. Applying Lemma 4.21, we see that

$$P(T)\mathbf{v} = P(T)\mathcal{Q}_s(T)^n\mathcal{Q}_s(T)^{-n}\mathbf{v} = \mathcal{Q}_s(T)^n P(T)\mathcal{Q}_s(T)^{-n}\mathbf{v} = \mathcal{Q}_s(T)^n f(T)\mathbf{v}$$

for $\mathbf{v} \in \operatorname{dom}(T^m)$. Since its inverse is bounded, the operator $\mathcal{Q}_s(T)^n$ is closed and in turn $P(T)$ is closed as it is the composition of a closed and a bounded operator.

□

Corollary 4.23. *Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \neq \emptyset$. If $f \in \mathcal{SH}(\sigma_S(T) \cup \{\infty\})$ has no zeros on $\sigma_S(T)$ and a zero of even order n at infinity, then $\operatorname{ran}(f(T)) = \operatorname{dom}(T^n)$ and $f(T)$ is invertible in the sense of closed operators. If $\rho_S(T) \cap \mathbb{R} \neq \emptyset$, this holds true for any order $n \in \mathbb{N}$.*

Proof. Let $P \in \rho_S(T)$ and set $k = n/2$. The function $h(s) = f(s)\mathcal{Q}_p(s)^k$ with $\mathcal{Q}_p(s) = s^2 - 2p_0 s + |p|^2$ belongs to $\mathcal{SH}(\sigma_S(T) \cup \{\infty\})$ and does not have any zeros in $\sigma_S(T)$. Furthermore, $h(\infty) = \lim_{s\to\infty} h(s)$ is finite and nonzero. Hence, the function $s \mapsto h(s)^{-1}$ belongs to $\mathcal{SH}(\sigma_S(T) \cup \{\infty\})$ and we deduce from Theorem 4.19 that $h(T)$ is invertible in $\mathcal{B}(V)$ with $h(T)^{-1} = h^{-1}(T)$. Theorem 4.19 moreover implies $f(T) = \mathcal{Q}_p(T)^{-k}h(T)$. Now observe that $h(T)$ maps V bijectively onto V and that $\mathcal{Q}_p(T)^{-k}$ maps V onto $\operatorname{dom}(T^{2k}) = \operatorname{dom}(T^n)$. Thus $\operatorname{ran}(f(T)) = \operatorname{dom}(T^n)$.

Finally, $f(T)^{-1} := h^{-1}(T)\mathcal{Q}_p(T)^k$ is a closed operator because h is bijective and continuous and $\mathcal{Q}_p(T)^k$ is closed by Corollary 4.22: It satisfies $f(T)^{-1}f(T)\mathbf{v} = \mathbf{v}$ for $\mathbf{v} \in V$ and $f(T)f(T)^{-1}\mathbf{v} = \mathbf{v}$ for $\mathbf{v} \in \operatorname{dom}(T^n)$. Thus it is the inverse of $f(T)$.

In the case there exists a point $a \in \rho_S(T) \cap \mathbb{R}$, similar arguments hold with $P(s) = (s-a)^n$ instead of $\mathcal{Q}_p(s)^k$. In particular, this allows us to include functions with a zero of odd order at infinity too.

□

4.4 The Spectral Mapping Theorem and Composite Functions

We recall that the extended spectrum $\sigma_{SX}(T)$ equals $\sigma_S(T)$ if T is bounded and it equals $\sigma_S(T) \cup \{\infty\}$ if T is unbounded.

Theorem 4.24 (Spectral Mapping Theorem)**.** *Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \neq \emptyset$. For any function $f \in \mathcal{SH}(\sigma_S(T) \cup \{\infty\})$, we have $\sigma_S(f(T)) = f(\sigma_{SX}(T))$.*

Proof. Let us first show the relation $\sigma_S(f(T)) \supset f(\sigma_{SX}(T))$. For $p \in \sigma_S(T)$ consider the function

$$g(s) := (f(s)^2 - 2\mathrm{Re}(f(p))f(s) - |f(p)|^2)(s^2 - 2\mathrm{Re}(p)s + |p|^2)^{-1},$$

which is defined on $\mathcal{D}(f) \setminus [p]$. If we set $p_{\mathsf{i}_s} = p_0 + \mathsf{i}_s p_1$, then p_{i_s} and s commute. Since f is intrinsic, it maps $\mathbb{C}_{\mathsf{i}}$ into $\mathbb{C}_{\mathsf{i}}$ and hence $f(p_{\mathsf{i}_s})$ and $f(s)$ commute, too. Thus

$$g(s) = \frac{(f(s) - f(p_{\mathsf{i}_s}))(f(s) - \overline{f(p_{\mathsf{i}_s})})}{(s - p_{\mathsf{i}_s})(s - \overline{p_{\mathsf{i}_s}})}$$

and we can extend g to all of $\mathcal{D}(f)$ by setting

$$g(s) = \begin{cases} \partial_S f(s) \left(\underline{f(p)}\,\underline{p}^{-1} \right) & s \in [p] \text{ if } p \notin \mathbb{R} \\ (\partial_S f(s))^2 & s = p, \text{ if } p \in \mathbb{R} \end{cases}, \tag{4.7}$$

where $\underline{p} = \frac{1}{2}(p - \overline{p})$ denotes the vectorial part of p. Now observe that

$$(s^2 - 2\mathrm{Re}(p)s + |p|^2)g(s) = f(s)^2 + 2\mathrm{Re}(f(p))f(s) + |f(p)|^2$$

and that g has zero of order greater or equal to 2 at infinity. Hence, we can apply the S-functional calculus to deduce from Lemma 4.21, Theorem 4.19 and Example 4.8 that

$$(T^2 - 2\mathrm{Re}(p)T + |p|^2\mathcal{I})g(T)\mathbf{v} = (f(T)^2 + 2\mathrm{Re}(f(p))f(T) + |f(p)|\mathcal{I})\mathbf{v}$$

for any $\mathbf{v} \in V$ and

$$g(T)(T^2 - 2\mathrm{Re}(p)T + |p|^2\mathcal{I}g(T))\mathbf{v} = (f(T)^2 + 2\mathrm{Re}(f(p))f(T) + |f(p)|\mathcal{I})\mathbf{v}$$

for $\mathbf{v} \in \mathrm{dom}(T^2)$. If $f(p) \in \rho_S(T)$, then

$$\mathcal{Q}_{f(p)}(f(T)) = f(T)^2 - 2\mathrm{Re}(f(p))f(T) + |f(p)|\mathcal{I}$$

is invertible and $\mathcal{Q}_{f(p)}(f(T))^{-1}g(T) = g(T)\mathcal{Q}_{f(p)}(f(T))^{-1}$ is the inverse of the operator $\mathcal{Q}_p(T) = T^2 - 2\mathrm{Re}(p)T + |p|^2\mathcal{I}$. Hence, $f(p) \notin \sigma_S(f(T))$ implies $p \notin \sigma_S(T)$ and as a consequence $p \in \sigma_S(T)$ implies $f(p) \in \sigma_S(T)$, that is $f(\sigma_S(T)) \subset \sigma_S(f(T))$.

Finally, observe that $f(\infty) = \lim_{p\to\infty} f(p)$ is real because f is intrinsic and thus takes real values on the real line. If T is unbounded and $f(\infty) \neq f(p)$ for any point $p \in \sigma_S(T)$ (otherwise we already have $f(\infty) \in f(\sigma_S(T)) \subset \sigma_S(f(T))$), then the function $h(s) = (f(s) - f(\infty))^2$ belongs to $\mathcal{SH}(\sigma_S(T) \cup \{\infty\})$ and has a zero of even order n at infinity but no zero in $\sigma_S(T)$. By Corollary 4.23, the range of $h(T) = \mathcal{Q}_{f(\infty)}(f(T))$ is $\mathrm{dom}(T^n)$. Thus, it does not admit a bounded inverse and we obtain $f(\infty) \in \sigma_S(f(T))$. Altogether, we have $f(\sigma_{SX}(T)) \subset \sigma(f(T))$.

In order to show the relation $\sigma_S(f(T)) \subset f(\sigma_{SX}(T))$, we first consider a point $c \in \sigma_S(f(T))$ such that $c \neq f(\infty)$. We want to show $c \in f(\sigma_S(T))$ and assume the converse, i.e. $f(s) - c$ has no zeros on $\sigma_S(T)$.

If c is real, then the function $h(s) = f(s) - c$ is intrinsic, has no zeros on $\sigma_S(T)$ and $\lim_{s\to\infty} h(s) = f(\infty) - c \neq 0$. Hence, $h^{-1}(s) = (f(s) - c)^{-1}$ belongs to $\mathcal{SH}(\sigma_S(T) \cup \{\infty\})$. Applying the S-functional calculus, we deduce from Theorem 4.19 that $h^{-1}(T)$ is the inverse of $f(T) - c\mathcal{I}$ and hence $\mathcal{Q}_c(f(T))^{-1} = (h^{-1}(T))^2$, which is a contradiction as $c \in \sigma_S(f(T))$. Thus, $c = f(p)$ for some $p \in \sigma_S(T)$.

If on the other hand c is not real, then $f - c_{\mathsf{i}} \neq 0$ for any $c_{\mathsf{i}} = c_0 + \mathsf{i}_c c_1 \in [c]$. Indeed, $f(p) = \alpha(p_0, p_1) + \mathsf{i}_p\beta(p_0, p_1) = c_0 + \mathsf{i}c_1$ would imply $\mathsf{i}_p = \mathsf{i}$ and $\alpha(p_0, p_1) = c_0$ and $\beta(p_0, p_1) = c_1$ as α and β are real-valued because f is intrinsic. This would in turn imply $f(p_{\mathsf{i}_c}) = \alpha(p_0, p_1) + \mathsf{i}_c\beta(p_0, p_1) = c$, which would contradict our assumption. Therefore, the function

$$h(s) = (f(s)^2 - 2\mathrm{Re}(c)f(s) + |c|^2) = (f(s) - c_{\mathsf{i}_s})(f(s) - \overline{c_{\mathsf{i}_s}})$$

does not have any zeros on $\sigma_S(T)$. Moreover, since $f(\infty)$ is real, we have

$$h(\infty) = (f(\infty) - c)\overline{(f(\infty) - c)} = |f(\infty) - c|^2 \neq 0$$

and hence $h^{-1}(s) = (f(s)^2 - 2\mathrm{Re}(c)f(s) + |c|^2)^{-1}$ belongs to $\mathcal{SH}(\sigma_S(T) \cup \{\infty\})$. Applying the S-functional calculus, we deduce again from Theorem 4.19 that the operator $h^{-1}(T)$ is the inverse of $\mathcal{Q}_c(T)$, which contradicts $c \in \sigma_S(f(T))$. Hence, there must exist some $p \in \sigma_S(T)$ such that $c = f(p)$.

Altogether, we obtain $\sigma_S(f(T)) \setminus \{f(\infty)\}$ is contained in $f(\sigma_S(T))$.

Finally, let us consider the case that the point $c = f(\infty)$ belongs to $\sigma_S(f(T))$. If T is unbounded, then $\infty \in \sigma_{SX}(T)$ and hence $c \in f(\sigma_{SX}(T))$. If on the other hand T is bounded, then there exists a function $g \in \mathcal{SH}(\sigma_S(T) \cup \{\infty\})$ that coincides on an axially symmetric neighborhood $\sigma_S(T)$ with f but satisfies $c \neq g(\infty)$. In this case $f(T) = g(T)$, as pointed out in Remark 4.7, and we can apply the above argumentation with g instead of f to see that $c \in g(\sigma_S(T)) = f(\sigma_S(T))$.

□

Theorem 4.25. *If $T \in \mathcal{K}(V)$ with $\sigma_S(T) \neq \emptyset$, then $P(\sigma_S(T)) = \sigma_S(P(T))$ for any intrinsic polynomial P.*

Proof. The arguments are similar to those in the proof of Theorem 4.24: in order to show that $P(\sigma_S(T)) \subset \sigma_S(P(T))$, we consider the polynomial $\mathcal{Q}_{P(p)}(P(s))$, which is given by $\mathcal{Q}_{P(p)}(P(s)) = P(s)^2 - 2\mathrm{Re}(P(p))P(s) + |P(p)|^2$ for any $p \in \sigma_S(T)$. As p and $\overline{p}$ are both zeros of $\mathcal{Q}_{P(p)}(P(s))$ resp. as p is a zero of even order of $\mathcal{Q}_{P(p)}(P(s)) = (P(s) - P(p))^2$ if p is real, there exists an intrinsic polynomial $R(s)$ such that

$$\mathcal{Q}_{P(p)}(P(s)) = \mathcal{Q}_p(s)R(s).$$

If $P(p) \notin \sigma_S(P(T))$, then $\mathcal{Q}_{P(p)}(P(T))$ is invertible and Lemma 4.21 and Example 4.8 imply that $\mathcal{Q}_{P(p)}(P(T))^{-1}R(T)$ is the inverse of $\mathcal{Q}_p(T)$, which is a contradiction because we assumed $p \in \sigma_S(T)$. Therefore $P(p) \in \sigma_S(P(T))$.

Conversely assume that $p \notin P(\sigma_S(T))$. Then the function

$$\mathcal{Q}_p(P(s)) = P(s)^2 - 2\mathrm{Re}(p)P(s) + |p|^2$$

does not take any zero on $\sigma_S(T)$ and we conclude from Corollary 4.23 that $\mathcal{Q}_p(P(T))$ has a bounded inverse. Thus $p \notin \sigma_S(P(T))$ and in turn $\sigma_S(P(T)) \subset P(\sigma_S(T))$.

□

Theorem 4.26. *Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \neq \emptyset$. If $f \in \mathcal{SH}(\sigma_S(T) \cup \{\infty\})$ and $g \in \mathcal{SH}_L(f(\sigma_{SX}(T))$ or $g \in \mathcal{SH}_R(f(\sigma_{SX}(T))$, then*

$$(g \circ f)(T) = g(f(T)).$$

Proof. Because of Remark 4.7, we can assume that $f(\infty)$ belongs to $f(\sigma_{SX}(T))$. We apply Theorem 4.1 in order to choose an unbounded slice Cauchy domain U_p such that $\sigma_S(f(T)) = f(\sigma_{SX}(T)) \subset U_p$ and $cl(U_p) \subset \mathcal{D}(g)$ and a second unbounded slice Cauchy domain U_s such that $\sigma_S(T) \subset U_s$ and $cl(U_s) \subset f^{-1}(U_p) \cap \mathcal{D}(f)$. The subscripts are chosen in order to indicate the respective variable of integration in the following computation.

After choosing an imaginary unit $\mathsf{i} \in \mathbb{S}$, we deduce from Theorem 2.30, Cauchy's integral formula, that

$$\begin{aligned}&(g \circ f)(T) - (g \circ f)(\infty)\mathcal{I}\\ =&\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,(g\circ f)(s)\\ =&\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\left(\frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(p,f(s))\,dp_{\mathsf{i}}\,g(p)\right).\end{aligned}$$

Changing the order of integration by applying Fubini's theorem, we obtain

$$\begin{aligned}&(g \circ f)(T) - (g \circ f)(\infty)\mathcal{I}\\ =&\frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})}\left(\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,S_L^{-1}(p,f(s))\right)dp_{\mathsf{i}}\,g(p)\\ =&\frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(p,f(T))\,dp_{\mathsf{i}}\,g(p)\\ &-\frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(p,f(\infty))\,dp_{\mathsf{i}}\,g(p)\mathcal{I}\\ =&g(f(T)) - g(f(\infty))\mathcal{I}\end{aligned}$$

and hence $(g \circ f)(T) = g(f(T))$.

□

4.5 Spectral Sets and Projections onto Invariant Subspaces

As in the complex case, the S-functional calculus allows to associate subspaces of V that are invariant under T to certain subsets of $\sigma_S(T)$.

Definition 4.27. A subset σ of $\sigma_{SX}(T)$ is called a spectral set if it is open and closed in $\sigma_{SX}(T)$.

Just as $\sigma_S(T)$ and $\sigma_{SX}(T)$, every spectral set is axially symmetric: if $s \in \sigma$ then the entire sphere $[s]$ is contained in σ. Indeed, the set $\sigma \cap [s]$ is then a nonempty, open and closed subset of $\sigma_{SX}(T) \cap [s] = [s]$. Since $[s]$ is connected this implies $\sigma \cap [s] = [s]$. Moreover, if σ is a spectral set, then $\sigma' = \sigma_{SX}(T) \setminus \sigma$ is a spectral set too.

If σ is a spectral set of T, then σ and σ' can be separated in $\mathbb{H}_\infty$ by axially symmetric open sets and hence Theorem 4.1 implies the existence of two slice Cauchy domains U_σ and U'_σ containing σ and σ' respectively such that one of them is unbounded and $cl(U) \cap cl(U_{\sigma'}) = \emptyset$. We define

$$\chi_\sigma(x) := \begin{cases} 1 & \text{if } x \in U_\sigma \\ 0 & \text{if } x \in U'_\sigma. \end{cases}$$

The function $\chi_\sigma(x)$ obviously belongs to $\mathcal{SH}(\sigma_S(T) \cup \{\infty\})$.

Definition 4.28. Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \neq \emptyset$ and let $\sigma \subset \sigma_S(T)$ be a spectral set of T. The spectral projection associated with σ is the operator $E_\sigma := \chi_\sigma(T)$ obtained by applying the S-functional calculus to the function χ_σ. Furthermore, we define $V_\sigma := E_\sigma V$ and $T_\sigma = T|_{\mathrm{dom}(T_\sigma)}$ with $\mathrm{dom}(T_\sigma) = \mathrm{dom}(T) \cap V_\sigma$.

Explicit formulas for the operator E_σ are for bounded σ

$$E_\sigma = \frac{1}{2\pi}\int_{\partial(U_\sigma\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}} = \frac{1}{2\pi}\int_{\partial(U_\sigma\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\,S_R^{-1}(s,T)$$

and for unbounded σ

$$E_\sigma = \mathcal{I} + \frac{1}{2\pi}\int_{\partial(U_\sigma\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}} = \mathcal{I} + \frac{1}{2\pi}\int_{\partial(U_\sigma\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\,S_R^{-1}(s,T),$$

where the imaginary unit $\mathsf{i}\in\mathbb{S}$ can be chosen arbitrarily.

Corollary 4.29. *Let $T\in\mathcal{K}(V)$ such that $\rho_S(T)\neq\emptyset$ and let σ be a spectral set of T.*

(i) The operator E_σ is actually a projection, i.e. $E_\sigma^2 = E_\sigma$.

(ii) Set $\sigma' = \sigma_{SX}(T)\setminus\sigma$. Then $E_\sigma + E_{\sigma'} = \mathcal{I}$ and $E_\sigma E_{\sigma'} = E_{\sigma'}E_\sigma = 0$.

Proof. This follows immediately from the algebraic properties of the S-functional calculus shown in Corollary 4.9 and Theorem 4.19 as $\chi_\sigma^2 = \chi_\sigma$ and $\chi_\sigma + \chi_{\sigma'} = 1$ and $\chi_\sigma\chi_{\sigma'} = \chi_{\sigma'}\chi_\sigma = 0$.

□

The following Lemma 4.30 is a special case of [16, Chapter II §1.9, Proposition 14] and Lemma 4.31 is an immediate consequence of the fact that any projection with closed range is continuous.

Lemma 4.30. *Let A, B, M and N be right linear subspaces of a quaternionic right vector space V_R such that $A\subset M$ and $B\subset M$. If $A\oplus B = M\oplus N$, then $A = M$ and $B = N$.*

Lemma 4.31. *Let A, B, M and N be right linear subspaces of a quaternionic Banach vector space V_R such that $A\subset M$, $B\subset N$ and such that M, N and $M\oplus N$ are closed. Then $A\oplus B$ is dense in $M\oplus N$ if and only if A is dense in M and B is dense in N.*

Theorem 4.32. *Let $T\in\mathcal{K}(V)$ with $\rho_S(T)\neq\emptyset$ and let $E_1,E_2\in\mathcal{B}(V)$ be projections such that $E_1+E_2=\mathcal{I}$ (and hence $E_1E_2 = E_2E_1 = 0$). Denote $V_\ell := E_\ell(V)$ and $\operatorname{dom}(T_\ell) := E_\ell(\operatorname{dom}(T))$ and assume that $T(\operatorname{dom}(T_\ell))\subset V_\ell$ such that $T_\ell := T|_{\operatorname{dom}(T_\ell)}$ is a closed operator on the right Banach space V_ℓ for $\ell = 1,2$. Then*

(i) $E_\ell T\mathbf{v} = TE_\ell\mathbf{v}$ for $\mathbf{v}\in\operatorname{dom}(T)$,

(ii) $\operatorname{dom}(T_\ell^2) = E_\ell(\operatorname{dom}(T^2))$ for $\ell = 1,2$,

(iii) $\operatorname{ran}(\mathcal{Q}_s(T)) = \operatorname{ran}(\mathcal{Q}_s(T_1))\oplus\operatorname{ran}(\mathcal{Q}_s(T_2))$ for any $s\in\mathbb{H}$,

(iv) $\sigma_S(T) = \sigma_S(T_1)\cup\sigma_S(T_2)$ and

(v) $\sigma_{Sp}(T) = \sigma_{Sp}(T_1)\cup\sigma_{Sp}(T_2)$.

If moreover $\sigma_S(T_1)\cap\sigma_S(T_2)=\emptyset$, then

(vi) $\sigma_{Sc}(T) = \sigma_{Sc}(T_1)\cup\sigma_{Sc}(T_2)$ and

(vii) $\sigma_{Sr}(T) = \sigma_{Sr}(T_1)\cup\sigma_{Sr}(T_2)$.

Proof. The assertions (i) to (iii) are obvious. Now assume that $s \in \rho_S(T)$. Then $\operatorname{ran}(\mathcal{Q}_s(T)) = V$ and from (iii) we deduce

$$V_1 \oplus V_2 = V = \operatorname{ran}(\mathcal{Q}_s(T)) = \operatorname{ran}(\mathcal{Q}_s(T_1)) \oplus \operatorname{ran}(\mathcal{Q}_s(T_2)).$$

As $\operatorname{ran}(\mathcal{Q}_s(T_\ell)) \subset V_\ell$, Lemma 4.30 implies $\operatorname{ran}(\mathcal{Q}_s(T_\ell)) = V_\ell$ and hence $\mathcal{Q}_s(T_\ell)^{-1} = \mathcal{Q}_s(T)^{-1}|_{V_\ell}$ as $\mathcal{Q}_s(T_\ell) = \mathcal{Q}_s(T)|_{\operatorname{dom}(T_\ell^2)}$. Indeed, we have

$$\mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T_\ell)\mathbf{v} = \mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T)\mathbf{v} = \mathbf{v} \qquad \text{for } \mathbf{v} \in \operatorname{dom}(T_\ell^2)$$

and, since $\mathcal{Q}_s(T)^{-1}\mathbf{v} \in \operatorname{dom}(T_\ell^2)$ for $\mathbf{v} \in V_\ell$, also

$$\mathcal{Q}_s(T_\ell)\mathcal{Q}_s(T)^{-1}\mathbf{v} = \mathcal{Q}_s(T)\mathcal{Q}_s(T)^{-1}\mathbf{v} = \mathbf{v} \qquad \text{for } \mathbf{v} \in V_\ell.$$

Thus, $s \in \rho_S(T_1) \cap \rho_S(T_2)$. Conversely, if $s \in \rho_S(T_1) \cap \rho_S(T_2)$, then the operator $\mathcal{Q}_s(T_1)^{-1}E_1 + \mathcal{Q}_s(T_2)^{-1}E_2$ is the inverse of $\mathcal{Q}_s(T)$ and hence $s \in \rho_S(T)$. Altogether, $\rho_S(T) = \rho_S(T_1) \cap \rho_S(T_2)$, which is equivalent to $\sigma_S(T) = \sigma_S(T_1) \cup \sigma_S(T_2)$ and hence (iv) holds true.

Obviously, $\sigma_{Sp}(T_\ell) \subset \sigma_{Sp}(T)$ as any S-eigenvector of T_ℓ is also an S-eigenvector of T associated with the same eigensphere. Conversely, if $\mathbf{v} \neq \mathbf{0}$ is an S-eigenvector of T associated with the eigensphere $[s] = s_0 + \mathbb{S}s_1$, then set $\mathbf{v}_\ell = E_\ell\mathbf{v}$ and we observe that

$$\mathbf{0} = \mathcal{Q}_s(T)\mathbf{v} = \mathcal{Q}_s(T_1)\mathbf{v}_1 + \mathcal{Q}_s(T_2)\mathbf{v}_2.$$

As $\mathcal{Q}_s(T_\ell)\mathbf{v}_\ell \in V_\ell$ and $V_1 \cap V_2 = \{\mathbf{0}\}$, this implies $\mathcal{Q}_s(T_\ell)\mathbf{v}_\ell = \mathbf{0}$ for $\ell = 1, 2$. As $\mathbf{v} \neq \mathbf{0}$, at least one of the vectors $\mathbf{v}_\ell$ is nonzero and therefore an S-eigenvalue of T_ℓ associated with the eigensphere $[s]$. Thus $[s] \subset \sigma_{Sp}(T_1) \cup \sigma_{Sp}(T_2)$ and in turn $\sigma_{Sp}(T) = \sigma_{Sp}(T_1) \cup \sigma_{Sp}(T_2)$ so that (v) holds true.

We assume now that $\sigma_S(T_1) \cap \sigma_S(T_2) = \emptyset$. Then assertions (iv) and (v) imply that $s \in \sigma_{Sc}(T) \cup \sigma_{Sr}(T)$ if and only if $s \in \sigma_{Sc}(T_\ell) \cup \sigma_{Sr}(T_\ell)$ for either $\ell = 1$ or $\ell = 2$. We assume w.l.o.g. $s \in \sigma_{Sc}(T_1) \cup \sigma_{Sr}(T_1)$ and thus $s \in \rho_S(T_2)$. As $\operatorname{ran}(\mathcal{Q}_s(T_2)) = V_2$, we deduce from (iii) that and Lemma 4.31 that $\operatorname{ran}(\mathcal{Q}_s(T))$ is dense in $V = V_1 \oplus V_2$ if and only if $\operatorname{ran}(\mathcal{Q}_s(T_1))$ is dense in V. In other words: $s \in \sigma_{Sc}(T)$ if and only if $s \in \sigma_{Sc}(T_1)$ and in turn $s \in \sigma_{Sr}(T)$ if and only if $s \in \sigma_{Sr}(T_1)$.

□

Theorem 4.33. *Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \neq \emptyset$ and let $\sigma \subset \sigma_S(T)$ be a spectral set of T. Then*

(i) $E_\sigma(\operatorname{dom}(T)) \subset \operatorname{dom}(T)$

(ii) $T(\operatorname{dom}(T) \cap V_\sigma) \subset V_\sigma$

(iii) $\sigma = \sigma_{SX}(T_\sigma)$

(iv) $\sigma \cap \sigma_{Sp}(T) = \sigma_{Sp}(T_\sigma)$

(v) $\sigma \cap \sigma_{Sc}(T) = \sigma_{Sc}(T_\sigma)$

(vi) $\sigma \cap \sigma_{Sr}(T) = \sigma_{Sr}(T_\sigma)$

If the spectral set σ is bounded, then we further have

(vii) $V_\sigma \subset \operatorname{dom}(T^\infty)$

(viii) T_σ *is a bounded operator on* V_σ*.*

Proof. Assertion (i) follows from the definition of E_σ and Lemma 4.20. In order to prove (ii), we observe that if $\mathbf{v} \in \operatorname{dom}(T) \cap V_\sigma$, then $E_\sigma \mathbf{v} = \mathbf{v}$. Hence, we deduce from Lemma 4.20 that $E_\sigma T\mathbf{v} = TE_\sigma \mathbf{v} = T\mathbf{v}$, which implies $T\mathbf{v} \in V_\sigma$.

If σ is bounded, then we can choose U_σ bounded and hence χ_σ has a zero of infinite order at infinity. We conclude from Lemma 4.21 that $\mathbf{v} = E_\sigma \mathbf{v} = \chi_\sigma(T)\mathbf{v} \in \operatorname{dom}(T^\infty)$ for any $\mathbf{v} \in V_\sigma$ and hence (vii) holds true. In particular, $V_\sigma \subset \operatorname{dom}(T)$. Therefore T_σ is a bounded operator on V_σ as it is closed and everywhere defined.

We show now assertion (iii) and consider first a point $s \in \mathbb{H} \setminus \sigma$. We show that s belongs to $\rho_S(T_\sigma)$. For an appropriately chosen slice Cauchy domain U_σ, the function $f(s) := \mathcal{Q}_s(p)^{-1}\chi_{U_\sigma}(s)$ belongs to $\mathcal{SH}(\sigma_S(T) \cup \{\infty\})$. By Lemma 4.21 and Lemma 4.20, we have

$$f(T)\mathcal{Q}_s(T)\mathbf{v} = \chi_{U_\sigma}(T)\mathbf{v} = E_\sigma \mathbf{v}, \qquad \text{for } \mathbf{v} \in \operatorname{dom}(T^2) \cap V_\sigma$$

and

$$\mathcal{Q}_s(T)f(T)\mathbf{v} = \chi_{U_\sigma}(T)\mathbf{v} = E_\sigma \mathbf{v} = \mathbf{v} \qquad \text{for } \mathbf{v} \in V_\sigma.$$

Hence, $\mathcal{Q}_s(T_\sigma) = \mathcal{Q}_s(T)|_{V_\sigma \cap \operatorname{dom}(T^2)}$ has the inverse $f(T)|_{V_\sigma} \in \mathcal{B}(X_\sigma)$. Thus, we find $s \in \rho_S(T_\sigma)$ and in turn $\sigma_S(T_\sigma) \subset \sigma \cap \mathbb{H} =: \sigma_1$. The same argumentation applied to $T_{\sigma'}$ with $\sigma' = \sigma_{SX}(T) \setminus \sigma$ shows that $\sigma_S(T_{\sigma'}) \subset \sigma' \cap \mathbb{H} := \sigma_2$. But by (iv) in Theorem 4.32, we have

$$\sigma_S(T_\sigma) \cup \sigma_S(T_\sigma) = \sigma_S(T) = \sigma_1 \cup \sigma_2$$

and hence $\sigma_S(T_\sigma) = \sigma_1 = \sigma \cap \mathbb{H}$ and $\sigma_S(T_{\sigma'}) = \sigma_2 = \sigma' \cap \mathbb{H}$. If σ is bounded, then this is equivalent to (iii) because of (viii). If σ is not bounded, then $\infty \in \sigma$ and T is not bounded on X. However, in this case σ' is bounded and hence $T_\sigma \in \mathcal{B}(V_\sigma)$. But as $T = T_\sigma E_\sigma + T_{\sigma'}E_{\sigma'}$, we conclude that T_σ is unbounded as T is unbounded. Hence $\infty \in \sigma_{SX}(T_\sigma)$ and (viii) holds true also in this case.

Finally, (iv) to (vi) are direct consequences of (v) to (vii) in Theorem 4.32 as we know now that $\sigma_S(T_\sigma)$ and $\sigma_S(T_{\sigma'})$ are disjoint.

□

Example 4.34. We choose a generating basis $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k} = \mathbf{ij}$ of $\mathbb{H}$ and consider the quaternionic right-linear operator T on $V = \mathbb{H}^2$ that is defined by its action on the two right linearly independent right eigenvectors $\mathbf{v}_1 = (1, \mathbf{i})^T$ and $\mathbf{v}_2 = (\mathbf{j}, -\mathbf{k})^T$, namely

$$\begin{pmatrix} 1 \\ \mathbf{i} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \mathbf{j} \\ -\mathbf{k} \end{pmatrix} \mapsto \begin{pmatrix} -\mathbf{k} \\ -\mathbf{j} \end{pmatrix} = \begin{pmatrix} \mathbf{j} \\ -\mathbf{k} \end{pmatrix} \mathbf{i}.$$

Its matrix representation is

$$T = \frac{1}{2}\begin{pmatrix} -\mathbf{i} & 1 \\ -1 & -\mathbf{i} \end{pmatrix}.$$

Since, for operators on finite-dimensional spaces, the S-spectrum coincides with the set of right-eigenvalues by Theorem 2.56, we have $\sigma_S(T) = \sigma_R(T) = \{0\} \cup \mathbb{S}$. Indeed,

we have

$$\begin{aligned}\mathcal{Q}_s(T) &= \frac{1}{2}\begin{pmatrix}-1 & -\mathbf{i}\\ \mathbf{i} & -1\end{pmatrix} - s_0\begin{pmatrix}-\mathbf{i} & 1\\ -1 & -\mathbf{i}\end{pmatrix} + |s|^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}-\frac{1}{2}+|s|^2+s_0\mathbf{i} & -s_0-\frac{1}{2}\mathbf{i}\\ s_0+\frac{1}{2}\mathbf{i} & -\frac{1}{2}+|s|^2+s_0\mathbf{i}\end{pmatrix}\end{aligned}$$

and hence

$$\mathcal{Q}_s(T)^{-1} = |s|^{-2}(-1+2\mathbf{i}s_0+|s|^2)^{-1}\begin{pmatrix}-\frac{1}{2}+|s|^2+\mathbf{i}s_0 & \frac{1}{2}\mathbf{i}+s_0\\ -\frac{1}{2}\mathbf{i}-s_0 & -\frac{1}{2}+|s|^2+\mathbf{i}s_0\end{pmatrix},$$

which is defined for any $s \notin \{0\}\cup\mathbb{S}$. For any $s \in \rho_S(T)$, the left S-resolvent is therefore given by

$$\begin{aligned}S_L^{-1}(s,T) = \frac{1}{2}&|s|^{-2}(-1+|s|^2+2\mathbf{i}s_0)^{-1}\\ &\cdot\begin{pmatrix}|s|^2(\mathbf{i}+2\overline{s})+\overline{s}(-1+2\mathbf{i}s_0) & -|s|^2+\overline{s}(\mathbf{i}+2s_0)\\ |s|^2-\overline{s}(\mathbf{i}+2s_0) & |s|^2(\mathbf{i}+2\overline{s})+\overline{s}(-1+2\mathbf{i}s_0)\end{pmatrix}.\end{aligned}$$

Since $\sigma_S(T)\cap\mathbb{C}_{\mathbf{i}} = \{0,\mathbf{i},-\mathbf{i}\}$, we choose $U_{\{0\}} = B_{1/2}(0)$ and set $U_{\mathbb{S}} = B_2(0)\setminus B_{2/3}(0)$. For $s = \frac{1}{2}e^{\mathbf{i}\varphi} \in \partial U_{\{0\}}(0)\cap\mathbb{C}_{\mathbf{i}}$, we have

$$S_L^{-1}(s,T) = 2e^{-\mathbf{i}\varphi}\left(3\mathbf{i}+4\mathrm{Re}\left(e^{\mathbf{i}\varphi}\right)\right)^{-1}\begin{pmatrix}\mathbf{i}+e^{\mathbf{i}\varphi}+2\cos(\varphi) & 2+\mathbf{i}e^{\mathbf{i}\varphi}+2\mathbf{i}\cos\varphi\\ -2-\mathbf{i}e^{\mathbf{i}\varphi}+2\mathbf{i}\cos\varphi & \mathbf{i}+e^{\mathbf{i}\varphi}+2\cos\varphi\end{pmatrix}$$

and so

$$\begin{aligned}E_{\{0\}} =&\frac{1}{2\pi}\int_{\partial(U_{\{0\}}\cap\mathbb{C}_{\mathbf{i}})} S_L^{-1}(s,T)\,ds_{\mathbf{i}}\\ =&\frac{1}{2\pi}\int_0^{2\pi} 2e^{-\mathbf{i}\varphi}\left(3\mathbf{i}+4\mathrm{Re}\left(e^{\mathbf{i}\varphi}\right)\right)^{-1}\cdot\\ &\qquad\cdot\begin{pmatrix}\mathbf{i}+e^{\mathbf{i}\varphi}+2\cos(\varphi) & 2+\mathbf{i}e^{\mathbf{i}\varphi}-2\mathbf{i}\cos\varphi\\ -2-\mathbf{i}e^{\mathbf{i}\varphi}+2\mathbf{i}\cos\varphi & \mathbf{i}+e^{\mathbf{i}\varphi}+2\cos\varphi\end{pmatrix}\frac{1}{2}e^{\mathbf{i}\varphi}\mathbf{i}(-\mathbf{i})\,d\varphi\\ =&\frac{1}{2}\begin{pmatrix}1 & -\mathbf{i}\\ \mathbf{i} & 1\end{pmatrix}.\end{aligned}$$

A similar computation shows that

$$E_{\mathbb{S}} = \frac{1}{2\pi}\int_{\partial(U_{\mathbb{S}}\cap\mathbb{C}_{\mathbf{i}})} S_L^{-1}(s,T)\,ds_{\mathbf{i}} = \frac{1}{2}\begin{pmatrix}1 & \mathbf{i}\\ -\mathbf{i} & 1\end{pmatrix}.$$

Basic calculations show that these matrices actually define projections on $\mathbb{H}^2$ with $E_{\{0\}}+E_{\mathbb{S}} = \mathcal{I}$. Moreover, we have $E_{\{0\}}\mathbf{v}_1 = \mathbf{v}_1$ and $E_{\mathbb{S}}\mathbf{v}_1 = \mathbf{0}$ as well as $E_{\{0\}}\mathbf{v}_2 = \mathbf{0}$ and $E_{\mathbb{S}}\mathbf{v}_2 = \mathbf{v}_2$. Thus, the invariant subspace $E_{\{0\}}V$ associated with the spectral set $\{0\}$ is the right linear span of $\mathbf{v}_1$, which consist of all eigenvectors with respect to the real eigenvalue 0 as $T(\mathbf{v}_1a) = T(\mathbf{v}_1)a = \mathbf{0}$ for all $a \in \mathbb{H}$. The invariant subspace $E_{\mathbb{S}}$ associated with the spectral set $\mathbb{S}$ consists of the right linear span of $\mathbf{v}_2$. For $a \in \mathbb{H}\setminus\{0\}$, we

have $T(\mathbf{v}_2 a) = T(\mathbf{v}_2)a = \mathbf{v}_2\mathsf{i}a = (\mathbf{v}_2 a)(a^{-1}\mathsf{i}a)$. Thus, as $a^{-1}\mathsf{i}a \in \mathbb{S}$, the subspace $E_{\mathbb{S}}$ consists of all right eigenvectors associated with an eigenvalues in $\mathbb{S}$. (This is true only because the associated subspace is one-dimensional! Otherwise the subspace would consists of sums of eigenvectors associated with possibly different eigenvalues in the sphere $\mathbb{S}$, which do no have to be eigenvectors again, cf. Lemma 9.3.

Finally, we can construct functions, which are left and right slice hyperholomorphic on $\sigma_S(T)$, but for which the S-functional calculi for left and right slice hyperholomorphic functions yield different operators: consider the function

$$f(s) = c_1\chi_{U_{\{0\}}}(s) + c_2\chi_{U_{\mathbb{S}}}(s)$$

such that c_1 or c_2 does not belong to $\mathbb{C}_{\mathsf{i}}$. Choose for instance $c_1 = \mathsf{j}$ and $c_2 = 0$ for the sake of simplicity. This function is a locally constant slice function on $U = U_{\{0\}} \cup U_{\mathbb{S}}$ and thus left and right slice hyperholomorphic by Lemma 4.15. Then

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}} f(s) &= \left(\frac{1}{2\pi}\int_{\partial(B_{1/2}(0)\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\right)\mathsf{j}\\ &= \frac{1}{2}\begin{pmatrix}1 & -\mathsf{i}\\ \mathsf{i} & 1\end{pmatrix}\mathsf{j} = \frac{1}{2}\begin{pmatrix}\mathsf{j} & -\mathsf{k}\\ \mathsf{k} & \mathsf{j}\end{pmatrix},\end{aligned}$$

but

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} f(s)\,ds_{\mathsf{i}}\, S_R^{-1}(s,T) &= \mathsf{j}\left(\frac{1}{2\pi}\int_{\partial(B_{1/2}(0)\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\, S_R^{-1}(s,T)\right)\\ &= \frac{1}{2}\mathsf{j}\begin{pmatrix}1 & -\mathsf{i}\\ \mathsf{i} & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\mathsf{j} & \mathsf{k}\\ -\mathsf{k} & \mathsf{j}\end{pmatrix}.\end{aligned}$$

As pointed in Remark 4.17, the reason for why we obtain different operators is that the spectral projections $E_{\S}$ and $E_{\{0\}}$ cannot commute with arbitrary scalars because the respective invariant subspaces are not two-sided. Indeed, $-\mathsf{j}\mathbf{v}_2 = (1,\mathsf{i}) = \mathbf{v}_1$, which does obviously not belong to $E_{\mathbb{S}}V$.

4.6 Taylor Expansion in the Operator Variable for Bounded Operators

If we consider a bounded operator T and N is a small perturbation operator that furthermore commutes with T, then $f(T+N)$ can be represented as a power series in N that formally corresponds to a Taylor series expansion in the operator. This result is quaternionic analogue of [38, Theorem VII.10] and it was shown in [22] in the more general setting of paravector operators on a two-sided Clifford module.

Before we are able to show the Taylor expansion in the operator, we need to determine the slice derivatives of the S-resolvents. We start by finding explicit formulas for the functions

$$S_L^n(s,x) := (s-x)^{*_L n} \quad \text{and} \quad S_R^n(s,x) := (s-x)^{*_R n},$$

which were introduced in Definition 2.26.

Lemma 4.35. *Let $s \in \mathbb{H}$. For $n \geq 0$, we have*

$$S_L^n(s,x) = \sum_{k=0}^{n} \binom{n}{k} (-x)^k s^{n-k} \quad \text{and} \quad S_R^n(s,x) = \sum_{k=0}^{n} \binom{n}{k} s^{n-k}(-x)^k. \tag{4.8}$$

With $\mathcal{Q}_s(x) = x^2 - 2s_0 x + |s|^2$, we moreover have

$$S_L^{-n}(s,x) = \mathcal{Q}_s(x)^{-n}(\overline{s} - x)^{*_L n} \quad \text{and} \quad S_R^{-n}(s,x) = (\overline{s} - x)^{*_R n}\mathcal{Q}_s(x)^{-n}. \tag{4.9}$$

Furthermore, for $m, n \geq 0$, we have

$$S_L^{-n}(s,x) *_L S_L^{-m}(\overline{s},x) = \mathcal{Q}_s(x)^{-(n+m)} \left[(\overline{s} - x)^{*_L n} *_L (s-x)^{*_L m}\right]$$

and

$$S_R^{-n}(s,x) *_R S_R^{-m}(\overline{s},x) = \left[(\overline{s} - x)^{*_R n} *_R (s-x)^{*_R m}\right] \mathcal{Q}_s(x)^{-(n+m)}.$$

Proof. For $n = 0$, we have $(s-x)^{*_L 0} \equiv 1$, and hence (4.8) is obviously true. Assume that it holds true for $n-1$. Then (2.13) implies

$$\begin{aligned}
S_L^n(s,x) &= (s-x)^{*_L n} = (s-x)^{*_L (n-1)} *_L (s-x) \\
=&(s-x)^{*_L (n-1)} *_L s + (s-x)^{*_L (n-1)} *_L (-x) \\
=&\sum_{k=0}^{n-1} \binom{n-1}{k} (-x)^k s^{n-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} (-x)^{k+1} s^{n-1-k} \\
=&\sum_{k=0}^{n-1} \binom{n-1}{k} (-x)^k s^{n-k} + \sum_{k=1}^{n} \binom{n-1}{k-1} (-x)^k s^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-x)^k s^{n-k}
\end{aligned}$$

and (4.8) follows by induction.

We also show the identity (4.9) by induction. It is obviously true for $n = 0$. Assume that it holds true for $n-1$ and observe that $\mathcal{Q}_s(x)^{-1} \in \mathcal{SH}(\mathbb{H} \setminus [s])$. Then (2.11) and Corollary 2.25 imply

$$\begin{aligned}
S_L^{-n}(s,x) &= (s-x)^{-*_L (n-1)} *_L (s-x)^{-*_L} \\
&= \left[\mathcal{Q}_s(x)^{-(n-1)}(\overline{s} - x)^{*_L (n-1)}\right] *_L \left[\mathcal{Q}_s(x)^{-1}(\overline{s} - x)\right] \\
&= \mathcal{Q}_s(x)^{-(n-1)} *_L (\overline{s} - x)^{*_L (n-1)} *_L \mathcal{Q}_s(x)^{-1} *_L (\overline{s} - x) \\
&= \mathcal{Q}_s(x)^{-(n-1)} *_L \mathcal{Q}_s(x)^{-1} *_L (\overline{s} - x)^{*_L (n-1)} *_L (\overline{s} - x) \\
&= \mathcal{Q}_s(x)^{-n}(\overline{s} - x)^{*_L n}.
\end{aligned}$$

Finally, (2.11) also implies for $m, n \geq 0$ that

$$\begin{aligned}
&S_L^{-n}(s,x) *_L S_L^{-m}(\overline{s},x) = \\
&= \left[\mathcal{Q}_s(x)^{-n}(\overline{s} - x)^{*_L n}\right] *_L \left[\mathcal{Q}_s(x)^{-m}(s-x)^{*_L m}\right] \\
&= \mathcal{Q}_s(x)^{-n} *_L (\overline{s} - x)^{*_L n} *_L \mathcal{Q}_s(x)^{-m} *_L (s-x)^{*_L m} \\
&= \mathcal{Q}_s(x)^{-n} *_L \mathcal{Q}_s(x)^{-m} *_L (\overline{s} - x)^{*_L n} *_L (s-x)^{*_L m} \\
&= \mathcal{Q}_s(x)^{-(n+m)} \left[(\overline{s} - x)^{*_L n} *_L (s-x)^{*_L m}\right].
\end{aligned}$$

The right slice hyperholomorphic case can be shown by similar computations.

□

Corollary 4.36. *Let $s = s_0 + \mathsf{i}_s s_1 \in \mathbb{H}$ and $n, m \in \mathbb{N}_0$. If $x \in \mathbb{C}_{\mathsf{i}_s}$, then*

$$(s-x)^{*_L m} *_L (\overline{s}-x)^{*_L n} = (s-x)^m(\overline{s}-x)^n \tag{4.10}$$

and

$$S_L^{-m}(s,x) *_L S_L^{-n}(\overline{s},x) = (s-x)^{-m}(\overline{s}-x)^{-n}. \tag{4.11}$$

Moreover, for any $n \in \mathbb{N}_0$, the function

$$P(x) := \sum_{k=0}^{n} (\overline{s}-x)^{*_L(k+1)} *_L (s-x)^{*_L(n-k+1)} \tag{4.12}$$

is a polynomial with real coefficients. Analogous statements hold for right slice hyperholomorphic powers $S_R^m(s,x)$ of $s-x$.

Proof. If $x \in \mathbb{C}_{\mathsf{i}_s}$, then s, $\overline{s}$ and x commute. Hence, it follows from (4.8) and the binomial theorem that $(s-x)^{*_L m} = (s-x)^m$ and $(\overline{s}-x)^{*_L n} = (\overline{s}-x)^n$. From (2.11), we deduce that (4.10) holds true. Since x and s commute, we also find that $\mathcal{Q}_s(x)^{-1} = (x-s)^{-1}(x-\overline{x})^{-1}$ and so (4.9) implies

$$S_L^{-m}(s,x) = (s-x)^{-m}(\overline{s}-x)^{-m}(\overline{s}-x)^m = (s-x)^{-m}.$$

An analogous computation shows $S_L^{-n}(\overline{s},x) = (\overline{s}-x)^{-n}$.

For arbitrary left slice hyperholomorphic functions f and g, it is because of (2.16) immediate that $(f *_L g)(x) = f(x)g(x)$ at a point x if $f(x) \in \mathbb{C}_{\mathsf{i}_x}$. Since $(s-x)^{-m}$ belongs to $\mathbb{C}_{\mathsf{i}_x}$ if $x \in \mathbb{C}_{\mathsf{i}_s}$, we furthermore find that

$$S_L^{-m}(s,x) *_L S_L^{-n}(\overline{s},x) = (s-x)^{-m} *_L (\overline{s}-x)^{-n} = (s-x)^{-m}(\overline{s}-x)^{-n}.$$

Finally, we consider $P(x)$. The restriction P_{i_s} of this function to the plane $\mathbb{C}_{\mathsf{i}_s}$ is the complex polynomial $P_{\mathsf{i}_s}(z) = \sum_{k=0}^{n}(\overline{s}-z)^{k+1}(s-z)^{n-k+1}$. From the relation

$$P_{\mathsf{i}_s}(\overline{x}) = \sum_{k=0}^{n}(\overline{s}-\overline{x})^{k+1}(s-\overline{x})^{n-k+1} = \overline{\sum_{k=0}^{n}(s-x)^{k+1}(\overline{s}-x)^{n-k+1}} = \overline{P_{\mathsf{i}_s}(x)},$$

we deduce that its coefficients are real. Consequently, $P = \operatorname{ext}_L(P_{\mathsf{i}})$ is a polynomial with real coefficients on $\mathbb{H}$. We can show the analogous statement for right slice hyperholomorphic powers $S_R^m(s,x)$ of $s-x$ with similar arguments. □

We need now to formally replace the scalar variable x in the functions introduced above by the operator T in a way that is consistent with the S-functional calculus. Recall however that the product rule $(fg)(T) = f(T)g(T)$ holds only if $f \in \mathcal{SH}(\sigma_S(T))$ and $g \in \mathcal{SH}_L(\sigma_S(T))$ or if $f \in \mathcal{SH}_R(\sigma_S(T))$ and $g \in \mathcal{SH}(\sigma_S(T))$. This is due to the fact that, for $f, g \in \mathcal{SH}_L(\sigma_S(T))$ or for $f, g \in \mathcal{SH}_R(\sigma_S(T))$, the product fg does in general not belong to $\mathcal{SH}_L(\sigma_S(T))$ resp. $\mathcal{SH}_R(\sigma_S(T))$.

If on the other hand one considers the left slice hyperholomorphic product $f *_L g$ of two left slice hyperholomorphic functions (or equivalently, the right slice hyperholomorphic of two right slice hyperholomorphic functions), then it is not clear to which operation between operators it corresponds. The considerations in Section 8.3 actually suggest, that such operation does not exist.

However, for power series of an operator variable, we can use the formulas (2.13) and (2.14) to define a their $*_L$- resp. $*_R$-product.

Definition 4.37. Let $T \in \mathcal{B}(V)$. For $F = \sum_{n=0}^{+\infty} T^n a_n$ and $G = \sum_{n=0}^{+\infty} T^n b_n$ with $a_\ell, b_\ell \in \mathbb{H}$ for $\ell \in \mathbb{N}$, we define

$$F *_L G := \sum_{n=0}^{+\infty} T^n \left(\sum_{k=0}^{n} a_k b_{n-k} \right).$$

For $\tilde{F} = \sum_{n=0}^{+\infty} a_n T^n$ and $\tilde{G} = \sum_{n=0}^{+\infty} b_n T^n$, we define

$$\tilde{F} *_R \tilde{G} := \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) T^n.$$

Remark 4.38. For $F = \sum_{n=0}^{+\infty} T^n a_n$ and $G = \sum_{n=0}^{+\infty} T^n b_n$ note that $F *_L G = FG$ if $a_n \in \mathbb{R}$ for any $n \in \mathbb{N}$. In this case, the coefficients a_n commute with the operator T, and hence,

$$F *_L G = \sum_{n=0}^{+\infty} T^n \left(\sum_{k=0}^{n} a_k b_{n-k} \right) = \sum_{n=0}^{+\infty} \sum_{k=0}^{n} T^k a_k T^{n-k} b_{n-k} = FG.$$

Similarly, $\tilde{F} *_R \tilde{G} = \tilde{F}\tilde{G}$ if $b_n \in \mathbb{R}$ for any $n \in \mathbb{N}$.

Corollary 4.39. *Let $T \in \mathcal{B}(V)$ and let $f(x) = \sum_{n=0}^{+\infty} x^n a_n$ and $g(x) = \sum_{n=0}^{+\infty} x^n b_n$ be two left slice hyperholomorphic power series that converge on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$. Then*

$$f(T) *_L g(T) = (f *_L g)(T).$$

Similarly, for two right slice hyperholomorphic power series $\tilde{f}(x) = \sum_{n=0}^{+\infty} a_n x^n$ and $\tilde{g}(x) = \sum_{n=0}^{+\infty} b_n x^n$ that converge on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$, we have

$$\tilde{f}(T) *_R \tilde{g}(T) = (\tilde{f} *_R \tilde{g})(T).$$

Proof. By the properties of the S-functional calculus, we have $f(T) = \sum_{n=0}^{+\infty} T^n a_n$ and $g(T) = \sum_{n=0}^{+\infty} T^n b_n$. Hence,

$$\begin{aligned} f(T) *_L g(T) &= \sum_{n=0}^{+\infty} T^n \left(\sum_{k=0}^{n} a_k b_{n-k} \right) \\ &= \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}} \sum_{n=0}^{+\infty} s^n \left(\sum_{k=0}^{n} a_k b_{n-k} \right) \\ &= \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f *_L g(s) = (f *_L g)(T). \end{aligned}$$

An analogous computation shows the right slice hyperholomorphic case. □

Observe that $S_L^n(s,T)$ and $S_R^n(s,T)$ and slice hyperholomorphic products of such expressions are well defined because of Definition 4.37. In analogy with (4.35), we furthermore give the following definition.

Definition 4.40. Let $T \in \mathcal{B}(V)$ and let $s \in \rho_S(T)$. For $n, m \geq 0$, we define

$$S_L^{-n}(s,T) := \mathcal{Q}_s(T)^{-n}(\overline{s}\mathcal{I} - T)^{*_L n}$$

and

$$S_L^{-n}(s,T) *_L S_L^{-m}(\overline{s},T) := \mathcal{Q}_s(T)^{-(n+m)}\left[(\overline{s}\mathcal{I} - T)^{*_L n} *_L (s\mathcal{I} - T)^{*_L m}\right].$$

Similarly, we define

$$S_R^{-n}(s,T) := (\overline{s}\mathcal{I} - T)^{*_R n}\mathcal{Q}_s(T)^{-n}$$

and

$$S_R^{-n}(s,T) *_R S_R^{-m}(\overline{s},T) := \left[(\overline{s}\mathcal{I} - T)^{*_R n} *_R (s\mathcal{I} - T)^{*_R m}\right]\mathcal{Q}_s(T)^{-(n+m)}.$$

Remark 4.41. Since the function $Q_s(x)^{-n}$ is intrinsic, the above definitions are due to the product rule compatible with the S-functional calculus, that is

$$\left[S_L^{-n}(s,\cdot)\right](T) = S_L^{-n}(s,T) \quad \text{and} \quad \left[S_R^{-n}(s,\cdot)\right](T) = S_R^{-n}(s,T)$$

as well as

$$\left[S_L^{-n}(s,\cdot) *_L S_L^{-m}(\overline{s},\cdot)\right](T) = S_L^{-n}(s,T) *_L S_L^{-m}(\overline{s},T)$$

and

$$\left[S_R^{-n}(s,\cdot) *_L S_R^{-m}(\overline{s},\cdot)\right](T) = S_R^{-n}(s,T) *_L S_R^{-m}(\overline{s},T)$$

Proposition 4.42. *Let $T \in \mathcal{B}(V)$ and let $s \in \rho_S(T)$. Then*

$$\partial_S{}^m S_L^{-1}(s,T) = (-1)^m m!\, S_L^{-(m+1)}(s,T) \tag{4.13}$$

and

$$\partial_S{}^m S_R^{-1}(s,T) = (-1)^m m!\, S_R^{-(m+1)}(s,T), \tag{4.14}$$

for any $m \geq 0$.

Proof. Recall that the slice derivative defined in Definition 2.12 coincides with the partial derivative with respect to the real part s_0 of s. We show only (4.13), since (4.14) follows by analogous computations.

We prove the statement by induction. For $m = 0$, the identity (4.13) is obvious. We assume that $\partial_S{}^{m-1} S_L^{-1}(s,T) = (-1)^{m-1}(m-1)!\, S_L^{-m}(s,T)$ and we compute $\partial_S{}^m S_L^{-1}(s,T)$. We represent $S_L^{-m}(s,T)$ using the S-functional calculus. If we choose the path of integration in the complex plane $\mathbb{C}_{\mathsf{i}_s}$, then we find because of (4.11) that

$$\begin{aligned}
\partial_S S_L^{-m}(s,T) &= \partial_S \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}_s})} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, S_L^{-m}(s,p) \\
&= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}_s})} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, \frac{\partial}{\partial s_0}(s-p)^{-m} \\
&= -m\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}_s})} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, (s-p)^{-(m+1)} = -m\, S_L^{-(m+1)}(s,T),
\end{aligned}$$

and in turn,

$$\begin{aligned}
\partial_S{}^m S_L^{-1}(s,T) &= \partial_S\left(\partial_S{}^{m-1} S_L^{-1}(s,T)\right) \\
&= (-1)^{m-1}(m-1)!\partial_S S_L^{-m}(s,T) = (-1)^m m!\, S_L^{-(m+1)}(s,T).
\end{aligned}$$

□

Remark 4.43. We point out that Proposition 4.42 also holds true for unbounded closed operators. In this case, we have have to modify the definition of $S_L^{-m}(s,T)$ by commuting every occurrence of T with $\mathcal{Q}_s(T)^{-m}$ just as we did it in the definition of the left S-resolvent operator. Otherwise $S_L^{-m}(s,T)$ is only defined on $\operatorname{dom}(T^m)$ and not on the entire space V.

Let us now turn our attention to the Taylor series expansion of $f(T+N)$ in the operator variable. In order for such an expansion to hold, it is essential that adding a somewhat small operator N does not perturb the S-spectrum of T a lot. The following result clarifies how one has to measure the distance between a point $s \in \rho_S(T)$ and the S-spectrum of T.

Lemma 4.44. *Let $A \subset \mathbb{H}$ be axially symmetric and let $s = s_0 + \mathsf{i}s_1 \in \mathbb{H}$. Then*

$$\operatorname{dist}(s,A) = \operatorname{dist}(s, A \cap \mathbb{C}_{\mathsf{i}}) = \operatorname{dist}\left(s, A \cap \mathbb{C}_{\mathsf{i}}^{\geq}\right),$$

where $\operatorname{dist}(s,A) := \inf\{|s-x| : x \in A\}$ *and* $\mathbb{C}_{\mathsf{i}}^{\geq} = \{x_0 + \mathsf{i}x_1 : x_0 \in \mathbb{R}, x_1 \geq 0\}$.

Proof. For $x = x_0 + \mathsf{i}_x x_1 \in A$ denote $x_{\mathsf{i}} = x_0 + \mathsf{i}x_1$. We choose $\mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$ and set $\mathsf{k} = \mathsf{ij}$. Then $x = x_0 + \tilde{x}_1\mathsf{i} + \tilde{x}_2\mathsf{j} + \tilde{x}_3\mathsf{k}$ with $\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2 = |\underline{x}|^2 = x_1^2$, and in turn

$$\begin{aligned}
&|s - x_{\mathsf{i}}|^2 = (s_0 - x_0)^2 + (s_1 - x_1)^2 \\
=&(s_0 - x_0)^2 + s_1^2 - 2s_1x_1 + x_1^2 \\
=&(s_0 - x_0)^2 + s_1^2 - 2s_1\sqrt{\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2} + \tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_2^3 \\
\leq&(s_0 - x_0)^2 + s_1^2 - 2s_1\tilde{x}_1 + \tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3{}^2 \\
=&(s_0 - x_0)^2 + (s_1 - \tilde{x}_1)^2 + \tilde{x}_2^2 = |s - x|^2.
\end{aligned}$$

Since A is axially symmetric, we have $\{x_{\mathsf{i}} : x \in A\} = A \cap \mathbb{C}_{\mathsf{i}}^{\geq}$. Consequently,

$$\inf_{x\in A} |s-x| \leq \inf_{x \in A\cap\mathbb{C}_{\mathsf{i}}^{\geq}} |s-x| \leq \inf_{x\in A} |s - x_{\mathsf{i}}| \leq \inf_{x\in A} |s-x|,$$

and in turn,

$$\operatorname{dist}(s,A) = \inf_{x\in A}|s-x| = \inf_{x\in A\cap\mathbb{C}_{\mathsf{i}}^{\geq}} |s-x| = \operatorname{dist}(s, A\cap\mathbb{C}_{\mathsf{i}}^{\geq}).$$

□

Proposition 4.45. *Let $T \in \mathcal{B}(V)$ and let $C \subset \mathbb{H}$ with $\operatorname{dist}(C, \sigma_S(T)) > \varepsilon$ for some $\varepsilon > 0$. Then there exists a positive constant K_T such that*

$$\left\| S_L^{-m}(s,T) *_L S_L^{-n}(\overline{s},T) \right\| \leq \frac{K_T}{\varepsilon^{m+n}} \tag{4.15}$$

and

$$\left\| S_R^{-m}(s,T) *_L S_R^{-n}(\overline{s},T) \right\| \leq \frac{K_T}{\varepsilon^{m+n}}, \tag{4.16}$$

for any $s \in C$ and any $m, n \geq 0$.

Proof. Let U be a bounded slice Cauchy domain with $\sigma_S(T) \subset U$ with $\operatorname{dist}(C, \overline{U}) > \varepsilon$. We choose $s = s_0 + \mathbf{i}s_1 \in C$. By Corollary 4.36, we have

$$S_L^{-m}(s,x) *_L S_L^{-n}(\overline{s},x) = (s-x)^{-m}(\overline{s}-x)^{-n}$$

for any $x \in \mathbb{C}_{\mathbf{i}}$. Lemma 4.44 implies $\operatorname{dist}(s, cl(U) \cap \mathbb{C}_{\mathbf{i}}) = \operatorname{dist}(s, cl(U)) > \varepsilon$. Since $cl(U) \cap \mathbb{C}_{\mathbf{i}}$ is symmetric with respect to the real axis, we also have $\operatorname{dist}(\overline{s}, cl(U) \cap \mathbb{C}_{\mathbf{i}}) > \varepsilon$ and we deduce

$$\begin{aligned}
&\left\| S_L^{-m}(s,T) *_L S_L^{-n}(\overline{s},T) \right\| = \\
=&\left\| \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_{\mathbf{i}})} S_L^{-1}(p,T)\, dp_{\mathbf{i}}\, S_L^{-m}(s,p) *_L S_L^{-n}(\overline{s},p) \right\| \\
=&\left\| \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_{\mathbf{i}})} S_L^{-1}(p,T)\, dp_{\mathbf{i}}\, (s-p)^{-m}(\overline{s}-p)^{-n} \right\| \\
\leq&\frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_{\mathbf{i}})} \left\| S_L^{-1}(p,T) \right\|\, d|p|\, \left| (s-p)^{-m}(\overline{s}-p)^{-n} \right| \\
\leq&\frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_{\mathbf{i}})} \left\| S_L^{-1}(p,T) \right\|\, d|p|\, \frac{1}{\varepsilon^{m+n}}.
\end{aligned}$$

Hence, if we set

$$K_T := \sup_{\mathbf{j}\in\mathbb{S}} \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_{\mathbf{j}})} \left\| S_L^{-1}(p,T) \right\|\, d|p|,$$

which depends neither on the point $s \in C$ nor on the numbers $n, m \geq 0$, then

$$\left\| S_L^{-m}(s,T) *_L S_L^{-n}(\overline{s},T) \right\| \leq \frac{K_T}{\varepsilon^{m+n}}.$$

□

Theorem 4.46. *Let $T \in \mathcal{B}(V)$ and let $N \in \mathcal{B}(V)$ such that T and N commute and such that $\sigma_S(N)$ is contained in the open ball $B_\varepsilon(0)$. If $\operatorname{dist}(s, \sigma_S(T)) > \varepsilon$, then $s \in \rho_S(T+N)$ and*

$$\mathcal{Q}_s(T)^{-1} = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T) \right) N^n,$$

where the series converges in the operator norm.

Proof. We first show the convergence of the series

$$\Sigma(s,T,N) := \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T) \right) N^n.$$

Since $\sigma_S(N)$ is compact, there exists $\theta \in (0,1)$ such that $\sigma_S(N) \subset B_{\theta\varepsilon}(0) \subset B_\varepsilon(0)$.

Applying the S-functional calculus, we obtain

$$\begin{aligned}\|N^m\| &= \left\|\frac{1}{2\pi}\int_{\partial(B_{\theta\varepsilon}(0)\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,N)\,ds_{\mathsf{i}}\,s^m\right\| \\ &\leq \frac{1}{2\pi}\int_{\partial(B_{\theta\varepsilon}(0)\cap\mathbb{C}_{\mathsf{i}})} \left\|S_L^{-1}(s,N)\right\|\,d|s|\,|s|^m \\ &= \frac{1}{2\pi}\int_{\partial(B_{\theta\varepsilon}(0)\cap\mathbb{C}_{\mathsf{i}})} \left\|S_L^{-1}(s,N)\right\|\,d|s|\,(\theta\varepsilon)^m\end{aligned}$$

for any $m \geq 0$. Hence,

$$\|N^m\| \leq K_N(\theta\varepsilon)^m \tag{4.17}$$

with

$$K_N := \frac{1}{2\pi}\int_{\partial(B_{\theta\varepsilon}(0)\cap\mathbb{C}_{\mathsf{i}})} \left\|S_L^{-1}(s,N)\right\|\,d|s|.$$

From Proposition 4.45, we deduce

$$\begin{aligned}&\sum_{n=0}^{+\infty}\left\|\left(\sum_{k=0}^{n} S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T)\right)N^n\right\| \\ \leq&\sum_{n=0}^{+\infty}\sum_{k=0}^{n}\left\|S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T)\right\|\,\|N^n\| \\ \leq&\sum_{n=0}^{+\infty}(n+1)\frac{K_T}{\varepsilon^{n+2}}K_N(\theta\varepsilon)^n \leq \frac{K_TK_N}{\varepsilon^2}\sum_{n=0}^{+\infty}(n+1)\theta^n.\end{aligned}$$

By the root test, this last series converges because $0 < \theta < 1$. The comparison test yields the convergence of the original series $\Sigma(s,T,N)$ in the operator norm.

From Definition 4.40 and the fact that T and N commute, we deduce

$$\begin{aligned}\mathcal{Q}_s(T+N) &= T^2 + 2TN + N^2 - 2s_0T - 2s_0N + |s|^2\mathcal{I} \\ &= \mathcal{Q}_s(T) + (2T - 2s_0)N + |s|^2\mathcal{I}.\end{aligned}$$

If we denote

$$\varLambda_T(k,n,s) := (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)}$$

for neatness, we therefore have

$$\begin{aligned}&\Sigma(s,T,N)\mathcal{Q}_s(T+N) = \\ =&\left(\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T)\right)N^n\right)\mathcal{Q}_s(T+N) \\ =&\sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+2)}\left(\sum_{k=0}^{n}\varLambda_T(k,n,s)\right)N^n\mathcal{Q}_s(T) \\ &+\sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+2)}\left(\sum_{k=0}^{n}\varLambda_T(k,n,s)\right)N^{n+1}(2T - 2s_0\mathcal{I}) \\ &+\sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+2)}\left(\sum_{k=0}^{n}\varLambda_T(k,n,s)\right)N^{n+2}.\end{aligned}$$

Applying Corollary 4.36 and the S-functional calculus, we see that any of the coefficients $\sum_{k=0}^{n} \Lambda_T(k,n,s) = \sum_{k=0}^{n} (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)}$ is a polynomial in T with real coefficients and hence commutes with the operator $\mathcal{Q}_s(T)$. Remark 4.38 implies

$$
\begin{aligned}
&\Sigma(s,T,N)\mathcal{Q}_s(T+N) = \\
=&\sum_{n=0}^{+\infty} \mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=0}^{n} \Lambda_T(k,n,s) \right) N^n \\
&+ \sum_{n=0}^{+\infty} \mathcal{Q}_s(T)^{-(n+2)} \left(\sum_{k=0}^{n} \Lambda_T(k,n,s) *_L (2T - 2s_0\mathcal{I}) \right) N^{n+1} \\
&+ \sum_{n=0}^{+\infty} \mathcal{Q}_s(T)^{-(n+2)} \left(\sum_{k=0}^{n} \Lambda_T(k,n,s) \right) N^{n+2} \\
=&\sum_{n=0}^{+\infty} \mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=0}^{n} \Lambda_T(k,n,s) \right) N^n \\
&+ \sum_{n=1}^{+\infty} \mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=0}^{n-1} \Lambda_T(k,n-1,s) *_L (2T - 2s_0\mathcal{I}) \right) N^n \\
&+ \sum_{n=2}^{+\infty} \mathcal{Q}_s(T)^{-n} \sum_{k=0}^{n-2} \Lambda_T(k,n-2,s) N^n.
\end{aligned}
$$

The identity

$$
\begin{aligned}
&\mathcal{Q}_s(T)^{-n} \left(\sum_{k=0}^{n-2} \Lambda_T(k,n-2,s) \right) \\
=&\mathcal{Q}_s(T)^{-n} \left(\sum_{k=0}^{n-2} (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k-1)} \right) \\
=&\mathcal{Q}_s(T)^{-n} \left(\sum_{k=1}^{n-1} (\overline{s}\mathcal{I} - T)^{*_L k} *_L (s\mathcal{I} - T)^{*_L(n-k)} \right) \\
=&\mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=1}^{n-1} (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)} \right) \\
=&\mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=1}^{n-1} \Lambda_T(k,n,s) \right),
\end{aligned}
$$

finally yields

$$\begin{aligned}
&\Sigma(s,T,N)\mathcal{Q}_s(T+N) = \mathcal{Q}_s(T)^{-1}\Lambda_T(0,0,s)N^0\\
&+ \mathcal{Q}_s(T)^{-2}\left(\sum_{k=0}^{1}\Lambda_T(k,1,s) + \Lambda(0,0,s) *_L (2T-2s_0\mathcal{I})\right)N\\
&+\sum_{n=2}^{+\infty}\mathcal{Q}_s(T)^{-(n+1)}\left(\sum_{k=0}^{n}\Lambda_T(k,n,s)\right.\\
&\qquad\left.+\sum_{k=0}^{n-1}\Lambda_T(k,n-1,s) *_L (2T-2s_0\mathcal{I}) + \sum_{k=1}^{n-1}\Lambda_T(k,n,s)\right)N^n.
\end{aligned}$$

Now observe that

$$\begin{aligned}
\mathcal{Q}_s(T)^{-1}\Lambda_T(0,0,s)N^0 &= \mathcal{Q}_s(T)^{-1}\left((\overline{s}\mathcal{I}-T) *_L (s\mathcal{I}-T)\right)\\
&= \mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T) = \mathcal{I}.
\end{aligned}$$

Because of $2T-2s_0\mathcal{I} = -(s\mathcal{I}-T)-(\overline{s}\mathcal{I}-T)$, we have

$$\begin{aligned}
&\sum_{k=0}^{1}\Lambda_T(k,1,s) + \Lambda(0,0,s) *_L (2T-2s_0\mathcal{I})\\
=&(\overline{s}\mathcal{I}-T) *_L (s\mathcal{I}-T)^{*_L 2} + (\overline{s}\mathcal{I}-T)^{*_L 2} *_L (s\mathcal{I}-T)\\
&-(\overline{s}\mathcal{I}-T)^{*_L 2} *_L (s\mathcal{I}-T) - (\overline{s}\mathcal{I}-T) *_L (s\mathcal{I}-T)^{*_L 2} = 0.
\end{aligned}$$

Finally, we also find again because of $2T-2s_0\mathcal{I} = -(s\mathcal{I}-T)-(\overline{s}\mathcal{I}-T)$ that

$$\begin{aligned}
\sum_{k=0}^{n}\Lambda_T(k,n,s) + \sum_{k=0}^{n-1}\Lambda_T(k,n-1,s) *_L (2T-2s_0\mathcal{I}) + \sum_{k=1}^{n-1}\Lambda_T(k,n,s) =\\
=\sum_{k=0}^{n}(\overline{s}\mathcal{I}-T)^{*_L(k+1)} *_L (s\mathcal{I}-T)^{*_L(n-k+1)}\\
-\sum_{k=0}^{n-1}(\overline{s}\mathcal{I}-T)^{*_L(k+2)} *_L (s\mathcal{I}-T)^{*_L(n-k)}\\
-\sum_{k=0}^{n-1}(\overline{s}\mathcal{I}-T)^{*_L(k+1)} *_L (s\mathcal{I}-T)^{*_L(n-k+1)}\\
+\sum_{k=1}^{n-1}(\overline{s}\mathcal{I}-T)^{*_L(k+1)} *_L (s\mathcal{I}-T)^{*_L(n-k+1)} = 0,
\end{aligned}$$

where the last identity follows after an index-shift k to $k+1$ in the second sum. Altogether, we find

$$\Sigma(s,T,N)\mathcal{Q}_s(T+N) = \mathcal{I}.$$

From Corollary 4.36 and the S-functional calculus, we already concluded that each of the coefficients $\sum_{k=0}^{n}\Lambda_T(k,n,s)$ in $\Sigma(s,T,N)$ is a polynomial in T with real coefficients and thus commutes with both T and N. Hence, it also commutes with $\mathcal{Q}_s(T+N)$

and so also

$$\mathcal{Q}_s(T+N)\Sigma(s,T,N) = \Sigma(s,T,N)\mathcal{Q}_s(T+N) = \mathcal{I}.$$

Hence, $\mathcal{Q}_s(T+N)$ is invertible, which implies $s \in \rho_S(T+N)$.

$\square$

Theorem 4.47. *Let $T, N \in \mathcal{B}(V)$ such that $\sigma_S(N) \subset B_\varepsilon(0)$ and such that T and N commute. For any $s \in \rho_S(T)$ with* $\operatorname{dist}(s, \sigma_S(T)) > \varepsilon$*, the identities*

$$S_L^{-1}(s, T+N) = \sum_{n=0}^{+\infty} N^n\, S_L^{-(n+1)}(s,T$$

and

$$S_R^{-1}(s, T+N) = \sum_{n=0}^{+\infty} S_R^{-(n+1)}(s,T)\, N^n$$

hold true, where these series converge uniformly on any set C with $\operatorname{dist}(C, \sigma_S(T)) > \varepsilon$.

Proof. In (4.17), we showed the existence of two constants $K_N \geq 0$ and $\theta \in (0,1)$ such that $\|N\|^m \leq K_N(\theta\varepsilon)^m$ for any $m \in \mathbb{N}_0$. Moreover, for any $C \subset \mathbb{H}$ with $\operatorname{dist}(C, \sigma_S(T)) > \varepsilon$, Proposition 4.45 implies the existence of a constant K_T such that $\|S_L^{-m}(s,T)\| \leq K_T/\varepsilon^m$ for any $s \in C$ and any $m \in \mathbb{N}_0$. Therefore, the estimate

$$\begin{aligned}
\sum_{n=n_0}^{\infty} \left\| N^n\, S_L^{-(n+1)}(s,T) \right\| &\leq \sum_{n=n_0}^{+\infty} \|N^n\| \left\| S_L^{-(n+1)}(s,T) \right\| \\
&\leq \sum_{n=n_0}^{+\infty} K_N(\theta\varepsilon)^n \frac{K_T}{\varepsilon^{n+1}} = \frac{K_T K_N}{\varepsilon} \sum_{n=n_0}^{+\infty} \theta^n \overset{n_0\to\infty}{\longrightarrow} 0
\end{aligned}$$

holds true for any $s \in C$ and implies the uniform convergence of the series on C.

Let $s \in \rho_S(T)$ with $\operatorname{dist}(s, \sigma_S(T)) > \varepsilon$. We have

$$\begin{aligned}
&\left((T+N)^2 - 2s_0(T+N) + |s|^2\mathcal{I}\right) \sum_{n=0}^{+\infty} N^n\, S_L^{-(n+1)}(s,T) \\
&= \left(T^2 - 2s_0T + |s|^2\mathcal{I}\right) \sum_{n=0}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n+1)} (\overline{s}\mathcal{I} - T)^{*_L(n+1)} \\
&\quad + (2T - 2s_0\mathcal{I})\, N \sum_{n=0}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n+1)} (\overline{s}\mathcal{I} - T)^{*_L(n+1)} \\
&\quad + N^2 \sum_{n=0}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n+1)} (\overline{s}\mathcal{I} - T)^{*_L(n+1)} \\
&= \sum_{n=0}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-n} (\overline{s}\mathcal{I} - T)^{*_L(n+1)}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=0}^{+\infty} N^{n+1}\,(2T-2s_0\mathcal{I})\,(T^2-2s_0T+|s|^2\mathcal{I})^{-(n+1)}(\overline{s}\mathcal{I}-T)^{*_L(n+1)}\\
&+\sum_{n=0}^{+\infty} N^{n+2}(T^2-2s_0T+|s|^2\mathcal{I})^{-(n+1)}(\overline{s}\mathcal{I}-T)^{*_L(n+1)}.
\end{aligned}
$$

Shifting the indices yields

$$
\begin{aligned}
&\big((T+N)^2-2s_0(T+N)+|s|^2\mathcal{I}\big)\sum_{n=0}^{+\infty} N^n\, S_L^{-(n+1)}(s,T)\\
=&\sum_{n=0}^{+\infty} N^n(T^2-2s_0T+|s|^2\mathcal{I})^{-n}(\overline{s}\mathcal{I}-T)^{*_L(n+1)}\\
&+\sum_{n=1}^{+\infty} N^n\,(2T-2s_0\mathcal{I})\,(T^2-2s_0T+|s|^2\mathcal{I})^{-n}(\overline{s}\mathcal{I}-T)^{*_L n}\\
&+\sum_{n=2}^{+\infty} N^n(T^2-2s_0T+|s|^2\mathcal{I})^{-(n-1)}(\overline{s}\mathcal{I}-T)^{*_L(n-1)}\\
=&\overline{s}\mathcal{I}-T+N(T^2-s_0T+|s|^2\mathcal{I})^{-1}(\overline{s}\mathcal{I}-T)^{*_L 2}\\
&+N(T^2-2s_0T+|s|^2\mathcal{I})^{-1}(2T-2s_0\mathcal{I})(\overline{s}\mathcal{I}-T)+\\
&+\sum_{n=2}^{+\infty} N^n(T^2-2s_0T+|s|^2\mathcal{I})^{-n}\Big[(\overline{s}\mathcal{I}-T)^{*_L(n+1)}\\
&\qquad+(2T-2s_0\mathcal{I})\,(\overline{s}\mathcal{I}-T)^{*_L n}\\
&\qquad+(T^2-2s_0T+|s|^2\mathcal{I})(\overline{s}\mathcal{I}-T)^{*_L(n-1)}\Big].
\end{aligned}
$$

The last series equals 0 because Remark 4.38 and the identity

$$
(T^2-2s_0T+|s|^2\mathcal{I})=(s\mathcal{I}-T)*_L(\overline{s}\mathcal{I}-T)
$$

imply

$$
\begin{aligned}
&(\overline{s}\mathcal{I}-T)^{*_L(n+1)}+(2T-2s_0\mathcal{I})\,(\overline{s}\mathcal{I}-T)^{*_L n}\\
&\ +(T^2-2s_0T+|s|^2\mathcal{I})(\overline{s}\mathcal{I}-T)^{*_L(n-1)}\\
=&(\overline{s}\mathcal{I}-T)^{*_L(n+1)}+(2T-2s_0\mathcal{I})*_L(\overline{s}\mathcal{I}-T)^{*_L n}+(s\mathcal{I}-T)*_L(\overline{s}\mathcal{I}-T)^{*_L n}\\
=&(\overline{s}\mathcal{I}-T+2T-2s_0\mathcal{I}+s\mathcal{I}-T)*_L(\overline{s}\mathcal{I}-T)^{*_L(n-1)}=0.
\end{aligned}
$$

Hence, we finally obtain

$$
\begin{aligned}
&\big((T+N)^2-2s_0(T+N)+|s|^2\mathcal{I}\big)\sum_{n=0}^{+\infty} N^n S_L^{-(n+1)}(s,T)\\
=&\overline{s}\mathcal{I}-T+N(T^2-2s_0T+|s|^2\mathcal{I})^{-1}(\overline{s}^2\mathcal{I}-2T\overline{s}+T^2)\\
&+N(T^2-2s_0T+|s|^2\mathcal{I})^{-1}(2T\overline{s}-2s_0\overline{s}\mathcal{I}-2T^2+2s_0T)\\
=&\overline{s}\mathcal{I}-T+N(T^2-2s_0T+|s|^2\mathcal{I})^{-1}(-T^2+2s_0T-|s|^2\mathcal{I})=\overline{s}\mathcal{I}-T-N.
\end{aligned}
$$

Since $\mathcal{Q}_s(T+N) = (T+N)^2 - 2s_0(T+N) + |s|^2\mathcal{I}$ is invertible by Theorem 4.46, this is equivalent to

$$\sum_{n=0}^{+\infty} N^n\, S_L^{-(n+1)}(s,T) = \mathcal{Q}_s(T+N)^{-1}(\overline{s}\mathcal{I} - T - N) = S_L^{-1}(s, T+N).$$

The identity for the right S-resolvent can be shown with analogous computations. □

Theorem 4.48 (The Taylor formulas). *Let $T, N \in \mathcal{B}(V)$ with $\sigma_S(N) \subset B_\varepsilon(0)$ such that T and N commute and set*

$$C_\varepsilon(\sigma_S(T)) := \{s \in \mathbb{H} : \operatorname{dist}(s, \sigma_S(T)) \leq \varepsilon\}.$$

If $f \in \mathcal{SH}_L(C_\varepsilon(\sigma_S(T)))$, then $f \in \mathcal{SH}_L(\sigma_S(T+N))$ and

$$f(T+N) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!}\left(\partial_S{}^n f\right)(T).$$

Similarly, if $f \in \mathcal{SH}_R(C_\varepsilon(\sigma_S(T)))$, then $f \in \mathcal{SH}_R(\sigma_S(T+N))$ and

$$f(T+N) = \sum_{n=0}^{+\infty} \frac{1}{n!}\left(\partial_S{}^n f\right)(T) N^n.$$

Proof. We prove just the first Taylor formula, the second one is obtained with similar computations. By Theorem 4.46, we have $\sigma_S(T+N) \subset C_\varepsilon(\sigma_S(T))$ and so the function f belongs to $\mathcal{SH}_L(\sigma_S(T+N))$. If U is a bounded slice Cauchy domain with $C_\varepsilon(\sigma_S(T)) \subset U$ and $cl(U) \subset \operatorname{dom}(f)$, then we find due to Theorem 4.47 that

$$\begin{aligned} f(T+N) &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s, T+N)\, ds_{\mathsf{i}}\, f(s) \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} \sum_{n=0}^{+\infty} N^n S_L^{-(n+1)}(s,T)\, ds_{\mathsf{i}}\, f(s) \\ &= \sum_{n=0}^{+\infty} N^n \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-(n+1)}(s,T)\, ds_{\mathsf{i}}\, f(s). \end{aligned}$$

By Proposition 4.42, we have

$$S_L^{-(n+1)}(s,T) = (-1)^n \frac{1}{n!}\partial_S{}^n S_L^{-1}(s,T)$$

and so

$$f(T+N) = \sum_{n=0}^{+\infty} N^n \frac{(-1)^n}{n!}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} \partial_S{}^n S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s).$$

After integrating the n-th term in the sum n times by parts, we finally obtain

$$f(T+N) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, (\partial_S{}^n f)(s) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!}(\partial_S{}^n f)(T).$$

□

CHAPTER 5

Functions of the Generator of a Strongly Continuous Group

The S-functional calculus is the functional calculus based on slice hyperholomorphic functions that makes the weakest assumptions on the operator. It only requires the operator's S-resolvent set to be nonempty. As in the complex case, the price for these weak assumptions are the relatively strong assumptions that one has to make on the class of admissible functions, namely that they are slice hyperholomorphic on the S-spectrum of T and at infinity. Similar to the classical case, additional knowledge about the operator allows us to extend the class of admissible functions. In this chapter we shall assume that

$$T \text{ is the infinitesimal generator of a strongly continuous group } (\mathcal{U}_T(t))_{t\in\mathbb{R}}. \qquad \text{(G)}$$

One can then formally replace the exponential in the quaternionic Laplace-Stieltjes-transform

$$f(s) = \int_{\mathbb{R}} d\mu(t)\, e^{-st} \qquad -(\omega+\varepsilon) < \mathrm{Re}(s) < \omega + \varepsilon,$$

where ω is the constant in Theorem 2.74, by the group $\mathcal{U}_T(t)$, which formally corresponds to e^{tT}. Precisely, one defines

$$f(T) := \int_{\mathbb{R}} d\mu(t)\mathcal{U}_T(-t).$$

We point out that the function f is now slice hyperholomorphic on $\sigma_S(T)$ but not necessarily at infinity.

In the complex setting the above procedure yields the Phillips functional calculus, which was introduced in [13, 72]. In this chapter we introduce its quaternionic counter

part and study its properties and its relation with the S-functional calculus following the treatise in [38]. The presented results were published in [3].

5.1 Preliminaries on Quaternionic Measure Theory

Before we are able to define the Phillips functional calculus for quaternionic linear operators, we have to recall some facts about quaternion-valued measures and to investigate their product measures. These results will be essential, when we study the properties of the quaternionic Laplace-Stieltjes-transform.

In [2, Section 3], Agrawal and Kulkarni showed quaternion-valued measures have properties similar to the properties of complex-measures. In particular, we can define their variation, which has analogous properties as the variation of a complex measure, and find that the Radon-Nikodỳm theorem also holds true in this setting. We recall some results that we will need in the sequel, for more details see [2].

Definition 5.1. Let $(\Omega, \mathcal{A})$ be a measurable space. A quaternionic measure is a function $\mu\colon \mathcal{A} \to \mathbb{H}$ that satisfies

$$\mu\left(\bigcup\nolimits_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

for any sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. We denote the set of all quaternionic measures on $\mathcal{A}$ by $\mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ or simply by $\mathcal{M}(\Omega, \mathbb{H})$ or $\mathcal{M}(\Omega)$ if there is no possibility of confusion.

Corollary 5.2. *Let $(\Omega, \mathcal{A})$ be a measurable space. The set $\mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ is a two-sided quaternionic vector space with the operations*

$$(\mu + \nu)(A) := \mu(A) + \nu(A), \qquad (a\mu)(A) := a\mu(A), \qquad (\mu a)(A) := \mu(A)a$$

for $\mu, \nu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$, $a \in \mathbb{H}$ and $A \in \mathcal{A}$.

Remark 5.3. Let $\mathsf{i}, \mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$. Since $\mathbb{H} = \mathbb{C}_{\mathsf{i}} + \mathsf{j}\mathbb{C}_{\mathsf{i}}$, it is immediate that a mapping $\mu : \mathcal{A} \to \mathbb{H}$ is a quaternionic measure if and only if there exist two $\mathbb{C}_{\mathsf{i}}$-valued complex measures μ_1, μ_2 such that $\mu(A) = \mu_1(A) + \mathsf{j}\mu_2(A)$ for any $A \in \mathcal{A}$. Moreover, since also $\mathbb{H} = \mathbb{C}_{\mathsf{i}} + \mathbb{C}_{\mathsf{i}}\mathsf{j}$, there exist $\mathbb{C}_{\mathsf{i}}$-valued measures $\tilde{\mu}_1, \tilde{\mu}_2$ such that $\mu(A) = \tilde{\mu}_1(A) + \tilde{\mu}_2(A)\mathsf{j}$ for any $A \in \mathcal{A}$.

Definition 5.4. Let $\mu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$. For all $A \in \mathcal{A}$ denote by $\Pi(A)$ the set of all countable partitions π of A into pairwise disjoint, measurable sets A_ℓ, $\ell \in \mathbb{N}$. The total variation of μ is the set function

$$|\mu|(A) := \sup\left\{ \sum_{A_\ell\in\pi} |\mu(A_\ell)| : \pi \in \Pi(A) \right\} \qquad \text{for all } A \in \mathcal{A}.$$

Corollary 5.5. *The total variatation $|\mu|$ of a measure $\mu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ is a finite positive measure on Ω. Moreover, $|a\mu| = |\mu a| = |\mu||a|$ and $|\mu + \nu| \leq |\mu| + |\nu|$ for any $\mu, \nu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ and $a \in \mathbb{H}$.*

Recall that a measure μ is called absolutely continuous with respect to a positive measure ν, if $\mu(A) = 0$ for any $A \in \mathcal{A}$ with $\nu(A) = 0$. In this case, we write $\mu \ll \nu$. We denote by $L^1(\Omega, \mathcal{A}, \nu, \mathbb{H})$ the Banach space of all $\mathbb{H}$-valued functions on Ω that are integrable with respect to the positive measure ν.

Theorem 5.6 (Radon-Nikodỳm theorem for quaternionic measures)**.** *Let ν be a σ-finite positive measure on $(\Omega, \mathcal{A})$. A quaternionic measure $\mu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ is absolutely continuous with respect to ν if and only if there exists a function $f \in L^1(\Omega, \mathcal{A}, \nu, \mathbb{H})$ such that*

$$\mu(A) = \int_A f(x)\, d\nu(x) \qquad \text{for all } A \in \mathcal{A}.$$

Moreover, f is unique and

$$|\mu|(A) = \int_A |f(x)|\, d\nu(x) \qquad \text{for all } A \in \mathcal{A}. \tag{5.1}$$

The identity (5.1) follows as in the classical case, cf. [75, Theorem 6.13].

Corollary 5.7. *Let $\mu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$. Then there exists an $\mathcal{A}$-measurable function $h : \Omega \to \mathbb{H}$ such that $|h(x)| = 1$ for any $x \in \Omega$ and such that $\mu(A) = \int_A h(x)\, d|\mu|(x)$ for any $A \in \mathcal{A}$.*

In order to define the quaternionic Laplace-Stieltjes-transform and the Phillips functional calculus for quaternionic linear operators, we define integrals with respect to quaternionic-valued measures as in [3]. Let us again consider a quaternionic two-sided Banach space V. By Remark 2.34, V is a real Banach space if we restrict the scalar multiplication to the real numbers. Moreover, recall that $\mathbb{H}$ itself is a quaternionic two-sided Banach space so that the following discussion also holds in the scalar case.

Let ν be a positive measure on $(\Omega, \mathcal{A})$. Recall that in Bochner's integration theory, a function $f : \Omega \to V$ is called ν-measurable if there exists a sequence of functions $f_n(x) = \sum_{\ell=1}^n a_i \chi_{A_\ell}(x)$, where $a_\ell \in V$ and χ_{A_ℓ} is the characteristic function of a measurable set A_ℓ, such that $f_n(x) \to f(x)$ as $n \to +\infty$ for ν-almost all $x \in \Omega$.

The next lemma is a simple application of the Pettis measurability theorem, which states that f is ν-measurable if and only if

(i) f is essentially separably valued, i.e. there exist a separable subspace B of V and a subset $N \subset \Omega$ with $\nu(N) = 0$ such that $f(\Omega \setminus N) \subset B$, and

(ii) f is weakly measurable, i.e. the function $\varphi \circ f : \Omega \to \mathbb{R}$ is measurable for any continuous $\mathbb{R}$-linear functional $\varphi : V \to \mathbb{R}$.

Lemma 5.8. *Let X be a quaternionic Banach space and let ν be a positive measure on $(\Omega, \mathcal{A})$. If $f : \Omega \to X$ and $g : \Omega \to \mathbb{H}$ are ν-measurable, then the functions fg and gf are ν-measurable.*

Proof. Let $g = \sum_{\ell=0}^3 g_\ell e_\ell$, where $e_0 = 1$ and e_1, e_2, e_3 is the generating basis of $\mathbb{H}$ and where $g_\ell : \Omega \to \mathbb{R}$ are the real-valued components of g. The components are given by $g_\ell = \pi_\ell \circ g$, where π_ℓ is the projection onto the ℓ-th component of $\mathbb{R}^4 \cong \mathbb{H}$, and so they are measurable.

By the Pettis measurability theorem, the function f is essentially separably valued and weakly measurable. Let now $N \subset \Omega$ with $\nu(N) = 0$ such that $f(\Omega \setminus N) \subset B$ for some separable ($\mathbb{R}$-linear) subspace B of V. Since $g_\ell(x) \in \mathbb{R}$ and $f(x) \in B$, we find that also $(fg_\ell)(x) = f(x)g_\ell(x) \in B$ for any $x \in \Omega \setminus N$ and hence fg_ℓ is essentially separably valued for $\ell = 0, \ldots, 3$. Furthermore, for any continuous $\mathbb{R}$-linear functional $\varphi : V \to \mathbb{R}$, we find that $\varphi \circ (fg_\ell)(x) = \varphi(f(x)g_\ell(x)) = \varphi(f(x))g_\ell(x)$. Hence,

$\varphi \circ (fg_\ell)$ is the product of the two measurable functions $\varphi \circ f$ and g_ℓ from Ω to $\mathbb{R}$ and so measurable itself. We conclude that fg_ℓ is weakly measurable and so, due to the Pettis measurability theorem, it is even ν-measurable.

We further observe that the mappings $\mathbf{v} \mapsto \mathbf{v}e_\ell$ and $\mathbf{v} \mapsto e_\ell\mathbf{v}$ are continuous $\mathbb{R}$-linear mappings on V and hence not only the functions $x \mapsto f(x)g_\ell(x)$ but also the functions $x \mapsto f(x)g_\ell(x)e_\ell$ and $x \mapsto e_\ell f(x)g_\ell(x)$ are ν-measurable. So we finally conclude that fg and gf are ν-measurable since they are the sums of ν-measurable functions.

□

Let ν be a positive measure on $(\Omega, \mathcal{A})$. Recall that a ν-measurable function on Ω with values in a real Banach space is Bochner-integrable, if $\int_\Omega \|f(x)\| d\nu(x) < +\infty$.

Definition 5.9. Let V be a quaternionic Banach space, let $\mu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ and let $h : \Omega \to \mathbb{H}$ be the function with $|h| = 1$ such that $d\mu(x) = h(x)\, d|\mu|(x)$. We call two μ-measurable functions $f : \Omega \to X$ and $g : \Omega \to \mathbb{H}$ a *μ-integrable pair*, if

$$\int_\Omega \|f\|\|g\|\, d|\mu| < +\infty.$$

In this case, we define

$$\int_\Omega f\, d\mu, g := \int_\Omega fhg\, d|\mu| \tag{5.2}$$

and

$$\int_\Omega g\, d\mu\, f = \int_\Omega ghf\, d|\mu|, \tag{5.3}$$

as the integrals of a function with values in a real Banach space in the sense of Bochner.

Remark 5.10. Note that in the definition of the integrals in (5.2) and (5.3), we can replace the variation of μ by any σ-finite positive measure ν with $\mu \ll \nu$. If h_ν is the density of μ with respect to ν and $\rho_{|\mu|}$ and ρ_ν are the real-valued densities of $|\mu|$ and ν with respect to $|\mu| + \nu$, then

$$\mu = h|\mu| = h\rho_{|\mu|}(|\mu| + \nu) \quad \text{and} \quad \mu = h_\nu\nu = h_\nu\rho_\nu(|\mu| + \nu).$$

Theorem 5.6 implies that $h\rho_{|\mu|} = h_\nu\rho_\nu$ is in $L^1(|\mu| + \nu)$. Therefore

$$\begin{aligned}&\int_\Omega fh_\nu g\, d\nu = \int_\Omega\int_\Omega fh_\nu g\rho_\nu\, d(|\mu| + \nu)\\ &= \int_\Omega\int_\Omega fh_\nu\rho_\nu g\, d(|\mu| + \nu) = \int_\Omega fh\rho_{|\mu|}g\, d(|\mu| + \nu)\\ &= \int_\Omega fhg\rho_{|\mu|}\, d(|\mu| + \nu) = \int_\Omega fhg\, d|\mu| = \int_\Omega f\, d\mu\, g.\end{aligned}$$

Hence, the integral is linear in the measure: if $\mu, \nu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ then μ and ν are absolutely continuous with respect to $\tau = |\mu| + |\nu|$. If ρ_μ and ρ_ν are the densities of μ and ν with respect to τ, then

$$\begin{aligned}&\int_\Omega f\, d(\mu + \nu)\, g = \int_\Omega f(\rho_\mu + \rho_\nu)g\, d\tau\\ &= \int_\Omega f\rho_\mu g\, d\tau + \int_\Omega f\rho_\nu g\, d\tau = \int_\Omega f\, d\mu\, g + \int_\Omega f\, d\nu\, g.\end{aligned}$$

Similarly, if $a \in \mathbb{H}$ and $\mu = \rho|\mu|$ then $a\mu = a\rho|\mu|$ and so

$$\int_\Omega f\, d(a\mu)\, g = \int_\Omega f(a\rho) g\, d|\mu| = \int_\Omega (fa)\rho g\, d|\mu| = \int_\Omega (fa)\, d\mu\, g.$$

In the same way, one can see that $\int_\Omega f\, d(\mu a) g = \int_\Omega f\, d\mu\, (ag)$.

We finally define now the product measure and the convolution of two quaternionic measures as in [3]. Also these concepts will be essential when we discuss the product rule of the quaternionic Phillips functional calculus.

Lemma 5.11. *Let $\mu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ and $\nu \in \mathcal{M}(\Upsilon, \mathcal{B}, \mathbb{H})$. Then there exists a unique measure $\mu \times \nu$ on the product measurable space $(\Omega \times \Upsilon, \mathcal{A} \otimes \mathcal{B})$ such that*

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B) \tag{5.4}$$

for all $A \in \mathcal{A}, B \in \mathcal{B}$. We call $\mu \times \nu$ the product measure *of μ and ν.*

Proof. Let $\mathbf{i}, \mathbf{j} \in \mathbb{S}$ with $\mathbf{i} \perp \mathbf{j}$ and let $\mu = \mu_1 + \mathbf{j}\mu_2$ with $\mu_1, \mu_2 \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{C}_{\mathbf{i}})$ and $\nu = \nu_1 + \nu_2\mathbf{j}$ with $\nu_1, \nu_2 \in \mathcal{M}(\Upsilon, \mathcal{B}, \mathbb{C}_{\mathbf{i}})$. Then, there exist unique complex product measures $\mu_\ell \times \nu_\kappa \in M(\Omega_1 \times \Omega_2, \mathcal{A} \otimes \mathcal{B}, \mathbb{C}_{\mathbf{i}})$ of μ_ℓ and ν_ℓ for $\ell, \kappa = 1, 2$. If we set

$$\mu \times \nu = \mu_1 \times \nu_1 + \mathbf{j}\mu_2 \times \nu_1 + \mu_1 \times \nu_2\mathbf{j} + J\mu_2 \times \nu_2\mathbf{j},$$

then $\mu \times \nu$ is a quaternionic measure on $(\Omega \times \Upsilon, \mathcal{A} \otimes \mathcal{B})$ and

$$\begin{aligned}
\mu(A)\nu(B) =& \mu_1(A)\nu_1(B) + \mathbf{j}\mu_2(A)\nu_1(B) \\
&+ \mu_1(A)\nu_2(B)\mathbf{j} + \mathbf{j}\mu_2(A)\nu_2(B)\mathbf{j} \\
=& \mu_1 \times \nu_1(A \times B) + \mathbf{j}\mu_2 \times \nu_1(A \times B) \\
&+ \mu_1 \times \nu_2(A \times B)\mathbf{j} + \mathbf{j}\mu_2 \times \nu_2(A \times B)\mathbf{j} = (\mu \times \nu)(A \times B).
\end{aligned}$$

In order to prove the uniqueness of the product measure, assume that two quaternionic measures $\rho = \rho_1 + \rho_2\mathbf{j}$ and $\tau = \tau_1 + \tau_2\mathbf{j}$ on $(\Omega \times \Upsilon, \mathcal{A} \times \mathcal{B})$ satisfy $\rho(A \times B) = \tau(A \times B)$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then $\rho_1(A \times B) = \tau_1(A \times B)$ and $\rho_2(A \times B) = \tau_2(A \times B)$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Since two complex measures on the product space $(\Omega \times \Upsilon, \mathcal{A} \otimes \mathcal{B})$ are equal if and only if they coincide on sets of the form $A \times B$, we obtain $\rho_1 = \tau_1$ and $\rho_2 = \tau_2$ and, in turn, $\rho = \rho_1 + \rho_2\mathbf{j} = \tau_1 + \tau_2\mathbf{j} = \tau$. Therefore, $\mu \times \nu$ is uniquely determined by (5.4).

□

Remark 5.12. Note that it is also possible to define a commuted product measure $\mu \times_c \nu$ on $(\Omega \times \Upsilon, \mathcal{A} \otimes \mathcal{B})$ that satisfies

$$(\mu \times_c \nu)(A \times B) = \nu(B)\mu(A), \quad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

This measure is different from the measure $\nu \times \mu$ that is defined on $(\Upsilon \times \Omega, \mathcal{B} \otimes \mathcal{A})$ and satisfies

$$(\nu \times \mu)(B \times A) = \nu(B)\mu(A), \quad \forall B \in \mathcal{B}, A \in \mathcal{A}.$$

Lemma 5.13. *Let $(\Omega, \mathcal{A}, \mu)$ and $(\Upsilon, \mathcal{B}, \nu)$ be quaternionic measure spaces. Then*

$$|\mu \times \nu| = |\mu| \times |\nu|.$$

Moreover, if $d\mu(x) = f(x)\,d|\mu|(x)$ *and* $\nu(x) = g(x)\,d|\nu|(x)$ *as in Corollary 5.7, then for any* $C \in \mathcal{A} \times \mathcal{B}$

$$(\mu \times \nu)(C) = \int_C f(s)g(t)\,d|\mu \times \nu|(s,t).$$

Proof. Let $f\colon \Omega \to \mathbb{H}$ and $g\colon \Upsilon \to \mathbb{H}$ with $|f| = 1$ and $|g| = 1$ be functions as in Corollary 5.7 such that $\mu(A) = \int_A f(t)\,d|\mu|(t)$ and $\nu(B) = \int_B g(s)\,d|\nu|(s)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Moreover, let $r = (t,s)$ and let $h(r) = f(t)g(s)$. Then the function $C \mapsto \int_C h(r)\,d(|\mu| \times |\nu|)(r)$ defines a measure on $(\Omega \times \Upsilon, \mathcal{A} \times \mathcal{B})$ and Fubini's theorem for positive measures implies

$$\begin{aligned}\int_{A\times B} h(r)\,d|\mu| \times |\nu|(r) &= \int_A \int_B f(t)g(s)\,d|\mu|(t)\,d|\nu|(s)\\ &= \int_A f(t)\,d|\mu|(t) \int_B g(s)\,d|\nu|(s) = \mu(A)\nu(B).\end{aligned}$$

The uniqueness of the product measure implies $\mu \times \nu(C) = \int_C h(r)\,d(|\mu| \times |\nu|)(r)$ for any $C \in \mathcal{A} \times \mathcal{B}$. Since $|h| = |f|\,|g| = 1$, we deduce from (5.1) that

$$|\mu \times \nu|(C) = \int_C |h(r)|\,d(|\mu| \times |\nu|)(r) = (|\mu| \times |\nu|)(C)$$

for all $C \in \mathcal{A} \times \mathcal{B}$. □

Splitting the measure μ into two complex components and applying the respective result for complex measures, we obtain the transformation rule for integrals with respect to a push forward measure stated in the following lemma.

Lemma 5.14. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a quaternionic measure space, let* $(\Upsilon, \mathcal{B})$ *be a measurable space and let* $\phi : \Omega \to \Upsilon$ *be a measurable function. If a function* $f : \Upsilon \to X$ *with values in a quaternionic Banach space* X *is integrable with respect to the image measure* $\mu^\phi(B) := \mu\left(\phi^{-1}(B)\right)$ *and* $f \circ \phi$ *is integrable with respect to* μ*, then*

$$\int_\Upsilon f\,d\mu^\phi = \int_\Omega f \circ \phi\,d\mu. \tag{5.5}$$

Definition 5.15. We denote the Borel sets on a topological space X by $\mathsf{B}(X)$. In particular, we denote the Borel sets on $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ by $\mathsf{B}(\mathbb{R})$, $\mathsf{B}(\mathbb{C})$ and $\mathsf{B}(\mathbb{H})$, respectively.

We recall that, for any Borel set $E \in \mathsf{B}(\mathbb{R})$, the set

$$P(E) := \{(u,v) \in \mathbb{R}^2 : u + v \in E\}$$

is a Borel subset of $\mathbb{R}^2$.

Definition 5.16. Let μ, ν be quaternionic measures on $\mathsf{B}(\mathbb{R})$. The convolution $\mu * \nu$ of μ and ν is the image measure of $\mu \times \nu$ under the mapping $\phi : \mathbb{R}^2 \to \mathbb{R}$, $(u,v) \mapsto u + v$, that is, for any $E \in \mathsf{B}(\mathbb{R})$, we set

$$(\mu * \nu)(E) := (\mu \times \nu)(P(E)).$$

The following corollary is immediate.

Corollary 5.17. *Let* $\mu, \nu, \rho \in \mathcal{M}(\mathbb{R}, \mathsf{B}(\mathbb{R}), \mathbb{H})$ *and let* $a, b \in \mathbb{H}$. *Then*

(i) $(\mu + \nu) * \rho = \mu * \rho + \nu * \rho$ *and* $\mu * (\nu + \rho) = \mu * \nu + \mu * \rho$

(ii) $(a\mu) * \nu = a(\mu * \nu)$ *and* $\mu * (\nu a) = (\mu * \nu)a$.

Corollary 5.18. *Let* $\mu, \nu \in \mathcal{M}(\mathbb{R}, \mathsf{B}(\mathbb{R}), \mathbb{H})$. *Then the estimate*

$$|\mu * \nu|(E) \leq (|\mu| * |\nu|)(E)$$

holds true for all $E \in \mathsf{B}(\mathbb{R})$.

Proof. Let $E \in \mathsf{B}(\mathbb{R})$ and let $\pi \in \Pi(E)$ be a countable measurable partition of E. Then

$$\begin{aligned}\sum_{E_\ell \in \pi} |(\mu * \nu)(E_\ell)| &= \sum_{E_\ell \in \pi} |(\mu \times \nu)(P(E_\ell))| \\ &\leq \sum_{E_\ell \in \pi} |\mu \times \nu|(P(E_\ell)) = |\mu \times \nu|(P(E)),\end{aligned}$$

and taking the supremum over all possible partitions $\pi \in \Pi(E)$ yields

$$|\mu * \nu|(E) \leq |\mu \times \nu|(P(E)) = (|\mu| \times |\nu|)(P(E)) = (|\mu| * |\nu|)(E).$$

$\square$

Corollary 5.19. *Let* $\mu, \nu \in \mathcal{M}(\mathbb{R}, \mathsf{B}(\mathbb{R}), \mathbb{H})$ *and let* $F \colon \mathbb{R} \to V$ *be integrable with respect to* $\mu * \nu$ *and such that* $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \|F(s+t)\| \, d|\mu|(s) \, d|\nu|(t) < +\infty$. *Then*

$$\int_{\mathbb{R}} F(r) \, d(\mu * \nu)(r) = \int_{\mathbb{R}} \int_{\mathbb{R}} F(s+t) \, d\mu(s) \, d\nu(t).$$

Proof. Because of our assumptions and Definition 5.16 we can apply Lemma 5.14 with $\phi(s,t) = s + t$. If $\mu(A) = \int_A f(t) \, d|\mu|(t)$ and $\nu(A) = \int_A g(s) \, d|\nu|(s)$, then the product measures satisfies $(\mu \times \nu)(B) = \int_B f(s)g(t) \, d(|\mu| \times |\nu|)(s,t)$ by Lemma 5.13. Applying Fubini's theorem yields

$$\begin{aligned}&\int_{\mathbb{R}} F(r) \, d(\mu * \nu)(r) = \\ &= \int_{\mathbb{R}} F(\phi(s,t)) \; d(\mu \times \nu)(s,t) \\ &= \int_{\mathbb{R}} F(\phi(s,t)) f(s) g(t) \; d|\mu \times \nu|(s,t) \\ &= \int_{\mathbb{R}} F(s+t) f(s) g(t) \; d|\mu|(s) \, d|\nu|(t) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} F(s+t) \, d\mu(s) \, d\nu(t).\end{aligned}$$

$\square$

5.2 The Quaternionic Laplace-Stieltjes-Transform and Functions of the Generator of a Strongly Continuous Group

Recall that we made the assumption (G), that is that $T \in \mathcal{K}(V)$ is the infinitesimal generator of the strongly continuous group $(\mathcal{U}_T(t))_{t\in\mathbb{R}}$ on operators V. By Theorem 2.74, there exist positive constants $M > 0$ and $\omega \geq 0$ such that $\|\mathcal{U}_T(t)\| \leq Me^{\omega|t|}$ and such that the S-spectrum of the infinitesimal generator T lies in the strip

$$W_\omega := \{s \in \mathbb{H} : -\omega < \mathrm{Re}(s) < \omega\}.$$

Moreover, we have

$$S_R^{-1}(s,T) = \int_0^{+\infty} e^{-ts}\,\mathcal{U}_T(t)\,dt, \qquad \mathrm{Re}(s) > \omega$$

and

$$S_R^{-1}(s,T) = -\int_{-\infty}^{0} e^{-ts}\,\mathcal{U}_T(t)\,dt, \qquad \mathrm{Re}(s) < -\omega.$$

Definition 5.20. We denote by $\mathbf{S}(T)$ the family of all quaternionic measures μ on $\mathsf{B}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} d|\mu|(t)\, e^{(\omega+\varepsilon)|t|} < +\infty$$

for some $\varepsilon = \varepsilon(\mu) > 0$. The function

$$\mathcal{L}(\mu)(s) = \int_{\mathbb{R}} d\mu(t)\, e^{-st}$$

with domain

$$W_{\omega+\varepsilon} := \{\, s \in \mathbb{H} : -(\omega+\varepsilon) < \mathrm{Re}(s) < (\omega+\varepsilon)\} \tag{5.6}$$

is called the quaternionic bilateral (right) Laplace-Stieltjes-transform of μ.

Definition 5.21. We denote by $\mathbf{V}(T)$ the set of quaternionic bilateral Laplace-Stieltjes-transforms of measures in $\mathbf{S}(T)$.

Lemma 5.22. *Let $\mu, \nu \in \mathbf{S}(T)$ and $a \in \mathbb{H}$.*

(i) The measures $a\mu$ and $\mu + \nu$ belong to $\mathbf{S}(T)$ and

$$\mathcal{L}(a\mu) = a\mathcal{L}(\mu), \qquad \mathcal{L}(\mu+\nu) = \mathcal{L}(\mu) + \mathcal{L}(\nu).$$

(ii) The measures $\mu * \nu$ belongs to $\mathbf{S}(T)$. If ν is real-valued, then

$$\mathcal{L}(\mu * \nu) = \mathcal{L}(\mu) * \mathcal{L}(\nu).$$

Proof. Let $\varepsilon = \min\{\varepsilon(\mu), \varepsilon(\nu)\}$. Corollary 5.5 implies

$$\int_{\mathbb{R}} d|a\mu|\, e^{|t|(\omega+\varepsilon)} = |a| \int_{\mathbb{R}} d|\mu|\, e^{|t|(\omega+\varepsilon)} < +\infty$$

and

$$\int_{\mathbb{R}} d|\mu+\nu|\, e^{|t|(\omega+\varepsilon)} \leq \int_{\mathbb{R}} d|\mu|\, e^{|t|(\omega+\varepsilon)} + \int_{\mathbb{R}} d|\nu|\, e^{|t|(\omega+\varepsilon)} < +\infty.$$

Thus, $a\mu$ and $\mu+\nu$ belong to $\mathbf{S}(T)$. The relations $\mathcal{L}(a\mu) = a\mathcal{L}(\mu)$ and $\mathcal{L}(\mu+\nu) = \mathcal{L}(\mu)+\mathcal{L}(\nu)$ follow from the left linearity of the integral in the measure.

The variation of the convolution of μ and ν satisfies $|\mu * \nu|(E) \leq (|\mu| * |\nu|)(E)$ for any Borel set $E \in \mathsf{B}(\mathbb{R})$, cf. Corollary 5.18. In view of Corollary 5.19, we have

$$\begin{aligned}\int_{\mathbb{R}} d|\mu * \nu|(r) e^{(w+\varepsilon)|r|} &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} d|\mu|(s)\, d|\nu|(t) e^{(w+\varepsilon)|s+t|}\\ &\leq \int_{\mathbb{R}} d|\mu|(s)\, e^{(w+\varepsilon)|s|} \int_{\mathbb{R}} d|\nu|(t)\, e^{(w+\varepsilon)|t|} < +\infty.\end{aligned}$$

Therefore, $\mu * \nu \in \mathbf{S}(T)$. If ν is real-valued, then ν commutes with e^{-st} and Fubini's theorem implies for $s \in W_{\omega+\varepsilon}$ that

$$\begin{aligned}\mathcal{L}(\mu * \nu)(s) &= \int_{\mathbb{R}} d(\mu*\nu)(r)\, e^{-sr} = \int_{\mathbb{R}}\int_{\mathbb{R}} d\mu(t)\, d\nu(u)\, e^{-s(t+u)}\\ &= \int_{\mathbb{R}} d\mu(t)\, e^{-st} \int_{\mathbb{R}} d\nu(u)\, e^{-su} = \mathcal{L}(\mu)(s)\, \mathcal{L}(\nu)(s).\end{aligned}$$

□

Theorem 5.23. *Let $f \in \mathbf{V}(T)$ with $f(s) = \int_{\mathbb{R}} d\mu(t)\, e^{-st}$ for any s in the strip $W_{\omega+\varepsilon}$ defined in* (5.6)*.*

(i) The function f is right slice hyperholomorphic on the strip $W_{\omega+\varepsilon}$.

(ii) For any $n \in \mathbb{N}$, the measure μ^n defined by

$$\mu^n(E) = \int_E d\mu(t)\, (-t)^n \qquad \textit{for } E \in \mathsf{B}(\mathbb{R})$$

belongs to $\mathbf{S}(T)$ and, for $s \in W_{\omega+\varepsilon}$, we have

$$\partial_S{}^n f(s) = \int_{\mathbb{R}} d\mu^n(t)\, e^{-st} = \int_{\mathbb{R}} d\mu(t)\, (-t)^n e^{-st}, \tag{5.7}$$

where $\partial_S f$ denotes the slice derivative of f as in Definition 2.12.

Proof. In the proof of this results we will make use of the same kind of arguments as in [38, Lemma 2, p. 642]. For every $n \in \mathbb{N}$ and every $0 < \varepsilon_1 < \varepsilon$ there exists a constant K such that

$$|t|^n e^{(\omega+\varepsilon_1)|t|} \leq K e^{(\omega+\varepsilon)|t|}, \quad t \in \mathbb{R}.$$

Since $\mu \in \mathbf{S}(T)$, we have

$$\int_{\mathbb{R}} d|\mu^n|(t)\, e^{(\omega+\varepsilon_1)|t|} = \int_{\mathbb{R}} d|\mu|(t)\, |t|^n e^{(\omega+\varepsilon_1)|t|} \leq K \int_{\mathbb{R}} d|\mu|(t)\, e^{(\omega+\varepsilon)|t|} < +\infty$$

and so $\mu^n \in \mathbf{S}(T)$.

The function f is a right slice function as

$$f(s) = \int_{\mathbb{R}} d\mu(t) e^{-t(s_0+\mathbf{i}_s s_1)} = \int_{\mathbb{R}} d\mu(t) e^{-ts_0}\cos(s_1) - \int_{\mathbb{R}} d\mu(t) e^{-ts_0}\sin(s_1)\mathbf{i}_s$$

and

$$\alpha(s_0, s_1) := \int_{\mathbb{R}} d\mu(t) e^{-|s|t} e^{-ts_0} \cos(s_1)$$

and

$$\beta(s_0, s_1) := -\int_{\mathbb{R}} d\mu(t) e^{-|s|t} e^{-ts_0} \sin(s_1)$$

satisfy the compatibility conditions (2.4). For any $s = s_0 + \mathbf{i}_s s_1 \in W_{\omega+\varepsilon}$, we have

$$\lim_{\mathbb{C}_{\mathbf{i}_s} \ni p \to s} (f_{\mathbf{i}_s}(p) - f_{\mathbf{i}_s}(s))(p-s)^{-1} = \lim_{\mathbb{C}_{\mathbf{i}_s} \ni p \to s} \int_{\mathbb{R}} d\mu(t) \, \frac{e^{-pt} - e^{-st}}{p-s}.$$

If p is sufficiently close to s such that also $p \in W_{\omega+\varepsilon}$, then the simple calculation

$$|e^{-pt} - e^{-st}| = \left| \int_0^1 e^{-ts - t\xi(p-s)} t(p-s) \, d\xi \right| \leq |t| e^{(\omega+\varepsilon)|t|} |p-s|,$$

yields the estimate

$$\frac{|e^{-pt} - e^{-st}|}{|p-s|} \leq |t| e^{(\omega+\varepsilon)|t|},$$

which allows us to apply Lebesgue's theorem of dominated convergence in order to exchange limit and integration. We obtain

$$\lim_{\mathbb{C}_{\mathbf{i}_s} \ni \to s} (f_{\mathbf{i}_s}(p) - f_{\mathbf{i}_s}(s))(p-s)^{-1} = \int_{\mathbb{R}} d\mu(t) \, (-t) e^{-st} = \int_{\mathbb{R}} d\mu^1(t) \, e^{-st}. \tag{5.8}$$

Consequently, the restriction $f_{\mathbf{i}_s}$ of f to the complex plane $\mathbb{C}_{\mathbf{i}_s}$ is right holomorphic and, by Lemma 2.15, the function$f$ is in turn right slice hyperholomorphic on the strip $W_{\omega+\varepsilon}$. Moreover, (5.8) implies

$$\partial_S f(s) = \int_{\mathbb{R}} d\mu^1(t) \, e^{-st}$$

for $s \in W_{\omega+\varepsilon}$. By induction we get (5.7).

□

Definition 5.24. Let T be the quaternionic infinitesimal generator of the strongly continuous group $(\mathcal{U}_T(t))_{t\in\mathbb{R}}$ on a quaternionic Banach space V. For $f \in \mathbf{V}(T)$ with

$$f(s) = \int_{\mathbb{R}} d\mu(t) \, e^{-st} \quad \text{for } s \in W_{\omega+\varepsilon},$$

and $\mu \in \mathbf{S}(T)$, we define the right linear operator $f(T)$ on V by

$$f(T) = \int_{\mathbb{R}} d\mu(t) \, \mathcal{U}_T(-t). \tag{5.9}$$

Remark 5.25. In particular for $p \in \mathbb{H}$ with $\mathrm{Re}(p) < -\omega$ the function $s \mapsto S_R^{-1}(p, s)$ belongs to $\mathbf{S}(T)$. Set $d\mu_p(t) = -\chi_{[0,+\infty)}(t) e^{tp} \, dt$, where χ_A denotes the characteristic function of a set A. If $\mathrm{Re}(p) < \mathrm{Re}(s)$, then

$$\mathcal{L}(\mu_p)(s) = \int_{\mathbb{R}} d\mu_p(t) \, e^{-ts} = -\int_0^{+\infty} e^{tp} e^{-ts} \, dt = -S_L^{-1}(s, p) = S_R^{-1}(p, s)$$

and

$$\mathcal{L}(\mu_p)(T) = \int_{\mathbb{R}} d\mu_p(t)\,\mathcal{U}(-t)$$
$$= -\int_0^{+\infty} e^{tp}\,\mathcal{U}(-t)\,dt = -\int_{-\infty}^{0} e^{-tp}\,\mathcal{U}(t)\,dt = S_R^{-1}(p,T).$$

For $p \in \mathbb{H}$ with $\omega < \operatorname{Re}(p)$ set $d\mu_p(t) = \chi_{(-\infty,0]}(t)e^{tp}\,dt$. Similar computations show that also in this case $S_R^{-1}(p,s) = \mathcal{L}(\mu_p)(s) \in \mathbf{S}(T)$ if $\operatorname{Re}(s) < \operatorname{Re}(p)$ and $\mathcal{L}(\mu_p)(T) = S_R^{-1}(p,T)$.

Theorem 5.26. *For any $f \in \mathbf{V}(T)$, the operator $f(T)$ is bounded.*

Proof. Let $f(s) = \int_{\mathbb{R}} d\mu(t)\, e^{-st} \in \mathbf{V}(T)$ with $\mu \in \mathbf{S}(T)$. Since $\|\mathcal{U}_T(t)\| \leq Me^{w|t|}$, we have
$$\|f(T)\| \leq \int_{\mathbb{R}} d|\mu|(t)\,\|\mathcal{U}_T(-t)\| \leq M\int_{\mathbb{R}} d|\mu|(t)\,e^{w|t|} < +\infty.$$
□

Lemma 5.27. *Let $f = \mathcal{L}(\mu)$ and $g = \mathcal{L}(\nu)$ belong to $\mathbf{V}(T)$ and let $a \in \mathbb{H}$.*

(i) We have $(af)(T) = af(T)$ and $(f+g)(T) = f(T) + g(T)$.

(ii) If g is an intrinsic function, then ν is real-valued and $(fg)(T) = f(T)g(T)$.

Proof. The statement (i) follows immediately from Lemma 5.22 and the left linearity of the integral (5.9) in the measure. In order to show (ii), we assume that $g = \mathcal{L}(\nu)$ is intrinsic. Then the measure ν is real-valued and Lemma 5.22 gives $fg = \mathcal{L}(\mu * \nu) \in \mathbf{V}(T)$. We find
$$(fg)(T)v = \int_{\mathbb{R}} d(\mu*\nu)(r)\,\mathcal{U}_T(-r) = \int_{\mathbb{R}}\int_{\mathbb{R}} d\mu(s)\,d\nu(t)\,\mathcal{U}_T(-(s+t))$$
$$= \int_{\mathbb{R}} d\mu(s)\,\mathcal{U}_T(-s)\int_{\mathbb{R}} d\nu(t)\,\mathcal{U}_T(-t) = f(T)g(T),$$
where we use that $\mathcal{U}_T(-s)$ and ν commute because ν is real-valued.

□

5.3 Comparison with the S-Functional Calculus

A natural question that arises is the relation between the functional calculus introduced in Definition 5.24 and the S-functional calculus for closed operators. In this section we show that the two functional calculi coincide if the function f is slice hyperholomorphic at infinity. In order to prove this, we need a specialised version of the Residue theorem that fits into our setting.

Lemma 5.28. *Let $O \subset \mathbb{H}$ be axially symmetric and open, let $f : O \setminus [p] \to \mathbb{H}$ be right slice hyperholomorphic and let $g : O \to V$ be left slice hyperholomorphic such that $p = p_0 + \mathsf{i}p_1 \in O$ is a pole of order $n_f \geq 0$ of the $\mathbb{H}$-valued right holomorphic function $f_{\mathsf{i}} := f_{O\cap\mathbb{C}_{\mathsf{i}}}$. If $\varepsilon > 0$ is such that $cl(U_\varepsilon(p)\cap\mathbb{C}_{\mathsf{i}}) \subset O$, then*
$$\frac{1}{2\pi}\int_{\partial(U_\varepsilon(p)\cap\mathbb{C}_{\mathsf{i}})} f(s)\,ds_{\mathsf{i}}\,g(s) = \sum_{k=0}^{n_f-1}\frac{1}{k!}\operatorname{Res}_p\left(f_{\mathsf{i}}(s)(s-p)^k\right)\left(\partial_S{}^k g(p)\right).$$

Proof. Since f is right slice hyperholomorphic, its restriction f_{i} is a vector-valued holomorphic function on $O \cap \mathbb{C}_{\mathsf{i}}$ if we consider $\mathbb{H}$ as a vector space over $\mathbb{C}_{\mathsf{i}}$ by restricting the multiplication with quaternions on the right to $\mathbb{C}_{\mathsf{i}}$. Similarly, since g is left slice hyperholomorphic, its restriction $g_{\mathsf{i}} := g|_{O\cap\mathbb{C}_{\mathsf{i}}}$ is a V-valued holomorphic function if we consider V as a complex vector space over $\mathbb{C}_{\mathsf{i}}$ by restricting the left scalar multiplication to $\mathbb{C}_{\mathsf{i}}$. Consequently, if we set $\rho = \operatorname{dist}(p, \partial(O \cap \mathbb{C}_{\mathsf{i}}))$, then

$$f_{\mathsf{i}}(s) = \sum_{k=-n_f}^{+\infty} a_k (s-p)^k \quad \text{and} \quad g_{\mathsf{i}}(s) = \sum_{k=0}^{+\infty} (s-p)^k b_k \tag{5.10}$$

for $s \in (B_\rho(p) \cap \mathbb{C}_{\mathsf{i}}) \setminus \{p\}$ with $a_k \in \mathbb{H}$ and $b_k \in V$. These series converge uniformly on $\partial(B_\varepsilon(p) \cap \mathbb{C}_{\mathsf{i}})$ for any $0 < \varepsilon < \rho$. Thus,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(B_\varepsilon(p)\cap\mathbb{C}_{\mathsf{i}})} f(s)\, ds_{\mathsf{i}}\, g(s) = \\
=&\frac{1}{2\pi}\int_{\partial(B_\varepsilon(p)\cap\mathbb{C}_{\mathsf{i}})} \left(\sum_{k=0}^{+\infty} a_{k-n_f}(s-p)^{k-n_f}\right) ds_{\mathsf{i}} \left(\sum_{j=0}^{+\infty}(s-p)^j b_j\right) \\
=&\sum_{k=0}^{+\infty}\sum_{j=0}^{k} a_{k-j-n_f}\left(\frac{1}{2\pi}\int_{\partial(B_\varepsilon(p)\cap\mathbb{C}_{\mathsf{i}})} (s-p)^{k-j-n_f}\, ds_{\mathsf{i}}\,(s-p)^j\right) b_j \\
=&\sum_{j=0}^{n_f-1} a_{-(j+1)} b_j,
\end{aligned}$$

since $\frac{1}{2\pi}\int_{\partial(B_\varepsilon(p)\cap\mathbb{C}_{\mathsf{i}})}(s-p)^{k-n_f}\, ds_{\mathsf{i}}$ equals 1 if $k - n_f = -1$ and 0 otherwise. Finally, we observe that $a_{-k} = \operatorname{Res}_p\left(f_{\mathsf{i}}(s)(s-p)^{k-1}\right)$ and $b_k = \frac{1}{k!}\partial_S{}^k g_{\mathsf{i}}(p)$ by their definition in (5.10).

□

In order to compute the integral in the S-functional calculus, we recall the definition of the strip

$$W_c := \{s \in \mathbb{H} : -c < \operatorname{Re}(s) < c\} \qquad \text{for } c > 0$$

and we introduce the set $\partial(W_c \cap \mathbb{C}_{\mathsf{i}})$ for $\mathsf{i} \in \mathbb{S}$. It consists of the two lines $s = c + \mathsf{i}\tau$ and $s = -c - \mathsf{i}\tau$ with $\tau \in \mathbb{R}$.

Proposition 5.29. *Let α and c be real numbers such that $\omega < c < |\alpha|$. For any vector $\mathbf{v} \in \operatorname{dom}(T^2)$, we have*

$$\mathcal{U}_T(t)\mathbf{v} = \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_{\mathsf{i}})} e^{ts}(\alpha - s)^{-2}\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2\mathbf{v}. \tag{5.11}$$

Proof. We recall that

$$S_R^{-1}(s,T) = \int_0^{+\infty} e^{-ts}\,\mathcal{U}_T(t)\, dt, \quad \operatorname{Re}(s) > \omega.$$

Since $\|\mathcal{U}_T(t)\| \leq Me^{\omega|t|}$, we get a bound for the S-resolvent operator by

$$\|S_R^{-1}(s,T)\| = M\int_0^{+\infty} e^{(\omega - \operatorname{Re}(s))t}\, dt, \quad \operatorname{Re}(s) > \omega \tag{5.12}$$

which assures that $\|S_R^{-1}(s,T)\|$ is uniformly bounded on $\{s \in \mathbb{H} : \operatorname{Re}(s) > \omega + \varepsilon\}$ for any $\varepsilon > 0$. A similar consideration gives a uniform bound of $\|S_R^{-1}(s,T)\|$ on $\{s \in \mathbb{H} : \operatorname{Re}(s) < -(\omega + \varepsilon)\}$. Thanks to such bound the integral in (5.11) is well defined since the $(\alpha - s)^{-2}$ goes to zero with order $1/|s|^2$ as $s \to \infty$. We set

$$F(t)\mathbf{v} = \frac{1}{2\pi} \int_{\partial(W_c \cap \mathbb{C}_{\mathsf{i}})} e^{ts}(\alpha - s)^{-2}\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 \mathbf{v}$$

for $\mathbf{v} \in \operatorname{dom}(T^2)$. We show that $F(t)\mathbf{v} = \mathcal{U}_T(t)\mathbf{v}$ using the Laplace transform and we first assume $t > 0$. If $\operatorname{Re}(p) > c$, then

$$\begin{aligned}
&\int_0^{+\infty} e^{-pt} F(t)\mathbf{v}\, dt = \\
=&\frac{1}{2\pi} \int_0^{+\infty} e^{-pt} \int_{\partial(W_c \cap \mathbb{C}_{\mathsf{i}})} e^{ts}(\alpha - s)^{-2}\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 \mathbf{v}\, dt \\
=&\frac{1}{2\pi} \int_{\partial(W_c \cap \mathbb{C}_{\mathsf{i}})} \left(\int_0^{+\infty} e^{-pt} e^{ts} dt \right) (\alpha - s)^{-2}\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 \mathbf{v}.
\end{aligned}$$

Now observe that

$$\int_0^{+\infty} e^{-pt} e^{ts}\, dt = S_R^{-1}(p,s),$$

so we have

$$\int_0^{+\infty} e^{-pt} F(t)\mathbf{v}\, dt = \frac{1}{2\pi} \int_{\partial(W_c \cap \mathbb{C}_{\mathsf{i}})} S_R^{-1}(p,s)(\alpha - s)^{-2}\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 \mathbf{v}.$$

We point out that the function $s \mapsto S_R^{-1}(p,s)(\alpha - s)^{-2}$ is right slice hyperholomorphic for $s \notin [p] \cup \{\alpha\}$ and that the function $s \mapsto S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2\mathbf{v}$ is left slice hyperholomorphic on $\rho_S(T)$. Observe that the integrand is such that $(\alpha - s)^{-2}$ goes to zero with order $1/|s|^2$ as $s \to \infty$. By applying Cauchy's integral theorem, we can replace the path of integration by small negatively oriented circles of radius $\delta > 0$ around the singularities of the integrand in the plane $\mathbb{C}_{\mathsf{i}}$. These singularities are α, $p_{\mathsf{i}} = p_0 + \mathsf{i}p_1$ and $\overline{p}$ if $\mathsf{i} \neq \pm\mathsf{i}_p$. We obtain

$$\begin{aligned}
&\int_0^{+\infty} e^{-pt} F(t)\mathbf{v}\, dt = \\
= &- \frac{1}{2\pi} \int_{\partial(U_\delta(\alpha) \cap \mathbb{C}_{\mathsf{i}})} S_R^{-1}(p,s)(\alpha - s)^{-2}\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 \mathbf{v} \\
&- \frac{1}{2\pi} \int_{\partial(U_\delta(p_{\mathsf{i}}) \cap \mathbb{C}_{\mathsf{i}})} S_R^{-1}(p,s)(\alpha - s)^{-2}\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 \mathbf{v} \\
&- \frac{1}{2\pi} \int_{\partial(U_\delta(\overline{p_{\mathsf{i}}}) \cap \mathbb{C}_{\mathsf{i}})} S_R^{-1}(p,s)(\alpha - s)^{-2}\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 \mathbf{v}.
\end{aligned}$$

Observe that the integrand has a pole of order 2 at α and poles of order 1 at p_{i} and $\overline{p_{\mathsf{i}}}$ (except if $\mathsf{i} = \pm\mathsf{i}_p$). Applying Lemma 5.28 with $f(s) = S_R^{-1}(p,s)(\alpha - s)^{-2}$ and

$g(s) = S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2\mathbf{v}$ yields therefore

$$
\begin{aligned}
&\int_0^{+\infty} e^{-pt}F(t)\mathbf{v}\,dt = \\
&= -\operatorname{Res}_\alpha\left(S_R^{-1}(p,s)(\alpha-s)^{-2}\right)S_R^{-1}(\alpha,T)(\alpha\mathcal{I}-T)^2\mathbf{v}\\
&\quad -\operatorname{Res}_\alpha\left(S_R^{-1}(p,s)(s-\alpha)^{-1}\right)\left(\partial_S S_R^{-1}(\alpha,T)(\alpha\mathcal{I}-T)^2\mathbf{v}\right)\\
&\quad -\operatorname{Res}_{p_{\mathsf{i}}}\left(S_R^{-1}(p,s)(\alpha-s)^{-2}\right)S_R^{-1}(p_{\mathsf{i}},T)(\alpha\mathcal{I}-T)^2\mathbf{v}\\
&\quad -\operatorname{Res}_{\overline{p_{\mathsf{i}}}}\left(S_R^{-1}(p,s)(\alpha-s)^{-2}\right)S_R^{-1}(\overline{p_{\mathsf{i}}},T)(\alpha\mathcal{I}-T)^2\mathbf{v}.
\end{aligned}
$$

We calculate the residues of the function $f(s) = S_R^{-1}(p,s)(\alpha-s)^{-2}$. Since it has a pole of order two at α, we have

$$
\operatorname{Res}_\alpha(f_{\mathsf{i}}) = \lim_{\mathbb{C}_{\mathsf{i}}\ni s\to\alpha}\frac{\partial}{\partial s}f_{\mathsf{i}}(s)(s-\alpha)^2 = \lim_{\mathbb{C}_{\mathsf{i}}\ni s\to\alpha}\frac{\partial}{\partial s}S_R^{-1}(p,s) = (p-\alpha)^{-2},
$$

where the last identity holds because α is real, and

$$
\operatorname{Res}_\alpha(f_{\mathsf{i}}(s)(s-\alpha)) = \lim_{\mathbb{C}_{\mathsf{i}}\ni s\to\alpha} f_{\mathsf{i}}(s)(s-\alpha)^2 = S_R^{-1}(p,\alpha).
$$

The point $p_{\mathsf{i}} = p_0 + \mathsf{i}p_1$ is a pole of order 1. Thus, setting $s_{\mathsf{i}_p} = s_0 + \mathsf{i}_p s_1 \in \mathbb{C}_{\mathsf{i}_p}$ for $s = s_0 + \mathsf{i}s_1 \in \mathbb{C}_{\mathsf{i}}$, we deduce from Theorem 2.9 that

$$
\begin{aligned}
&\operatorname{Res}_{p_{\mathsf{i}}}(f_{\mathsf{i}}) = \lim_{\mathbb{C}_{\mathsf{i}}\ni s\to p_{\mathsf{i}}} f_{\mathsf{i}}(s)(s-p_{\mathsf{i}}) = \lim_{\mathbb{C}_{\mathsf{i}}\ni s\to p_{\mathsf{i}}} S_R^{-1}(p,s)(\alpha-s)^{-2}(s-p_{\mathsf{i}})\\
&= \lim_{\mathbb{C}_{\mathsf{i}}\ni s\to p_{\mathsf{i}}}\left[S_R^{-1}(p,s_{\mathsf{i}_p})(1-\mathsf{i}_p\mathsf{i})\frac{1}{2} + S_R^{-1}(p,\overline{s_{\mathsf{i}_p}})(1+\mathsf{i}_p\mathsf{i})\frac{1}{2}\right](s-p_{\mathsf{i}})(\alpha-s)^{-2}\\
&= \lim_{\mathbb{C}_{\mathsf{i}}\ni s\to p_{\mathsf{i}}}(p-s_{\mathsf{i}_p})^{-1}(1-\mathsf{i}_p\mathsf{i})(s-p_{\mathsf{i}})\frac{1}{2}(\alpha-p_{\mathsf{i}})^{-2}\\
&\quad + \lim_{\mathbb{C}_{\mathsf{i}}\ni s\to p_{\mathsf{i}}}(p-\overline{s_{\mathsf{i}_p}})^{-1}(1+\mathsf{i}_p\mathsf{i})(s-p_{\mathsf{i}})\frac{1}{2}(\alpha-p_{\mathsf{i}})^{-2}\\
&= \left[\lim_{\mathbb{C}_{\mathsf{i}}\ni s\to p_{\mathsf{i}}}(p-s_{\mathsf{i}_p})^{-1}(1-\mathsf{i}_p\mathsf{i})(s-p_{\mathsf{i}})\right]\frac{1}{2}(\alpha-p_{\mathsf{i}})^{-2}.
\end{aligned}
$$

We compute

$$
\begin{aligned}
&\lim_{\mathbb{C}_{\mathsf{i}}\ni s\to p_{\mathsf{i}}}(p-s_{\mathsf{i}_p})^{-1}(1-\mathsf{i}_p\mathsf{i})(s-p_{\mathsf{i}})\\
&= \lim_{\mathbb{C}_{\mathsf{i}}\ni s\to p_{\mathsf{i}}}(p-s_{\mathsf{i}_p})^{-1}(1-\mathsf{i}_p\mathsf{i})(s_0-p_0) + (p-s_{\mathsf{i}_p})^{-1}(1-\mathsf{i}_p\mathsf{i})\mathsf{i}(s_1-p_1)\\
&= \lim_{\mathbb{C}_{\mathsf{i}}\ni s\to p_{\mathsf{i}}}(p-s_{\mathsf{i}_p})^{-1}(s_0-p_0)(1-\mathsf{i}_p\mathsf{i}) + (p-s_{\mathsf{i}_p})^{-1}(s_1-p_1)(\mathsf{i}+\mathsf{i}_p)\\
&= \lim_{\mathbb{C}_{\mathsf{i}}\ni s\to p_{\mathsf{i}}}(p-s_{\mathsf{i}_p})^{-1}(s_0-p_0)(1-\mathsf{i}_p\mathsf{i}) + (p-s_{\mathsf{i}_p})^{-1}(s_1-p_1)\mathsf{i}_p(-\mathsf{i}_p\mathsf{i}+1)\\
&= \lim_{\mathbb{C}_{\mathsf{i}}\ni s\to p_{\mathsf{i}}}(p-s_{\mathsf{i}_p})^{-1}(s_0-p_0+\mathsf{i}_p(s_1-p_1))(1-\mathsf{i}_p\mathsf{i})\\
&= \lim_{\mathbb{C}_{\mathsf{i}}\ni s\to p_{\mathsf{i}}}(p-s_{\mathsf{i}_p})^{-1}(s_{\mathsf{i}_p}-p)(1-\mathsf{i}_p\mathsf{i}) = -(1-\mathsf{i}_p\mathsf{i})
\end{aligned}
$$

and finally obtain

$$\operatorname{Res}_{p_{\mathsf{i}}}(f_{\mathsf{i}}) = -\frac{1}{2}(1 - \mathsf{i}_p\mathsf{i})(\alpha - p_{\mathsf{i}})^{-2}.$$

Replacing i by $-\mathsf{i}$ in this formula yields

$$\operatorname{Res}_{\overline{p_{\mathsf{i}}}}(f_{\mathsf{i}}) = -\frac{1}{2}(1 + \mathsf{i}_p\mathsf{i})(\alpha - \overline{p_{\mathsf{i}}})^{-2}.$$

Note that these formulas also hold true if $\mathsf{i} = \pm\mathsf{i}_p$. In this case either $\operatorname{Res}_{p_{\mathsf{i}}}(f_{\mathsf{i}}) = -(\alpha - p_{\mathsf{i}})^{-2}$ and $\operatorname{Res}_{\overline{p_{\mathsf{i}}}}(f_{\mathsf{i}}) = 0$ because $\overline{p_{\mathsf{i}}}$ is a removable singularity of f_{i} or vice versa. Moreover,

$$S_R^{-1}(\alpha, T)(\alpha\mathcal{I} - T)^2\mathbf{v} = (\alpha\mathcal{I} - T)^{-1}(\alpha\mathcal{I} - T)^2\mathbf{v} = (\alpha\mathcal{I} - T)\mathbf{v}$$

and

$$\partial_S S_R^{-1}(\alpha, T)(\alpha\mathcal{I} - T)^2\mathbf{v} = -(\alpha\mathcal{I} - T)^{-2}(\alpha\mathcal{I} - T)^2\mathbf{v} = -\mathbf{v}$$

because α is real and so $S_R^{-1}(\alpha, T) = (\alpha\mathcal{I} - T)^{-1}$. Putting these pieces together, we get

$$\begin{aligned}
&\int_0^{+\infty} e^{-pt}F(t)\mathbf{v}\,dt = \\
=&-(p-\alpha)^{-2}S_R^{-1}(\alpha, T)(\alpha\mathcal{I} - T)^2\mathbf{v} \\
&+ S_R^{-1}(p, \alpha)S_R^{-2}(\alpha, T)(\alpha\mathcal{I} - T)^2\mathbf{v} \\
&+ \frac{1}{2}(1 - \mathsf{i}_p\mathsf{i})(\alpha - p_{\mathsf{i}})^{-2}S_R^{-1}(p_{\mathsf{i}}, T)(\alpha\mathcal{I} - T)^2\mathbf{v} \\
&+ \frac{1}{2}(1 + \mathsf{i}_p\mathsf{i})(\alpha - \overline{p_{\mathsf{i}}})^{-2}S_R^{-1}(\overline{p_{\mathsf{i}}}, T)(\alpha\mathcal{I} - T)^2\mathbf{v} = \\
=&-(p-\alpha)^{-2}(\alpha\mathcal{I} - T)\mathbf{v} + (p-\alpha)^{-1}\mathbf{v} \\
&+ (p-\alpha)^{-2}S_R^{-1}(p, T)(\alpha\mathcal{I} - T)^2\mathbf{v},
\end{aligned}$$

where that last identity follows from Theorem 2.9 because the mapping

$$p \mapsto (\alpha - p)^{-2}S_R^{-1}(p, T)(\alpha\mathcal{I} - T)^2\mathbf{v}$$

is left slice hyperholomorphic. We factor out $(p-\alpha)^{-2}$ on the left and obtain

$$\begin{aligned}
&\int_0^{+\infty} e^{-pt}F(t)\mathbf{v}\,dt = \\
=&(p-\alpha)^{-2}\left(-(\alpha\mathcal{I} - T)\mathbf{v} + (p-\alpha)\mathbf{v} + S_R^{-1}(p, T)(\alpha\mathcal{I} - T)^2\mathbf{v}\right) \\
=&(p-\alpha)^{-2}\left(p\mathbf{v} - 2\alpha\mathbf{v} + T\mathbf{v} + S_R^{-1}(p, T)(\alpha\mathcal{I} - T)^2\mathbf{v}\right).
\end{aligned}$$

Recall that we assumed that $\mathbf{v} \in \operatorname{dom}(T^2)$. Hence, $T\mathbf{v} \in \operatorname{dom}(T)$ and so we can apply the right S-resolvent equation (2.28) twice to obtain

$$\begin{aligned}
&S_R^{-1}(p, T)(\alpha\mathcal{I} - T)^2\mathbf{v} = S_R^{-1}(p, T)(T^2\mathbf{v} - 2\alpha T\mathbf{v} + \alpha^2\mathbf{v}) \\
&= pS_R^{-1}(p, T)T\mathbf{v} - T\mathbf{v} - 2\alpha pS_R^{-1}(p, T)\mathbf{v} + 2\alpha\mathbf{v} + \alpha^2 S_R^{-1}(p, T)\mathbf{v} \\
&= p^2S_R^{-1}(p, T)\mathbf{v} - p\mathbf{v} - T\mathbf{v} - 2\alpha pS_R^{-1}(p, T)\mathbf{v} + 2\alpha\mathbf{v} + \alpha^2 S_R^{-1}(p, T)\mathbf{v} \\
&= (p-\alpha)^2S_R^{-1}(p, T)\mathbf{v} - p\mathbf{v} + 2\alpha\mathbf{v} - T\mathbf{v}.
\end{aligned}$$

So finally

$$\int_0^{+\infty} e^{-pt} F(t)\mathbf{v}\, dt = (p-\alpha)^{-2}(p-\alpha)^2 S_R^{-1}(p,T)\mathbf{v} = S_R^{-1}(p,T)\mathbf{v}.$$

Hence,

$$\int_0^{+\infty} e^{-pt} F(t)\mathbf{v}\, dt = S_R^{-1}(p,T)\mathbf{v} = \int_0^{+\infty} e^{-pt}\,\mathcal{U}_T(t)\mathbf{v}\, dt,$$

for $\operatorname{Re}(p) > c$, which implies $F(t)\mathbf{v} = \mathcal{U}_T(t)\mathbf{v}$ for $\mathbf{v} \in D(T^2)$ and $t \geq 0$ as a consequence of the quaternionic version of the Hahn-Banach theorem [36, Theorem 4.10.1].

Applying the same reasoning to the semigroup $(\mathcal{U}(-t))_{t\geq 0}$, with infinitesimal generator $-T$, we see that

$$\begin{aligned}\mathcal{U}(-t)\mathbf{v} &= \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_{\mathbf{i}})} e^{ts}(\alpha-s)^{-2}\, ds_{\mathbf{i}}\, S_R^{-1}(s,-T)(\alpha\mathcal{I}+T)^2\mathbf{v}\\ &= \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_{\mathbf{i}})} e^{-ts}(\alpha+s)^{-2}\, ds_{\mathbf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I}+T)^2\mathbf{v},\end{aligned}$$

where the second equality follows by substitution of s by $-s$ because $S_R^{-1}(-s,-T) = -S_R^{-1}(s,T)$. Replacing α by $-\alpha$ and $-t$ by t, we finally find

$$\mathcal{U}(t)\mathbf{v} = \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_{\mathbf{i}})} e^{ts}(\alpha-s)^{-2}\, ds_{\mathbf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2\mathbf{v}$$

also for $t < 0$.

□

Proposition 5.30. *Let α and c be real numbers such that $\omega < c < |\alpha|$. If $f \in \mathbf{V}(T)$ is right slice hyperholomorphic on $cl(W_c)$, then for any $\mathbf{v} \in D(T^2)$ we have*

$$f(T)\mathbf{v} = \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_{\mathbf{i}})} f(s)(\alpha-s)^{-2}\, ds_{\mathbf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2\mathbf{v}. \tag{5.13}$$

Proof. We recall that f can be represented as

$$f(s) = \int_{\mathbb{R}} d\mu(t)\, e^{-st}$$

with $\mu \in \mathbf{S}(T)$. Using Proposition 5.29 we obtain

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_{\mathbf{i}})} f(s)(\alpha-s)^{-2}\, ds_{\mathbf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2\mathbf{v}\\ =&\frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_{\mathbf{i}})}\int_{\mathbb{R}} d\mu(t)\, e^{-st}(\alpha-s)^{-2}\, ds_{\mathbf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2\mathbf{v}\\ =&\int_{\mathbb{R}} d\mu(t)\left(\frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_{\mathbf{i}})} e^{-st}(\alpha-s)^{-2}\, ds_{\mathbf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2\mathbf{v}\right)\\ =&\int_{\mathbb{R}} d\mu(t)\,\mathcal{U}_T(-t)\mathbf{v} = f(T)\mathbf{v}.\end{aligned}$$

Note that Fubini's theorem allows us to exchange the order of integration as the S-resolvent $S_R^{-1}(s,T)$ is uniformly bounded on $\partial(W_c \cap \mathbb{C}_{\mathsf{i}})$ because of (5.12) and so there exists a constant $K > 0$ such that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_{\mathsf{i}})}\int_{\mathbb{R}} \left\| d\mu(t)\, e^{-st}(\alpha - s)^{-2}\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2\mathbf{v}\right\| \\
\leq &\frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_{\mathsf{i}})}\int_{\mathbb{R}} d|\mu|(t)\, e^{-\mathrm{Re}(s)t}\frac{1}{|\alpha - s|^{-2}}\|S_R^{-1}(s,T)\|\|(\alpha\mathcal{I} - T)^2\mathbf{v}\| ds \\
\leq &K\int_{\partial(W_c\cap\mathbb{C}_{\mathsf{i}})}\int_{\mathbb{R}} d|\mu|(t)\, e^{c|t|}\frac{1}{(1+|s|)^2}\, ds.
\end{aligned}$$

This integral is finite because, as $\mu \in \mathbf{S}(T)$, we have

$$\int_{\mathbb{R}} d|\mu|(t)\, e^{c|t|} < +\infty.$$

□

Theorem 5.31. *Let $f \in \mathbf{V}(T)$ and suppose that f is right slice hyperholomorphic at infinity. Then the operator $f(T)$ defined using the Laplace transform equals the operator $f[T]$ obtained from the S-functional calculus.*

Proof. Consider $\alpha \in \mathbb{R}$ with $c < |\alpha|$ and observe that the function $g(s) := f(s)(\alpha - s)^{-2}$ is right slice hyperholomorphic and, since f is right slice hyperholomorphic at infinity, tends to zero with order $1/|s|^2$ as s tends to infinity. The S-functional calculus for unbounded operators thus satisfies

$$g[T] = g(\infty)\mathcal{I} + \int_{\partial(W_c\cap\mathbb{C}_{\mathsf{i}})} g(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T).$$

By Theorem 4.19, we have for $\mathbf{v} \in V$ that

$$f[T](\alpha\mathcal{I} - T)^{-2}\mathbf{v} = \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_{\mathsf{i}})} f(s)(\alpha - s)^2\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)\mathbf{v}.$$

But by Proposition 5.30, it is

$$f(T)\mathbf{u} = \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_{\mathsf{i}})} f(s)(\alpha - s)^2\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2\mathbf{u},$$

for $\mathbf{u} \in D(T^2)$. Setting $\mathbf{v} = (\alpha\mathcal{I} - T)^2\mathbf{u}$, we conclude

$$f[T]\mathbf{v} = f(T)\mathbf{v}, \qquad \text{for } \mathbf{v} \in D(T^2).$$

Since $D(T^2)$ is dense in V and since the operators $f[T]$ and $f(T)$ are bounded we get $f[T] = f(T)$.

□

5.4 The Inversion of the Operator $f(T)$

We study the inversion of the operator $f(T)$ defined by the above functional calculus via approximation with polynomials P_n such that $\lim_{n\to+\infty} P_n(s)f(s) = 1$. In general the pointwise product $P_n(s)f(s)$ is not slice hyperholomorphic and therefore we must limit ourselves to intrinsic functions. The main goal of this section is to deduce sufficient conditions such that

$$\lim_{n\to+\infty} P_n(T)f(T)\mathbf{v} = \mathbf{v}, \quad \text{for every } \mathbf{v} \in V.$$

Lemma 5.32. *Let $T \in \mathcal{K}(V)$ such that $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. If $\mathrm{dom}(T)$ is closed, then $\mathrm{dom}(T^n)$ is dense in V for every $n \in \mathbb{N}$.*

Proof. If $\alpha \in \rho_S(T) \cap \mathbb{R}$, then $\mathrm{dom}(T^n) = \mathrm{dom}((\alpha\mathcal{I} - T)^n) = (\alpha\mathcal{I} - T)^{-n}V$. Therefore a continuous right linear functional $\mathbf{v}^* \in V^*$ on V vanishes on $\mathrm{dom}(T^n)$ if and only if the functional $\mathbf{v}^*(\alpha\mathcal{I} - T)^{-n}$, which is defined as $\langle \mathbf{v}^*(\alpha\mathcal{I} - T)^{-n}, \mathbf{v}\rangle := \langle \mathbf{v}^*, (\alpha\mathcal{I} - T)^{-n}\mathbf{v}\rangle$ for $\mathbf{v} \in V$, vanishes on the entire space V.

We show the statement by induction. It is obviously true for $n = 0$, so let us choose $n \in \mathbb{N}$ and let us assume that it holds for $n - 1$. By the above arguments, a functional $\mathbf{v}^* \in V^*$ vanishes on $\mathrm{dom}(T^n)$ if and only if the functional $\mathbf{v}^*(\alpha\mathcal{I}-T)^{-n} = \mathbf{v}^*(\alpha\mathcal{I}-T)^{-(n-1)}(\alpha\mathcal{I}-T)^{-1}$ vanishes on V, which is equivalent to $\mathbf{v}^*(\alpha\mathcal{I}-T)^{-(n-1)}$ vanishing on $\mathrm{dom}(T)$. Now observe that by assumption $\mathrm{dom}(T)$ is dense in V. Hence, Corollary 2.52 implies that a functional $\mathbf{u}^* \in V^*$ vanishes on $\mathrm{dom}(T)$ if and only if it vanishes on all of V. We conclude that $\mathbf{v}^* \in V^*$ vanishes on $\mathrm{dom}(T^n)$ if and only if $\mathbf{v}^*(\alpha\mathcal{I} - T)^{-(n-1)}$ vanishes on all of V^*, which is in turn equivalent to $\mathbf{v}^*$ vanishing on $\mathrm{dom}(T^{n-1})$. Since $\mathrm{dom}(T^{n-1})$ is dense in V by the induction hypothesis, Corollary 2.52 implies again that a functional $\mathbf{u}^* \in V^*$ vanishes on $\mathrm{dom}(T^{n-1})$ if and only if it vanishes on all of V. Therefore, we finally find that $\mathbf{v}^*$ vanishes on $\mathrm{dom}(T^n)$ if and only if it vanishes on all of V^* and a final application of Corollary 2.52 yields that $\mathrm{dom}(T^n)$ is dense in V.

□

Lemma 5.33. *Let P be an intrinsic polynomial of degree m and let f and P_nf both belong to $\mathbf{V}(T)$. Then $f(T)V \subseteq \mathrm{dom}(T^m)$ and*

$$P(T)f(T)\mathbf{v} = (Pf)(T)\mathbf{v}, \quad \textit{for all } \mathbf{v} \in V.$$

Proof. We first consider the case $\mathbf{v} \in \mathrm{dom}\,(T^{m+2})$. Let $\alpha, c \in \mathbb{R}$ with $w < c < |\alpha|$ and let $\mathbf{i} \in \mathbb{S}$. The function Pf is the product of two intrinsic functions and therefore intrinsic itself. By Proposition 5.30, Lemma 5.27 and Remark 5.25, we have

$$\begin{aligned}&(\alpha\mathcal{I} - T)^{-m}(Pf)(T)\mathbf{v} = \\ =&\frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_\mathbf{i})} (\alpha - s)^{-m}P(s)f(s)(\alpha - s)^{-2}\, ds_\mathbf{i}\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2\mathbf{v}.\end{aligned}$$

We write the polynomial P in the form $P(x) = \sum_{k=0}^m a_k(\alpha - x)^k$ with $a_k \in \mathbb{R}$. In view

ofProposition 5.30, Lemma 5.27 and Remark 5.25 we obtain again

$$
\begin{aligned}
&(\alpha\mathcal{I} - T)^{-m}(Pf)(T)\mathbf{v} \\
=&\sum_{k=0}^{m} a_k \frac{1}{2\pi} \int_{\partial(W_c\cap\mathbb{C}_\mathsf{i})} (\alpha - s)^{-m+k} f(s)(\alpha - s)^{-2}\, ds_\mathsf{i}\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2\mathbf{v} \\
=&\sum_{k=0}^{m} a_k (\alpha\mathcal{I} - T)^{-m+k} f(T)u = (\alpha\mathcal{I} - T)^{-m} \sum_{k=0}^{m} a_k (\alpha\mathcal{I} - T)^k f(T)\mathbf{v} \\
=&(\alpha\mathcal{I} - T)^{-m} P(T) f(T)\mathbf{v}.
\end{aligned}
$$

Consequently, $(Pf)(T)\mathbf{v} = P(T)f(T)\mathbf{v}$ for $\mathbf{v} \in \mathrm{dom}\,(T^{m+2})$.

Now let $\mathbf{v} \in V$ be arbitrary. Since $\mathrm{dom}(T^{m+2})$ is dense in V by Lemma 5.32, there exists a sequence $\mathbf{v}_n \in \mathrm{dom}(T^{m+2})$ with $\lim_{n\to+\infty} \mathbf{v}_n = \mathbf{v}$. Then $f(T)\mathbf{v}_n \to f(T)\mathbf{v}$ and $P(T)f(T)\mathbf{v}_n = (Pf)(T)\mathbf{v}_n \to (Pf)(T)\mathbf{v}$ as $n \to +\infty$. Since $P(T)$ is closed with domain $\mathrm{dom}(T^m)$, it follows that $f(T)\mathbf{v} \in \mathrm{dom}(T^m)$ and $P(T)f(T)\mathbf{v} = (Pf)(T)\mathbf{v}$.

□

Definition 5.34. A sequence of intrinsic polynomials $(P_n)_{n\in\mathbb{N}}$ is called an inverting sequence for an intrinsic function $f \in \mathbf{V}(T)$ if

(i) $P_n f \in \mathbf{V}(T)$,

(ii) $|P_n(s)f(s)| \leq M$, $n \in \mathbb{N}$ for some constant $M > 0$ and $\lim_{n\to+\infty} P_n(s)f(s) = 1$ in a strip $W_{\omega+\varepsilon} = \{s \in \mathbb{H} : -(\omega+\varepsilon) < \mathrm{Re}(s) \leq \omega + \varepsilon\}$,

(iii) $\|(P_n f)(T)\| \leq M$, $n \in \mathbb{N}$ for some constant $M > 0$.

Theorem 5.35. *If $(P_n)_{n\in\mathbb{N}}$ is an inverting sequence for an intrinsic function $f \in \mathbf{V}(T)$, then*

$$\lim_{n\to+\infty} P_n(T)f(T)\mathbf{v} = \mathbf{v} \qquad \forall \mathbf{v} \in V.$$

Proof. First consider $\mathbf{v} \in \mathrm{dom}(T^2)$ and choose $\alpha \in \mathbb{R}$ with $\omega < |\alpha|$. ThenProposition 5.30 and Lemma 5.33 imply

$$
\begin{aligned}
P_n(T)f(T)\mathbf{v} &= (P_n f)(T)\mathbf{v} \\
&= \frac{1}{2\pi} \int_{\partial(W_{c_n}\cap\mathbb{C}_\mathsf{i})} P_n(s)f(s)(\alpha - s)^{-2}\, ds_\mathsf{i}\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2\mathbf{v}
\end{aligned}
$$

for arbitrary $\mathsf{i} \in \mathbb{S}$ and $c_n \in \mathbb{R}$ with $w < c_n < |\alpha|$ such that $P_n f$ is right slice hyperholomorphic on $cl(W_{c_n})$. However, we have assumed that there exists a constant M such that $|P_n(s)f(s)| \leq M$ for any $n \in \mathbb{N}$ on a strip $-(\omega+\varepsilon) \leq \mathrm{Re}(s) \leq \omega + \varepsilon$. Moreover, because of (5.12), the right S-resolvent is uniformly bounded on any set $\{s \in \mathbb{C}_\mathsf{i} : |\mathrm{Re}(s)| > \omega + \varepsilon'\}$ with $\varepsilon' > 0$. Applying Cauchy's integral theorem we can therefore replace $\partial(W_{c_n}\cap\mathbb{C}_\mathsf{i})$ for any $n \in \mathbb{N}$ by $\partial(W_c\cap\mathbb{C}_\mathsf{i})$ where c is a real number with $\omega < c < \min\{|\alpha|, \omega+\varepsilon\}$. In particular, we can choose c independent of n. Lebesgue's dominated convergence theorem allows us to exchange limit and integration and we obtain

$$P_n(T)f(T)\mathbf{v} = \frac{1}{2\pi} \int_{\partial(W_c\cap\mathbb{C}_\mathsf{i})} (\alpha - s)^{-2}\, ds_I\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2\mathbf{v} = \mathbf{v}.$$

If $\mathbf{v} \in V$ does not belong to $\operatorname{dom}(T^2)$, then we can choose for any $\varepsilon > 0$ a vector $\mathbf{v}_\varepsilon \in \operatorname{dom}(T^2)$ with $\|\mathbf{v} - \mathbf{v}_\varepsilon\| < \varepsilon$. Since the mappings $(P_n f)(T)$ are uniformly bounded by a constant $M > 0$, we get

$$
\begin{aligned}
&(P_n f)(T)\mathbf{v} - \mathbf{v}\| \leq \\
&\leq \|(P_n f)(T)\mathbf{v} - (P_n f)(T)\mathbf{v}_\varepsilon\| + \|(P_n f)(T)\mathbf{v}_\varepsilon - \mathbf{v}_\varepsilon\| + \|\mathbf{v}_\varepsilon - \mathbf{v}\| \\
&\leq M\|\mathbf{v} - \mathbf{v}_\varepsilon\| + \|(P_n f)(T)\mathbf{v}_\varepsilon - \mathbf{v}_\varepsilon\| + \|\mathbf{v}_\varepsilon - \mathbf{v}\| \\
&\overset{n\to+\infty}{\longrightarrow} M\|\mathbf{v} - \mathbf{v}_\varepsilon\| + \|\mathbf{v}_\varepsilon - \mathbf{v}\| \leq (M+1)\varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, we deduce $\lim_{n\to+\infty} \|(P_n f)(T)\mathbf{v} - \mathbf{v}\| = \mathbf{0}$ even for arbitrary $\mathbf{v} \in V$.

□

Corollary 5.36. *Let V be reflexive and let P_n be an inverting sequence for an intrinsic function $f \in \mathbf{S}(T)$. A vector $\mathbf{v}$ belongs to the range of $f(T)$ if and only if it is in $\operatorname{dom}(P_n(T))$ for all $n \in \mathbb{N}$ and the sequence $(P_n(T)\mathbf{v})_{n\in\mathbb{N}}$ is bounded.*

Proof. If $\mathbf{v} \in \operatorname{ran} f(T)$ with $\mathbf{v} = f(T)\mathbf{u}$ then Lemma 5.33 implies $\mathbf{v} \in \operatorname{dom}(P_n(T))$ for all $n \in \mathbb{N}$. Theorem 5.35 states $\lim_{n\to+\infty} P_n(T)\mathbf{v} = \mathbf{u}$, which implies that the sequence $(P_n(T)\mathbf{v})_{n\in\mathbb{N}}$ is bounded.

To prove the converse statement consider $\mathbf{v} \in V$ such that $(P_n(T)\mathbf{v})_{n\in\mathbb{N}}$ is bounded. Since V is reflexive the set $\{P_n(T)\mathbf{v} : n \in \mathbb{N}\}$ is weakly sequentially compact. (The proof that a set E in a reflexive quaternionic Banach space V is weakly sequentially compact if and only if E is bounded can be completed just as in the classical case when V is a complex Banach space, see [38, Theorem II.28].) Hence, there exists a subsequence $(P_{n_k}(T)\mathbf{v})_{k\in\mathbb{N}}$ and a vector $\mathbf{u} \in V$ such that $\langle \mathbf{v}^*, P_{n_k}(T)\mathbf{v}\rangle \to \langle \mathbf{v}^*, \mathbf{u}\rangle$ as $k \to +\infty$ for any $\mathbf{v}^* \in V^*$. We show $\mathbf{v} = f(T)\mathbf{u}$.

For any functional $\mathbf{v}^* \in V^*$ the mapping $\mathbf{v}^* f(T)$, which is defined by

$$
\langle \mathbf{v}^* f(T), \mathbf{w}\rangle = \langle \mathbf{v}^*, f(T)\mathbf{w}\rangle,
$$

also belongs to V^*. Hence,

$$
\langle \mathbf{v}^*, f(T)P_{n_k}(T)\mathbf{v}\rangle = \langle \mathbf{v}^* f(T), P_{n_k}(T)\mathbf{v}\rangle \to \langle \mathbf{v}^* f(T), \mathbf{u}\rangle = \langle \mathbf{v}^*, f(T)\mathbf{u}\rangle.
$$

Recall that the measure μ with $f = \mathcal{L}(\mu)$ is real-valued since f is intrinsic. Therefore it commutes with the operator $P_{n_k}(T)$. Recall also that if $\mathbf{w} \in \operatorname{dom}(T^n)$ for some $n \in \mathbb{N}$ then $\mathcal{U}_T(t)\mathbf{w} \in \operatorname{dom}(T^n)$ for any $t \in \mathbb{R}$ and $\mathcal{U}_T(t)T^n\mathbf{w} = T^n\mathcal{U}_T(t)\mathbf{w}$. Thus, $P_{n_k}(T)\mathcal{U}_T(t)\mathbf{v} = \mathcal{U}_T(t)P_{n_k}(T)\mathbf{v}$ because P_{n_k} has real coefficients. Moreover, we can therefore exchange the integral with the unbounded operator $P_{n_k}(T)$ in the following computation

$$
\begin{aligned}
f(T)P_{n_k}(T)\mathbf{v} &= \int_{\mathbb{R}} d\mu(t)\,\mathcal{U}_T(-t)P_{n_k}(T)\mathbf{v} \\
&= P_{n_k}(T)\int_{\mathbb{R}} d\mu(t)\,\mathcal{U}_T(-t)\mathbf{v} = P_{n_k}(T)f(T)\mathbf{v}.
\end{aligned}
$$

Theorem 5.35 implies for any $\mathbf{v}^* \in V^*$

$$\langle \mathbf{v}^*, \mathbf{v} \rangle = \lim_{k\to\infty} \langle \mathbf{v}^*, P_{n_k}(T) f(T)\mathbf{v} \rangle = \lim_{k\to\infty} \langle \mathbf{v}^*, f(T) P_{n_k}(T)\mathbf{v} \rangle = \langle x^*, f(T)\mathbf{u} \rangle$$

and so $\mathbf{v} = f(T)\mathbf{u}$ follows from Corollary 2.52.

□

CHAPTER 6

The H^∞-Functional Calculus

The H^∞-functional calculus was originally introduced in [67] by McIntosh in 1986. His approach was generalized to quaternionic sectorial operators that are injective and have dense range in [8] by Alpay, Colombo, Qian, and Sabadini. The H^∞-functional calculus stands out among all holomorphic resp. slice hyperholomorphic functional calculuses because it allows to define functions of an operator such that $f(T)$ is unbounded.

We shall now define the H^∞-functional calculus for arbitrary sectorial operators following the strategy of [59], which will provide the techniques to introduce fractional powers of quaternionic linear operators in Section 7.2. This approach requires neither the injectivity of T nor that T has dense range. Several proofs do not need a lot of additional work and the strategies of the complex setting can be applied in a quite straightforward way. We shall therefore, in particular focus, on the proof of the chain rule and of the the spectral mapping theorem, where more severe technical difficulties appear. These results are part of the article [19].

6.1 The S-Functional Calculus for Sectorial Operators

In order to define the notion of a sectorial operator, we introduce the sector Σ_φ for $\varphi \in (0, \pi]$ as

$$\Sigma_\varphi := \{s \in \mathbb{H} : \arg(s) < \varphi\}.$$

Definition 6.1. Let $\omega \in [0, \pi)$. An operator $T \in \mathcal{K}(V)$ is called sectorial of angle ω if

(i) we have $\sigma_S(T) \subset cl(\Sigma_\omega)$ and

(ii) for every $\varphi \in (\omega, \pi)$ there exists a constant $C > 0$ such that for $s \notin cl(\Sigma_\varphi)$

$$\left\|S_L^{-1}(s,T)\right\| \leq \frac{C}{|s|} \quad \text{and} \quad \left\|S_R^{-1}(s,T)\right\| \leq \frac{C}{|s|}. \tag{6.1}$$

We denote the infimum of all these constants by C_φ resp. by $C_{\varphi,T}$ if we also want to stress its dependence on T.

We denote the set of all operators in $\mathcal{K}(V)$ that are sectorial of angle ω by $\mathrm{Sect}(\omega)$. Furthermore, if T is a sectorial operator, we call $\omega_T = \min\{\omega : T \in \mathrm{Sect}(\omega)\}$ the spectral angle of T.

Finally, a family of operators $(T_\ell)_{\ell\in\Lambda}$ is called uniformly sectorial of angle ω if $T_\ell \in \mathrm{Sect}(\omega)$ for all $\ell \in \Lambda$ and $\sup_{\ell\in\Lambda} C_{\varphi,T_\ell} < +\infty$ for all $\varphi \in (\omega, \pi)$.

Definition 6.2. We say that a slice hyperholomorphic function f has polynomial limit $c \in \mathbb{H}$ in Σ_φ at 0 if there exists $\alpha > 0$ such that $f(p) - c = O(|p|^\alpha)$ as $p \to 0$ in Σ_φ and that it has polynomial limit ∞ in Σ_φ at 0 if f^{-*_L} resp. f^{-*_R} has polynomial limit 0 at 0 in Σ_φ. (By (2.18) this is equivalent to $1/|f(p)| \in O(|p|^\alpha)$ for some $\alpha > 0$ as $p \to 0$ in Σ_φ.)

Similarly, we say that f has polynomial limit $c \in \mathbb{H}_\infty$ at ∞ in Σ_φ if $p \mapsto f(p^{-1})$ has polynomial limit c at 0. If a function has polynomial limit 0 at 0 or ∞, we say that it decays regularly at 0 resp. ∞.

Observe that the mapping $p \mapsto p^{-1}$ leaves Σ_φ invariant such that the above relation between polynomial limits at 0 and ∞ makes sense.

Definition 6.3. Let $\varphi \in (0, \pi]$. We define $\mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ as the set of all bounded functions in $\mathcal{SH}_L(\Sigma_\varphi)$ that decay regularly at 0 and ∞. Similarly, we define $\mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$ and $\mathcal{SH}^\infty_0(\Sigma_\varphi)$ as the set of all bounded functions in $\mathcal{SH}_R(\Sigma_\varphi)$ resp. $\mathcal{SH}(\Sigma_\varphi)$ that decay regularly at 0 and ∞.

The following Lemma is an immediate consequence of Corollary 2.7.

Lemma 6.4. *Let $\varphi \in (0, \pi]$.*

(i) If $f, g \in \mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ and $a \in \mathbb{H}$, then $fa + g \in \mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$. If in addition even $f \in \mathcal{SH}^\infty_0(\Sigma_\varphi)$, then also $fg \in \mathcal{SH}^\infty_0(\Sigma_\varphi)$.

(ii) If $f, g \in \mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$ and $a \in \mathbb{H}$, then $af + g \in \mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$. If in addition even $g \in \mathcal{SH}^\infty_0(\Sigma_\varphi)$, then also $fg \in \mathcal{SH}^\infty_0(\Sigma_\varphi)$.

(iii) The space $\mathcal{SH}^\infty_0(\Sigma_\varphi)$ is a real algebra.

Definition 6.5 (S-functional calculus for sectorial operators)**.** Let $T \in \mathrm{Sect}(\omega)$. For $f \in \mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ with $\omega < \varphi < \pi$, we choose φ' with $\omega < \varphi' < \varphi$ and $\mathsf{i} \in \mathbb{S}$ and define

$$f(T) := \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi'} \cap \mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s). \tag{6.2}$$

Similarly, for $f \in \mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$ with $\omega < \varphi < \pi$, we choose φ' with $\omega < \varphi' < \varphi$ and $\mathsf{i} \in \mathbb{S}$ and define

$$f(T) := \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi'} \cap \mathbb{C}_{\mathsf{i}})} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T). \tag{6.3}$$

Remark 6.6. Since T is sectorial of angle ω, the estimates in (6.1) assure the convergence of the above integrals. A standard argument using the slice hyperholomorphic version of Cauchy's integral theorem show that the integrals are independent of the choice of the angle φ' and standard slice hyperholomorphic techniques based on the representation formula as in the proof of Theorem 4.5 show that they are independent of the choice of the imaginary unit $\mathbf{i}$. For further details we also refer to the proof of Theorem 4.9 in [8]. Finally, computations as in the proof of Theorem 4.12 show that (6.2) and (6.3) yield the same operator if f is intrinsic.

If $T \in \mathrm{Sect}(\omega)$, then $f(T)$ in Definition 6.5 can be defined for any function that belongs to $\mathcal{SH}^{\infty}_{L,0}(\Sigma_\varphi)$ for some $\varphi \in (\omega, \pi]$. We thus introduce a notation for the space of all such functions.

Definition 6.7. For $\omega \in (0,\pi)$, we define $\mathcal{SH}^{\infty}_{L,0}[\Sigma_\omega] = \bigcup_{\omega<\varphi\leq\pi} \mathcal{SH}^{\infty}_{L,0}(\Sigma_\varphi)$ and similarly also $\mathcal{SH}^{\infty}_{R,0}[\Sigma_\omega] = \bigcup_{\omega<\varphi\leq\pi} \mathcal{SH}^{\infty}_{R,0}(\Sigma_\varphi)$ and $\mathcal{SH}^{\infty}_{0}[\Sigma_\omega] = \bigcup_{\omega<\varphi\leq\pi} \mathcal{SH}^{\infty}_{0}(\Sigma_\varphi)$.

The following properties of the S-functional calculus for sectorial operators can be proved by standard slice-hyperholomorphic techniques as, see. Theorem 4.19 or see also [8, Theorem 4.12].

Lemma 6.8. *If $T \in \mathrm{Sect}(\omega)$, then the following statements hold true.*

(i) If $f \in \mathcal{SH}^{\infty}_{L,0}[\Sigma_\omega]$ or $f \in \mathcal{SH}^{\infty}_{R,0}[\Sigma_\omega]$, then the operator $f(T)$ is bounded.

(ii) If $f, g \in \mathcal{SH}^{\infty}_{L,0}[\Sigma_\omega]$ and $a \in \mathbb{H}$, then $(fa+g)(T) = f(T)a + g(T)$. Similarly, if $f, g \in \mathcal{SH}^{\infty}_{R,0}[\Sigma_\omega]$ and $a \in \mathbb{H}$, then $(af+g)(T) = af(T) + g(T)$.

(iii) If $f \in \mathcal{SH}^{\infty}_{0}[\Sigma_\omega]$ and $g \in \mathcal{SH}^{\infty}_{L,0}[\Sigma_\omega]$, then $(fg)(T) = f(T)g(T)$. Similarly, if $f \in \mathcal{SH}^{\infty}_{R,0}[\Sigma_\omega]$ and $g \in \mathcal{SH}^{\infty}_{0}[\Sigma_\omega]$, then also $(fg)(T) = f(T)g(T)$.

We recall that a closed operator $A \in \mathcal{K}(V)$ is said to commute with $B \in \mathcal{B}(V)$, if $BA \subset AB$.

Lemma 6.9. *Let $T \in \mathrm{Sect}(\omega)$ and $A \in \mathcal{K}(V)$ commute with $\mathcal{Q}_s(T)^{-1}$ and $T\mathcal{Q}_s(T)^{-1}$ for any $s \in \rho_S(T)$. Then A commutes with $f(T)$ for any $f \in \mathcal{SH}^{\infty}_{0}[\Sigma_\omega]$. In particular $f(T)$ commutes with T for any $f \in \mathcal{SH}^{\infty}_{0}[\Sigma_\omega]$.*

Proof. If $f \in \mathcal{SH}^{\infty}_{0}[\Sigma_\omega]$, then for suitable $\varphi \in (\omega, \pi)$ and $\mathbf{i} \in \mathbb{S}$, we have

$$\begin{aligned}
f(T) =& \frac{1}{2\pi} \int_{\partial(\Sigma_\varphi \cap \mathbb{C}_{\mathbf{i}})} f(s)\, ds_{\mathbf{i}} S_R^{-1}(s,T) \\
=& \frac{1}{2\pi} \int_{-\infty}^{0} f\left(-te^{\mathbf{i}\varphi}\right)\left(-e^{\mathbf{i}\varphi}\right)(-\mathbf{i})\left(-te^{-\mathbf{i}\varphi} - T\right) \mathcal{Q}_{-te^{\mathbf{i}\varphi}}(T)^{-1}\, dt \\
&+ \frac{1}{2\pi} \int_{0}^{+\infty} f\left(te^{-\mathbf{i}\varphi}\right)\left(e^{-\mathbf{i}\varphi}\right)(-\mathbf{i})\left(te^{\mathbf{i}\varphi} - T\right) \mathcal{Q}_{te^{-\mathbf{i}\varphi}}(T)^{-1}\, dt.
\end{aligned}$$

After the changing $t \mapsto -t$ in the first integral, we find

$$\begin{aligned}
f(T) =& \frac{1}{2\pi}\int_0^{+\infty} f\left(te^{\mathbf{i}\varphi}\right)\left(e^{\mathbf{i}\varphi}\mathbf{i}\right)\left(te^{-\mathbf{i}\varphi}-T\right)\mathcal{Q}_{te^{\mathbf{i}\varphi}}(T)^{-1}\,dt\\
&+\frac{1}{2\pi}\int_0^{+\infty} f\left(te^{-\mathbf{i}\varphi}\right)\left(-e^{-\mathbf{i}\varphi}\mathbf{i}\right)\left(te^{\mathbf{i}\varphi}-T\right)\mathcal{Q}_{te^{-\mathbf{i}\varphi}}(T)^{-1}\,dt\\
=&\frac{1}{2\pi}\int_0^{+\infty} 2\mathrm{Re}\left[f\left(te^{\mathbf{i}\varphi}\right)\mathbf{i}t\right]\mathcal{Q}_{te^{\mathbf{i}\varphi}}(T)^{-1}\,dt\\
&-\frac{1}{2\pi}\int_0^{+\infty} 2\mathrm{Re}\left[f\left(te^{\mathbf{i}\varphi}\right)\mathbf{i}e^{\mathbf{i}\varphi}\right]T\mathcal{Q}_{te^{\mathbf{i}\varphi}}(T)^{-1}\,dt,
\end{aligned}$$

where the last identity holds because $f(\overline{s}) = \overline{f(s)}$ as f is intrinisic and $\mathcal{Q}_{te^{\mathbf{i}\varphi}}(T)^{-1} = \mathcal{Q}_{te^{-\mathbf{i}\varphi}}(T)^{-1}$.

If now $\mathbf{v} \in \mathrm{dom}(A)$, then the fact that A commutes with $\mathcal{Q}_s(T)^{-1}$ and $T\mathcal{Q}_s(T)^{-1}$ and any real scalar implies

$$\begin{aligned}
f(T)A\mathbf{v} =& \frac{1}{2\pi}\int_0^{+\infty} 2\mathrm{Re}\left[f\left(te^{\mathbf{i}\varphi}\right)\mathbf{i}t\right]\mathcal{Q}_{te^{\mathbf{i}\varphi}}(T)^{-1}A\mathbf{v}\,dt\\
&-\frac{1}{2\pi}\int_0^{+\infty} 2\mathrm{Re}\left[f\left(te^{\mathbf{i}\varphi}\right)\mathbf{i}e^{\mathbf{i}\varphi}\right]T\mathcal{Q}_{te^{\mathbf{i}\varphi}}(T)^{-1}A\mathbf{v}\,dt\\
=&A\frac{1}{2\pi}\int_0^{+\infty} 2\mathrm{Re}\left[f\left(te^{\mathbf{i}\varphi}\right)\mathbf{i}t\right]\mathcal{Q}_{te^{\mathbf{i}\varphi}}(T)^{-1}\mathbf{v}\,dt\\
&-A\frac{1}{2\pi}\int_0^{+\infty} 2\mathrm{Re}\left[f\left(te^{\mathbf{i}\varphi}\right)\mathbf{i}e^{\mathbf{i}\varphi}\right]T\mathcal{Q}_{te^{\mathbf{i}\varphi}}(T)^{-1}\mathbf{v}\,dt = Af(T)\mathbf{v}.
\end{aligned}$$

We thus find $\mathbf{v} \in \mathrm{dom}(Af(T))$ with $f(T)A\mathbf{v} = Af(T)\mathbf{v}$ and in turn $f(T)A \subset Af(T)$. □

Lemma 6.10. *Let $T \in \mathrm{Sect}(\omega)$. If $\lambda \in (-\infty, 0)$ and $f \in \mathcal{SH}^\infty_{L,0}[\Sigma_\omega]$, then $s \mapsto (\lambda - s)^{-1}f(s) \in \mathcal{SH}^\infty_{L,0}[\Sigma_\omega]$ and*

$$\left((\lambda - s)^{-1}f(s)\right)(T) = (\lambda - T)^{-1}f(T) = S_L^{-1}(\lambda, T)f(T).$$

Similarly, if $\lambda \in (-\infty, 0)$ and $f \in \mathcal{SH}^\infty_{R,0}[\Sigma_\omega]$, then $s \mapsto f(s)(\lambda - s)^{-1} \in \mathcal{SH}^\infty_{L,0}[\Sigma_\omega]$ and

$$\left(f(s)(\lambda - s)^{-1}\right)(T) = f(T)(\lambda - T)^{-1} = f(T)S_R^{-1}(\lambda, T).$$

Proof. Let $\lambda \in (-\infty, 0)$ and observe that, since λ is real, the S-resolvent equation (2.30) turns into

$$(\lambda - T)^{-1}S_L^{-1}(s, T) = S_R^{-1}(\lambda, T)S_L^{-1}(s, T) = \left(S_R^{-1}(\lambda, T) - S_L^{-1}(s, T)\right)(s - \lambda)^{-1}.$$

If now $f \in \mathcal{SH}^{\infty}_{L,0}[\Sigma_\omega]$, then for suitable $\varphi \in (\omega, \pi)$ and $\mathsf{i} \in \mathbb{S}$, we have

$$
\begin{aligned}
&(\lambda\mathcal{I} - T)^{-1} f(T) = \\
=&\frac{1}{2\pi}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_{\mathsf{i}})} (\lambda\mathcal{I} - T)^{-1} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s) \\
=&\frac{1}{2\pi}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_{\mathsf{i}})} \left(S_R^{-1}(\lambda,T) - S_L^{-1}(s,T)\right)(s-\lambda)^{-1}\, ds_{\mathsf{i}}\, f(s) \\
=&S_R^{-1}(\lambda,T)\frac{1}{2\pi}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\,(s-\lambda)^{-1} f(s) \\
&+ \frac{1}{2\pi}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\,(\lambda - s)^{-1} f(s) = \left((\lambda-s)^{-1} f(s)\right)(T),
\end{aligned}
$$

where the last equality holds because $\frac{1}{2\pi}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\,(s-\lambda)^{-1}f(s) = 0$ by Cauchy's integral theorem.

□

Similar to [59], we can extend the class of functions that are admissible to this functional calculus to the analogue of the extended Riesz class.

Definition 6.11. For $0 < \varphi < \pi$, we define

$$
\mathcal{E}_L(\Sigma_\varphi) = \left\{ f(p) = \tilde{f}(p) + (1+p)^{-1}a + b : \tilde{f} \in \mathcal{SH}^{\infty}_{L,0}(\Sigma_\varphi), a, b \in \mathbb{H} \right\}
$$

and similarly

$$
\mathcal{E}_R(\Sigma_\varphi) = \left\{ f(p) = \tilde{f}(p) + a(1+p)^{-1} + b : \tilde{f} \in \mathcal{SH}^{\infty}_{R,0}(\Sigma_\varphi), a, b \in \mathbb{H} \right\}.
$$

Finally, we define $\mathcal{E}(\Sigma_\varphi)$ as the set of all intrinsic functions in $\mathcal{E}_L(\Sigma_\varphi)$, i.e.

$$
\mathcal{E}(\Sigma_\varphi) = \left\{ f(p) = \tilde{f}(p) + (1+p)^{-1}a + b : \tilde{f} \in \mathcal{SH}^{\infty}_{0}(\Sigma_\varphi), a, b \in \mathbb{R} \right\}.
$$

Keeping in mind the product rule of slice-hyperholomorphic functions, simple calculations as in the classical case show the following two corollaries, cf. also [59, Lemma 2.2.3].

Corollary 6.12. *Let* $0 < \varphi < \pi$.

(i) The set $\mathcal{E}_L(\Sigma_\varphi)$ *is a quaternionic right vector space and it is closed under multiplication with functions in* $\mathcal{E}(\Sigma_\varphi)$ *from the left.*

(ii) The set $\mathcal{E}_R(\Sigma_\varphi)$ *is a quaternionic left vector space and it is closed under multiplication with functions in* $\mathcal{E}(\Sigma_\varphi)$ *from the right.*

(iii) The set $\mathcal{E}(\Sigma_\varphi)$ *is a real algebra.*

Corollary 6.13. *Let* $0 < \varphi < \pi$. *A function* $f \in \mathcal{SH}_L(\Sigma_\varphi)$ *(or* $f \in \mathcal{SH}_R(\Sigma_\varphi)$ *or* $f \in \mathcal{SH}(\Sigma_\varphi)$*) belongs to* $\mathcal{E}_L(\Sigma_\varphi)$ *(resp.* $\mathcal{E}_R(\Sigma_\varphi)$ *or* $\mathcal{E}(\Sigma_\varphi)$*) if and only if it is bounded and has finite polynomial limits at* 0 *and infinity.*

Definition 6.14. For $\omega \in (0,\pi)$, we denote $\mathcal{E}_L[\Sigma_\omega] = \bigcup_{\omega<\varphi<\pi} \mathcal{E}_L(\Sigma_\varphi)$ as well as $\mathcal{E}_R[\Sigma_\omega] = \bigcup_{\omega<\varphi<\pi} \mathcal{E}_R(\Sigma_\varphi)$ and $\mathcal{E}[\Sigma_\omega] = \bigcup_{\omega<\varphi<\pi} \mathcal{E}(\Sigma_\varphi)$.

Definition 6.15. Let $T \in \mathrm{Sect}(\omega)$. We define for any function $f \in \mathcal{E}_L[\Sigma_\omega]$ with $f(s) = \tilde{f}(s) + (1+s)^{-1}a + b$ the bounded operator

$$f(T) := \tilde{f}(T) + (1+T)^{-1}a + \mathcal{I}b$$

and for any function $f \in \mathcal{E}_R[\Sigma_\omega]$ with $f(s) = \tilde{f}(s) + a(1+s)^{-1} + b$ the bounded operator

$$f(T) := \tilde{f}(T) + a(1+T)^{-1} + b\mathcal{I},$$

where $\tilde{f}(T)$ is intended in the sense of Definition 6.5.

Lemma 6.16. *Let $T \in \mathrm{Sect}(\omega)$ and let $f \in \mathcal{E}_L[\Sigma_\omega]$. If f is left slice hyperholomorphic at 0 and decays regularly at infinity, then*

$$f(T) = \frac{1}{2\pi}\int_{\partial(U(r)\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s), \tag{6.4}$$

with $\mathsf{i} \in \mathbb{S}$ arbitrary and $U(r) = \Sigma_\varphi \cup B_r(0)$, where $\varphi \in (\omega,\pi)$ is such that $f \in \mathcal{E}_L(\Sigma_\varphi)$ and $r > 0$ is such that f is left slice hyperholomorphic on $cl(B_r(0))$. Moreover, if f is left slice hyperholomorphic both at 0 and at infinity, then

$$f(T) = f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U(r,R)\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s), \tag{6.5}$$

with $\mathsf{i} \in \mathbb{S}$ arbitrary and $U(r,R) = U(r) \cup (\mathbb{H} \setminus B_R(0))$, where $\varphi \in (\omega,\pi)$ is such that $f \in \mathcal{E}_L(\Sigma_\varphi)$, $r > 0$ is such that f is left slice hyperholomorphic on $cl(B_r(0))$ and $R > r$ is such that f is left slice-hyperholmorphic on $\mathbb{H} \setminus B_R(0)$.

Similarly, if $f \in \mathcal{E}_R[\Sigma_\omega]$, is right slice hyperholomorphic at 0 and decays regularly at infinity, then

$$f(T) = \frac{1}{2\pi}\int_{\partial(U(r)\cap\mathbb{C}_{\mathsf{i}})} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T),$$

with $\mathsf{i} \in \mathbb{S}$ arbitrary and $U(r)$ chosen as above. Moreover, if $f \in \mathcal{E}_R[\Sigma_\omega]$ is right slice hyperholomorphic both at 0 and at infinity, then

$$f(T) = f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U(r,R)\cap\mathbb{C}_{\mathsf{i}})} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T),$$

with $\mathsf{i} \in \mathbb{S}$ arbitrary and $U(r,R)$ is chosen as above.

Proof. Let us first assume that $f \in \mathcal{E}_L[\Sigma_\omega]$ is left slice hyperholomorphic at 0 and regularly decaying at infinity. Then $f(s) = \tilde{f}(s) + (1+s)^{-1}a$, where $\tilde{f} \in \mathcal{SH}^\infty_{L,0}(\Sigma_{\varphi'})$ with $\omega < \varphi < \varphi'$, and the function $\tilde{f}$ is moreover also left slice hyperholomorphic at 0. For $\mathsf{i} \in \mathbb{S}$ and $\omega < \varphi < \varphi'$, we therefore have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(U(r)\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s)\\
=&\frac{1}{2\pi}\int_{\partial(U(r)\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, \tilde{f}(s) + \frac{1}{2\pi}\int_{\partial(U(r)\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, (1+s)^{-1}a.
\end{aligned}$$

If $r' > r > 0$ is sufficiently small such that $\tilde{f}$ is left slice hyperholomorphic at $cl(B_{r'}(0))$, then Cauchy's integral theorem implies that the value of the first integral remains constant as r varies. Letting r tend to 0 we find that this integral equals $\tilde{f}(T)$ in the sense of Definition 6.5. For the second integral we find that

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial(U(r)\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,(1+s)^{-1}a\\ =&\lim_{R\to+\infty}\frac{1}{2\pi}\int_{\partial(U(r,R)\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,(1+s)^{-1}a=(1+T)^{-1}a,\end{aligned}$$

where the last identity can be deduced either from the compatibility of the S-functional calculus for closed operators with intrinsic polynomials in Lemma 4.21 and Theorem 4.19 or as in the complex case in [59, Lemma 2.3.2] from the residue theorem. Altogether, we obtain (6.4).

If $f \in \mathcal{E}_L[\omega]$ is left slice hyperholomorphic both at 0 and at infinity, then $f(s) = \tilde{f}(s) + (1+s)^{-1}a + b$ where $\tilde{f} \in \mathcal{SH}_{L,0}^{\infty}(\Sigma_{\varphi'})$ with $\omega < \varphi' < \pi$ is left slice hyperholomorphic both at 0 and infinity and $a, b \in \mathbb{H}$. We therefore have

$$\begin{aligned}&f(\infty)\mathcal{I}+\frac{1}{2\pi}\int_{\partial(U(r,R)\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,f(s)\\ =&\frac{1}{2\pi}\int_{\partial(U(r,R)\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,\tilde{f}(s)\\ &+f(\infty)\mathcal{I}+\frac{1}{2\pi}\int_{\partial(U(r,R)\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,\left((1+s)^{-1}a+b\right).\end{aligned}$$

As before, because of the left slice hyperholomorphicity of $\tilde{f}$ at 0 and infinity, Cauchy's integral theorem allows us to vary the values of r and R for sufficiently small r and sufficiently large R without changing the value of the first integral. Letting r tend to 0 and R tend to ∞, we find that this integral equals $\tilde{f}(T)$ in the sense of Definition 6.5. Since $f(\infty) = b$, the remaining terms however equal $(1+T)^{-1}a + \mathcal{I}b$, which can again either be deduced by a standard application of the the residue theorem and Cauchy's integral theorem as in [59, Corollary 2.3.5], or from the properties of the S-functional calculus for closed operators since the function $s \mapsto (1+s)^{-1}a+b$ is left slice hyperholomorphic on the spectrum of T and at infinity. Altogether, we find that also (6.5) holds true.

The right slice hyperholomorphic case finally follows by analogous arguments.

□

Corollary 6.17. *The S-functional calculus for closed operators and the S-functional calculus for sectorial operators are compatible.*

Proof. Let $T \in \mathrm{Sect}(\omega)$. If $f \in \mathcal{E}_L[\Sigma_\omega]$ is admissible for the S-functional calculus for closed operators, then it is left slice hyperholomorphic at infinity such that (6.5) holds true. The set $U(r, R)$ in this representation is however a slice Cauchy domain and therefore admissible as a domain of integration in the S-functional calculus for closed operators. Hence, both approaches yield the same operator.

□

Definition 6.15 is compatible with the algebraic structures of the underlying function classes.

Lemma 6.18. *If $T \in \mathrm{Sect}(\omega)$, then the following statements hold true.*

(i) If $f, g \in \mathcal{E}_L[\Sigma_\omega]$ and $a \in \mathbb{H}$, then $(fa+g)(T) = f(T)a + g(T)$. If $f, g \in \mathcal{E}_R[\Sigma_\omega]$ and $a \in \mathbb{H}$, then $(af+g)(T) = af(T) + g(T)$.

(ii) If $f \in \mathcal{E}[\Sigma_\omega]$ and $g \in \mathcal{E}_L[\Sigma_\omega]$, then $(fg)(T) = f(T)g(T)$. If $f \in \mathcal{E}_R[\Sigma_\omega]$ and $g \in \mathcal{E}[\Sigma_\omega]$, then also $(fg)(T) = f(T)g(T)$.

Proof. The compatibility with the respective vector space structure is trivial. In order to show the product rule, consider $f \in \mathcal{E}[\Sigma_\omega]$ and $g \in \mathcal{E}_L[\Sigma_\omega]$ with $f(s) = \tilde{f}(s) + (1+s)^{-1}a + b$ with $\tilde{f} \in \mathcal{SH}_0^\infty[\Sigma_\omega]$ and $a, b \in \mathbb{R}$ and $g(s) = \tilde{g}(s) + (1+s)^{-1}c + d$ with $\tilde{g} \in \mathcal{SH}_{L,0}^\infty[\Sigma_\omega]$ and $c, d \in \mathbb{H}$. By Lemma 6.8, Lemma 6.10 and the identity $(\mathcal{I}+T)^{-2} = (\mathcal{I}+T)^{-1} - T(\mathcal{I}+T)^{-2}$, we then have

$$\begin{aligned} f(T)g(T) =& \tilde{f}(T)\tilde{g}(T) + \tilde{f}(T)(\mathcal{I}+T)^{-1}c + \tilde{f}(T)d + (\mathcal{I}+T)^{-1}\tilde{g}(T)a \\ &+ (\mathcal{I}+T)^{-2}ac + (\mathcal{I}+T)^{-1}ad + \tilde{g}(T)b + (\mathcal{I}+T)^{-1}bc + bd\mathcal{I} \\ =& \left(\tilde{f}\tilde{g} + \tilde{f}(1+s)^{-1}c + \tilde{f}d + (1+s)^{-1}\tilde{g}a + \tilde{g}b\right)(T) \\ &- T(\mathcal{I}+T)^{-2}ac + (\mathcal{I}+T)^{-1}(ad+ac+bc) + bd\mathcal{I}. \end{aligned}$$

Since $-s(1+s)^{-2} \in \mathcal{E}_L[\Sigma_\omega]$ is left slice hyperholomorphic at zero and infinity, Corollary 6.17 and the properties of the S-functional calculus imply $(-s(1+s)^2)(T) = -T(\mathcal{I}+T)^{-2}$ such that

$$\begin{aligned} f(T)g(T) = &\left[\tilde{f}\tilde{g} + \tilde{f}(1+s)^{-1}c + \tilde{f}d + (1+s)^{-1}\tilde{g}a + \tilde{g}b - s(1+s)^{-2}ac\right](T) \\ &+ (\mathcal{I}+T)^{-1}(ad+ac+bc) + bd\mathcal{I} = (fg)(T) \end{aligned}$$

since

$$\begin{aligned} (fg)(s) =& \tilde{f}(s)\tilde{g}(s) + \tilde{f}(s)(1+s)^{-1}c + \tilde{f}(s)d + (1+s)^{-1}\tilde{g}(s)a \\ &+ \tilde{g}(s)b - s(1+s)^{-2}ac + (1+s)^{-1}(ad+ac+bc) + bd. \end{aligned}$$

The product rule in the right slice hyperholomorphic case can be shown with analogous arguments.

□

Lemma 6.19. *If $T \in \mathrm{Sect}(\omega)$, then the following statements hold true.*

(i) We have $(s(1+s)^{-1})(T) = T(\mathcal{I}+T)^{-1}$.

(ii) If A is closed and commutes with $\mathcal{Q}_s(T)^{-1}$ and $T\mathcal{Q}_s(T)^{-1}$ for all $s \in \rho_S(T)$, then A commutes with $f(T)$ for any $f \in \mathcal{E}[\Sigma_\omega]$. In particular T commutes with $f(T)$ for any $f \in \mathcal{E}[\Sigma_\omega]$.

(iii) If $\mathbf{v} \in \ker(T)$ and $f \in \mathcal{E}_R[\Sigma_\omega]$, then $f(A)\mathbf{v} = f(0)\mathbf{v}$. In particular this holds true if $f \in \mathcal{E}[\Sigma_\omega]$.

Proof. The first statement holds as

$$\left(s(1+s)^{-1}\right)(T) = (1-(1+s)^{-1})(T) = \mathcal{I} - (\mathcal{I}+T)^{-1} = T(\mathcal{I}+T)^{-1}$$

and the second one follows from Lemma 6.9. Finally, if $\mathbf{v} \in \ker(T)$, then

$$\mathcal{Q}_s(T)\mathbf{v} = \left(T^2 - 2s_0T + |s|^2\right)\mathbf{v} = |s|^2\mathbf{v}$$

and hence

$$S_R^{-1}(s,T)\mathbf{v} = (\overline{s}\mathcal{I} - T)\mathcal{Q}_s(T)^{-1}\mathbf{v} = \overline{s}|s|^{-2}\mathbf{v} = s^{-1}\mathbf{v}.$$

For $\tilde{f} \in \mathcal{SH}_{R,0}^\infty[\Sigma_\varphi]$, we hence have

$$\begin{aligned}\tilde{f}(T)\mathbf{v} =& \frac{1}{2\pi}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_{\mathsf{i}})} \tilde{f}(s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)\mathbf{v}\\ =& \frac{1}{2\pi}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_{\mathsf{i}})} \tilde{f}(s)\,ds_{\mathsf{i}}\,s^{-1}\mathbf{v} = \mathbf{0}\end{aligned}$$

by Cauchy's integral theorem such that for $f(s) = \tilde{f}(s) + a(1+s)^{-1} + b$ and $\mathbf{v} \in \ker(T)$

$$f(T)\mathbf{v} = \tilde{f}(T)\mathbf{v} + a(\mathcal{I}+T)^{-1}\mathbf{v} + b\mathcal{I}\mathbf{v} = a\mathbf{v} + b\mathbf{v} = f(0)\mathbf{v}.$$

□

Remark 6.20. If $f \in \mathcal{E}_L(\Sigma_\omega)$, then we cannot expect (iii) in Lemma 6.19 to hold true. In this case

$$\tilde{f}(T)\mathbf{v} = \frac{1}{2\pi}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,f(s)\mathbf{v},$$

but $\mathbf{v}$ and $ds_{\mathsf{i}}\,f(s)$ do not commute. So we cannot exploit the fact that $\mathbf{v} \in \ker(T)$ to simplify $S_L^{-1}(s,T)\mathbf{v} = s^{-1}\mathbf{v}$. Indeed, also this identity does not necessarily hold true as $S_L^{-1}(s,T) = \mathcal{Q}_s(T)^{-1}(\overline{s} - T)\mathbf{v} = \mathcal{Q}_s(T)^{-1}\overline{s}\mathbf{v}$ for $\mathbf{v} \in \ker(T)$. But the kernel of T is in general not a left linear subspace of T and hence we cannot assume $\overline{s}\mathbf{v} \in \ker(T)$. The simplification $\mathcal{Q}_s(T)^{-1}(s,T)\overline{s}\mathbf{v} = |s|^2\overline{s}\mathbf{v} = s^{-1}\mathbf{v}$ is not possible.

6.2 The H^∞-Functional Calculus

The H^∞-functional calculus for complex linear sectorial operators in [59] applies to meromorphic functions that are regularisable. Defining the orders of zeros and hence also of poles of slice-hyperholomorphic functions properly is a not our goal and goes beyond the scope of this thesis. We hence use the following simple definition, which is sufficient for our purposes.

Definition 6.21. Let $s \in \mathbb{H}$ and let f be left slice hyperholomorphic on an axially symmetric neighbourhood $[B_r(s)] \setminus \{s\}$ of s with $[B_r(s)] = \{x \in \mathbb{H} : \operatorname{dist}([s], x) < r\}$ and assume that f does not have a left slice hyperholomorphic continuation to all of $[B_r(s)]$. We say that f has a pole at the sphere $[s]$ if there exists $n \in \mathbb{N}$ such that $x \mapsto \mathcal{Q}_s(x)^n f(x)$ has a left slice hyperholomorphic continuation to $[B_r(s)]$ if $s \notin \mathbb{R}$ resp. if there exists $n \in \mathbb{N}$ such that $x \mapsto (x-s)^{-n}f(x)$ has a left slice hyperholomorphic continuation to $[B_r(s)]$ if $s \in \mathbb{R}$.

Remark 6.22. If $[s]$ is a pole of f and x_n is a sequence with $\lim_{n\to+\infty}\operatorname{dist}(x_n,[s]) = 0$, then not necessarily $\lim_{n\to+\infty}|f(x_n)| = +\infty$. One can see this easily if one restricts f to one of the complex planes $\mathbb{C}_{\mathsf{i}}$. If $\mathsf{i}, \mathsf{j} \in \mathbb{S}$ with $\mathsf{j} \perp \mathsf{i}$, then the function $f_{\mathsf{i}} := f|_{[B_r(s)]\cap\mathbb{C}_{\mathsf{i}}}$

a meromorphic function with values in the complex (left) vector space $\mathbb{H} \cong \mathbb{C}_{\mathsf{i}} + \mathbb{C}_{\mathsf{i}}\mathsf{j}$ over $\mathbb{C}_{\mathsf{i}}$. It must have a pole at $s_{\mathsf{i}} = s_0 + \mathsf{i}s_1$ or $\overline{s_{\mathsf{i}}} = s_0 - \mathsf{i}s_1$. Otherwise we could extend f_{i} to a holomorphic function on $B_r(s) \cap \mathbb{C}_{\mathsf{i}}$. The representation formula would allow us then to define a slice hyperholomorphic extension of f to $B_r(s)$. However, s_{i} and $\overline{s_{\mathsf{i}}}$ are not necessarily both poles of f_{i}. Consider for instance the function $f(x) = S_L^{-1}(s,x) = (x^2 - 2s_0x + |s|^2)^{-1}(\overline{s} - x)$, which is defined on $U = \mathbb{H} \setminus [s]$. If we choose $\mathsf{i} = \mathsf{i}_s$, then $f|_{U\cap\mathbb{C}_{\mathsf{i}}} = (s-x)^{-1}$, which does obviously not have a pole at $\overline{s}$. Hence, if $x_n \in \mathbb{C}_{\mathsf{i}}$ tends to $\overline{s}$, then $|f(x_n)|$ remains bounded.

However, the representation formula implies that there exists at most one complex plane $\mathbb{C}_{\mathsf{i}}$ such that only one of the points $\overline{s_{\mathsf{i}}}$ and s_{i} is a pole of f_{i}. Otherwise we could use it again to find a slice hyperholomorphic extension of f to $B_r(0)$. For intrinsic functions both points s_{i} and $\overline{s_{\mathsf{i}}}$ always need to be poles of f_{i} as in this case $f_{\mathsf{i}}(\overline{x}) = \overline{f_{\mathsf{i}}(x)}$.

In general we therefore do not have $\lim_{\operatorname{dist}(x,[s])\to 0} |f(x)| = +\infty$, but at least for the limit superior the equality

$$\limsup_{\operatorname{dist}(x,[s])\to 0} |f(x)| = +\infty$$

holds. If f is intrinsic, then even $\lim_{\operatorname{dist}(x,[s])\to 0} |f(x)| = +\infty$ holds true.

Definition 6.23. Let $U \subset \mathbb{H}$ be axially symmetric. A function f is said to be left meromorphic on U if there exist isolated spheres $[x_n] \subset U$ for $n \in \Theta$, where Θ is a subset of $\mathbb{N}$, such that $f|_{\tilde{U}} \in \mathcal{SH}_L(\tilde{U})$ with $\tilde{U} = U \setminus \bigcup_{n\in\Theta}[x_n]$ and such that each sphere $[p_n]$ is a pole of f. We denote the set of all such functions by $\mathcal{M}_L(U)$ and the set of all such functions that are intrinsic by $\mathcal{M}(U)$.

For $U = \Sigma_\omega$ with $0 < \omega < \pi$, we furthermore denote

$$\mathcal{M}_L[\Sigma_\omega]_T = \cup_{\omega<\varphi<\pi}\mathcal{M}_L(\Sigma_\varphi) \quad \text{and} \quad \mathcal{M}[\Sigma_\omega]_T = \cup_{\omega<\varphi<\pi}\mathcal{M}(\Sigma_\varphi).$$

Definition 6.24. Let $T \in \operatorname{Sect}(\omega)$. A left slice hyperholomorphic function f is said to be regularisable if $f \in \mathcal{M}_L(\Sigma_\varphi)$ for some $\omega < \varphi < \pi$ and there exists $e \in \mathcal{E}(\Sigma_\varphi)$ such that $e(T)$ defined in the sense of Definition 6.15 is injective and $ef \in \mathcal{E}_L(\Sigma_\varphi)$. In this case we call e a regulariser for f.

We denote the set of all such functions by $\mathcal{M}_L[\Sigma_\omega]_T$. Furthermore, we denote the subset of intrinsic functions in $\mathcal{M}_L[\Sigma_\omega]_T$ by $\mathcal{M}[\Sigma_\omega]_T$.

Lemma 6.25. *Let $T \in \operatorname{Sect}(\omega)$.*

(i) If $f, g \in \mathcal{M}_L[\Sigma_\omega]_T$ and $a \in \mathbb{H}$, then $fa + g \in \mathcal{M}_L[\Sigma_\omega]_T$. If furthermore $f \in \mathcal{M}[\Sigma_\omega]_T$, then also $fg \in \mathcal{M}_L[\Sigma_\omega]_T$.

(ii) The space $\mathcal{M}[\Sigma_\omega]_T$ is a real algebra.

Proof. If e_1 is a regulariser for f and e_2 is a regulariser for g, then $e = e_1e_2$ is a regulariser for $fa + g$ and also for fg if f is intrinsic. Hence the statement follows. □

Remark 6.26. If T is injective, then f does not need to have finite polynomial limit at 0 in Σ_ω. Indeed, the function $p \mapsto p(1+p)^{-2}$ or the function $p \mapsto p\left(1+p^2\right)^{-1}$ and their powers can then serve as regularisers that may compensate a singularity at 0. Choosing the latter as a specific regulariser yields exactly the approach chosen in [8], where the H^∞-functional calculus was first introduced for quaternionic linear operators.

Definition 6.27 (H^∞-functional calculus)**.** Let $T \in \mathrm{Sect}(\omega)$. For regularisable $f \in \mathcal{M}_L[\Sigma_\omega]_T$, we define
$$f(T) := e(T)^{-1}(ef)(T),$$
where $e(T)^{-1}$ is the closed inverse of $e(T)$ and $(ef)(T)$ is intended in the sense of Definition 6.15.

Remark 6.28. The operator $f(T)$ is independent of the regulariser e and hence well-defined. Indeed, if $\tilde{e}$ is a different regulariser, then e and $\tilde{e}$ commute because they both belong to $\mathcal{E}[\Sigma_\omega]$. Hence, $\tilde{e}(T)e(T) = (\tilde{e}e)(T) = (e\tilde{e})(T) = e(T)\tilde{e}(T)$ by Lemma 6.18. Inverting this equality yields $e(T)^{-1}\tilde{e}(T)^{-1} = \tilde{e}(T)^{-1}e(T)$ such that

$$\begin{aligned} &e(T)^{-1}(ef)(T) = e(T)^{-1}\tilde{e}(T)^{-1}\tilde{e}(T)(ef)(T) = e(T)^{-1}\tilde{e}(T)^{-1}(\tilde{e}ef)(T) \\ =&\tilde{e}(T)^{-1}e(T)^{-1}(e\tilde{e}f)(T) = \tilde{e}(T)^{-1}e(T)^{-1}e(T)(\tilde{e}f)(T) = \tilde{e}(T)^{-1}(\tilde{e}f)(T). \end{aligned}$$

If $f \in \mathcal{E}_L[\Sigma_\omega]$, then we can use the constant function 1 with $1(T) = \mathcal{I}$ as a regulariser in order to see that Definition 6.27 is consistent with Definition 6.15.

Remark 6.29. Since we are considering right linear operators, Definition 6.27 is not possible for right slice hyperholomorphic functions. Right slice hyperholomorphic functions maintain slice hyperholomorphicity under multiplication with intrinsic functions from the right. A regulariser of a function f would hence be a function e such that $e(T)$ is injective and $fe \in \mathcal{E}_R(\Sigma_\varphi)$. The operator $f(T)$ would then be defined as $(fe)(T)e(T)^{-1}$, but this operator is only defined on $\operatorname{ran} e(T)$ and can hence not be independent of the choice of e. If we consider left linear operators, the situation is of course vice versa, which is a common phenomenon in quaternionic operator theory, cf. for instance also Remark 6.20.

The next lemma shows that the function f needs to have a proper limit behaviour at 0 if T is not injective.

Lemma 6.30. *Let $T \in \mathrm{Sect}(\omega)$ and $f \in \mathcal{M}_L[\Sigma_\omega]_T$. If T is not injective, then f has finite polynomial limit $f(0) \in \mathbb{H}$ in Σ_ω at 0. If furthermore f is intrinsic, then $f(T)\mathbf{v} = f(0)\mathbf{v}$ for any $\mathbf{v} \in \ker(T)$.*

Proof. Assume that T is not injective and let e be a regulariser for f. Since $e(T)\mathbf{v} = e(0)\mathbf{v}$ for all $\mathbf{v} \in \ker(T)$ because of (iii) in Lemma 6.19, we have $e(0) \neq 0$ as $e(T)$ is injective. The limit $e(0)f(0) := \lim_{p\to 0} e(p)f(p)$ of $e(p)f(p)$ as p tends to 0 in Σ_ω exists and is finite because $ef \in \mathcal{E}_L(\Sigma_\omega)$. Hence the respective limit of $f(p) = e(p)^{-1}(e(p)f(p))$ exists too and is finite. Indeed, it is $f(0) = \lim_{p\to 0} f(p) = e(0)^{-1}(e(0)f(0))$. We find that

$$f(p) - f(0) = e(p)^{-1}\left[(e(p)f(p) - e(0)f(0)) - (e(p) - e(0))\, f(0)\right] = O(|p|^\alpha)$$

as p tends to 0 in Σ_ω because both ef and e have polynomial limit at 0. Hence, f has polynomial limit $f(0)$ at 0 in Σ_ω.

If f is intrinsic, then ef is intrinsic too and $e(0)$, $(ef)(0)$ and $f(0)$ are all real. Hence, for any $\mathbf{v} \in \ker(T)$, we have $(ef)(0)\mathbf{v} = \mathbf{v}(ef)(0) \in \ker(T)$. As $\ker(T)$ is a right linear subspace of V, we conclude that also $(ef)(0)\mathbf{v} \in \ker(T)$ and so (iii) in Lemma 6.19 yields

$$f(T)\mathbf{v} = e(T)^{-1}(ef)(T)\mathbf{v} = e(T)^{-1}(ef)(0)\mathbf{v} = e(0)^{-1}(ef)(0)\mathbf{v} = f(0)\mathbf{v}.$$

□

The proof of the following lemma is analogue to the one of the corresponding complex results, Proposition 1.2.2 and Corollary 1.2.4 in [59], and does not employ any specific quaternionic techniques. For the convenience of the reader, we nevertheless give the detailed proof as this result turns out to be crucial for what follows.

Lemma 6.31. *Let $T \in \operatorname{Sect}(\omega)$.*

(i) If $A \in \mathcal{B}(V)$ commutes with T, then A commutes with $f(T)$ for any function $f \in \mathcal{M}[\Sigma_\omega]_T$. Moreover, if $f \in \mathcal{M}[\Sigma_\omega]_T$ and $f(T) \in \mathcal{B}(V)$, then $f(T)$ commutes with T.

(ii) If $f, g \in \mathcal{M}_L[\Sigma_\omega]_T$, then

$$f(T) + g(T) \subset (f+g)(T).$$

If furthermore $f \in \mathcal{M}[\Sigma_\omega]_T$, then

$$f(T)g(T) \subset (fg)(T)$$

with $\operatorname{dom}(f(T)g(T)) = \operatorname{dom}((fg)(T)) \cap \operatorname{dom}(g(T))$. In particular, the above inclusion turns into an equality if $g(T) \in \mathcal{B}(V)$.

(iii) Let $f \in \mathcal{M}[\Sigma_\omega]_T$ and $g \in \mathcal{M}[\Sigma_\omega]$ be such that $fg \equiv 1$. Then $g \in \mathcal{M}[\Sigma_\omega]_T$ if and only if $f(T)$ is injective. In this case $f(T) = g(T)^{-1}$.

Proof. If $A \in \mathcal{B}(V)$ commutes with T, then it commutes with $\mathcal{Q}_s(T)^{-1}$ and $T\mathcal{Q}_s(T)^{-1}$ for any $s \in \rho_S(T)$. Hence it also commutes with $e(T)$ for any $e \in \mathcal{E}[\Sigma_\omega]$ by Lemma 6.19. If $f \in \mathcal{M}[\Sigma_\varphi]_T$ and e is a regulariser for f, we thus have

$$Af(T) = Ae(T)^{-1}(ef)(T) \subset e(T)^{-1}A(ef)(T) = e(T)^{-1}(ef)(T)A = f(T)A$$

such that the first assertion in (i) holds true. Because of (i) in Lemma 6.19, the function $(1+p)^{-1}$ regularizes the identity function $p \mapsto p$ and we have $p(T) = T$. Once we have shown (ii), we can hence obtain the second assertion in (i) from

$$f(T)T \subset (f(p)p)(T) = (pf(p))(T) = Tf(T).$$

In order to show (ii) assume that $f, g \in \mathcal{M}_L[\Sigma_\omega]_T$ and let e_1 be a regulariser for f and e_2 be a regulariser for g. Then $e = e_1e_2$ regularises both f and g and hence also $f+g$ such that

$$\begin{aligned} f(T) + g(T) &= e(T)^{-1}(ef)(T) + e(T)^{-1}(eg)(T) \subset e(T)^{-1}\left[(ef)(T) + (eg)(T)\right] \\ &= e(T)^{-1}(e(f+g))(T) = (f+g)(T). \end{aligned}$$

Applying this relation to the functions $f+g$ and $-g$, we find that $(f+g)(T) - g(T) \subset f(T)$ and so $(f+g)(T) = f(T) + g(T)$ if $g(T)$ is bounded.

If even $f \in \mathcal{E}[\Sigma_\omega]_T$, then f and e_2 are both intrinsic and hence commute. Thus $e(fg) = (e_1f)(e_2g) \in \mathcal{E}_L[\Sigma_\omega|_T$ by Corollary 6.12 and so e regularises fg. Because of

(ii) in Lemma 6.19 the operator $(e_1 f)(T)$ commutes with $e_2(T)$ and hence also with the inverse $e_2(T)^{-1}$. Because of (6.18), we thus find that

$$\begin{aligned} f(T)g(T) &= e_1(T)^{-1}(e_1 f)(T)e_2(T)^{-1}(e_2 g)(T) \\ &\subset e_1(T)^{-1}e_2(T)^{-1}(e_1 f)(T)(e_2 g)(T) \\ &= \left[e_2(T)e_1(T)\right]^{-1}(e_1 f e_2 g)(T) = e(T)^{-1}(efg)(T) = (fg)(T). \end{aligned}$$

In order to prove the statement about the domains, we consider

$$\mathbf{v} \in \operatorname{dom}((fg)(T)) \cap \operatorname{dom}(g(T)).$$

Then $\mathbf{w} := (e_2 g)(T)\mathbf{v} \in \operatorname{dom}(e_2(T)^{-1})$. Since $(e_1 f)(T)$ commutes with $e_2(T)^{-1}$, we conclude that also $(e_1 f)(T)\mathbf{w} \in \operatorname{dom}(e_2(T)^{-1})$. Since $\mathbf{v} \in \operatorname{dom}((fg)(T))$ and $(fg)(T)\mathbf{v} = e(T)^{-1}(efg)(T)\mathbf{v}$, we further have $(efg)(T)\mathbf{v} \in \operatorname{dom}(e(T)^{-1})$. As $e(T)^{-1} = e_1(T)^{-1}e_2(T)^{-1}$ this implies $e_2(T)^{-1}(efg)(T)\mathbf{v} \in \operatorname{dom}(e_1(T)^{-1})$. From the identity

$$\begin{aligned} (e_1 f)(T)g(T)\mathbf{v} &= (e_1 f)(T)e_2(T)^{-1}\mathbf{w} \\ =& e_2(T)^{-1}(e_1 f)(T)\mathbf{w} = e_2(T)^{-1}(efg)(T)\mathbf{v} \end{aligned}$$

we conclude that $(e_1 f)(T)g(T)\mathbf{v} \in \operatorname{dom}(e_1(T)^{-1})$. Thus, $g(T)\mathbf{v} \in \operatorname{dom}(f(T))$ and in turn $\mathbf{v} \in \operatorname{dom}(f(T)g(T))$. Therefore

$$\operatorname{dom}(f(T)g(T)) \supset \operatorname{dom}((fg)(T)) \cap \operatorname{dom}(g(T)).$$

The other inclusion is trivial such that altogether we find equality. If $g(T)$ is bounded, then $\operatorname{dom}(g(T)) = V$ and we find $\operatorname{dom}(f(T)g(T)) = \operatorname{dom}((fg)(T))$ such that both operators agree.

We show now the statement (iii) and assume that $f, g \in \mathcal{M}[\Sigma_\omega]$ with $fg \equiv 1$ and that f is regularisable. If g is regularisable too, then (iii) implies $g(T)f(T) \subset (gf)(T) = 1(T) = \mathcal{I}$ with $\operatorname{dom}(g(T)f(T)) = \operatorname{dom}(\mathcal{I}) \cap \operatorname{dom}(f(T)) = \operatorname{dom}(f(T))$. Hence $f(T)$ is injective and interchanging the role of f and g shows that $f(T)g(T) = \mathcal{I}$ on $\operatorname{dom}(g(T))$ such that actually $f(T) = g(T)^{-1}$. Conversely, if $f(T)$ is injective and e is a regulariser for f, then $(fe)g = e(fg) = e \in \mathcal{E}[\Sigma_\omega]_T$. Moreover $(fe)(T)$ is injective as $f(T)$ and $e(T)$ are both injective and $(fe)(T) = f(T)e(T)$ by (ii). Thus fe is a regulariser for g, i.e. $g \in \mathcal{M}[\Sigma_\omega]_T$.

□

Intrinsic polynomials of an operator T are defined as $P[T] = \sum_{k=0}^{n} T^k a_k$ with $\operatorname{dom}(P[T]) = \operatorname{dom}(T^n)$ for any polynomial $P(x) = \sum_{k=0}^{n} x^k a_k$. We use the squared brackets to indicate that the operator $P[T]$ is defined via this functional calculus and not via the H^∞-functional calculus. However, as the next lemma shows, both approaches are consistent.

Lemma 6.32. *The H^∞-functional calculus is compatible with intrinsic rational functions. More precisely, if $r(p) = P(p)Q(p)^{-1}$ is an intrinsic rational function with intrinsic polynomials P and Q such that the zeros of Q lie in $\rho_S(T)$, then $r \in \mathcal{M}[\Sigma_\omega]_T$ and the operator $r(T)$ is given by $r(T) = P[T]Q[T]^{-1}$.*

Proof. We first prove compatibility with intrinsic polynomials. For intrinsic polynomials of degree 1 this follows from the linearity of the H^∞-functional calculus and from (i) in Lemma 6.19, which shows that $(1+p)^{-1}$ regularises the identity function $p \mapsto p$ and that

$$p(T) = \left((1+p)^{-1}(T)\right)^{-1}(p(1+p)^{-1})(T) = (\mathcal{I}+T)T(\mathcal{I}+T)^{-1} = T.$$

Let us now generalise the statement by induction and let us assume that it holds for intrinsic polynomials of degree n. If P is a polynomial of degree $n+1$, let us write $P(x) = Q(x)x + a$ with $a \in \mathbb{R}$ and an intrinsic polynomial Q of degree n. The induction hypothesis implies that $Q \in \mathcal{M}[\Sigma_\omega]_T$, that $Q(T) = Q[T]$, and that $\operatorname{dom}(Q(T)) = \operatorname{dom}(T^n)$. Since $\mathcal{M}[\Sigma_\omega]_T$ is a real algebra, we find that P also belongs to $\mathcal{M}[\Sigma_\omega]_T$ and we deduce from (iii) in Lemma 6.31 that

$$P(T) \supset Q(T)T + a\mathcal{I} = Q[T]T + a\mathcal{I} = P[T]$$

with $\operatorname{dom}(P[T]) = \operatorname{dom}\left(T^{n+1}\right) = \operatorname{dom}(Q(T)T) = \operatorname{dom}(P(T)) \cap \operatorname{dom}(T)$. Hence, if we show that $\operatorname{dom}(T) \subset \operatorname{dom}(P(T))$, the induction is complete. In order to do this, we consider $\mathbf{v} \in \operatorname{dom}(P(T))$. Then $(\mathcal{I}+T)^{-1}\mathbf{v}$ does also belong to $\operatorname{dom}(P(T))$ because $(\mathcal{I}+T)^{-1}P(T) \subset P(T)(\mathcal{I}+T)^{-1}$ by (i) in Lemma 6.31. But obviously also $(\mathcal{I}+T)^{-1}\mathbf{v} \in \operatorname{dom}(T)$ and hence $(\mathcal{I}+T)^{-1}\mathbf{v} \in \operatorname{dom}(P(T)) \cap \operatorname{dom}(T) = \operatorname{dom}\left(T^{n+1}\right)$, which implies $\mathbf{v} \in \operatorname{dom}(T^n) \subset \operatorname{dom}(T)$. We conclude $\operatorname{dom}(T) \subset \operatorname{dom}(P(T))$.

Let us now turn to arbitrary intrinsic rational functions. If $s \in \rho_S(T)$ is not real, then $\mathcal{Q}_s(T)$ is injective because $\mathcal{Q}_s(T)^{-1} \in \mathcal{B}(V)$ and hence $\mathcal{Q}_s(p)^{-1} \in \mathcal{M}[\Sigma_\omega]_T$ by (iii) in Lemma 6.31. Similarly, if $s \in \rho_S(T)$ is real, then $x \mapsto (s-x)^{-1} \in \mathcal{M}[\Sigma_\omega]_T$ because $(s-x)(T) = (s\mathcal{I} - T)$ is injective as $(s\mathcal{I}-T)^{-1} = S_L^{-1}(s,T) \in \mathcal{B}(V)$. If now $r(x) = P(x)Q(x)^{-1}$ is an intrinsic rational function with poles in $\rho_S(T)$, then we can write $Q(x)$ as product of such factors, namely

$$Q(x) = \prod_{\ell=1}^{N}(\lambda_\ell - x)^{n_\ell}\prod_{\kappa=1}^{M}\mathcal{Q}_{s_\kappa}(x)^{m_\kappa},$$

where $\lambda_1,\ldots,\lambda_N \in \rho_S(T)$ are the real zeros of Q and $[s_1],\ldots,[s_M] \subset \rho_S(T)$ are the spherical zeros of Q and n_ℓ and m_κ are the orders of λ_ℓ resp $[s_\kappa]$. Since $\mathcal{M}[\Sigma_\omega]_T$ is a real algebra, we conclude that $Q \in \mathcal{M}[\Sigma_\omega]_T$ and because of (iii) we find $Q^{-1}(T) = Q(T)^{-1} = Q[T]^{-1}$. Moreover, (ii) in Lemma 6.31 implies

$$Q^{-1}(T) = \prod_{\ell=1}^{N}(\lambda_\ell\mathcal{I} - T)^{-n_\ell}\prod_{\kappa=1}^{M}\mathcal{Q}_{s_\kappa}(T)^{-m_\kappa} \in \mathcal{B}(V)$$

because each of the factors in this product is bounded. Finally, we deduce from the boundedness of $Q^{-1}(T)$ and (ii) in Lemma 6.31 that

$$r(T) = \left(PQ^{-1}\right)(T) = P(T)Q^{-1}(T) = P[T]Q[T]^{-1} = r[T].$$

□

6.3 The Composition Rule

Let us now turn our attention to the composition rule, which will occur at several occasions when we consider fractional powers of sectorial operators. As always in the

quaternionic setting, we can only expect such a rule to hold true if the inner function is intrinsic since the composition of two slice hyperholomorphic functions is only slice hyperholomorphic if the inner function is intrinsic.

Theorem 6.33. *Let $T \in \operatorname{Sect}(\omega)$ and $g \in \mathcal{M}[\Sigma_\omega]_T$ be such that $g(T) \in \operatorname{Sect}(\omega')$. Furthermore assume that for any $\varphi' \in (\omega', \pi)$ there exists some $\varphi \in (\omega, \pi)$ such that $g \in \mathcal{M}(\Sigma_\varphi)$ and $g(\Sigma_\varphi) \subset cl(\Sigma_{\varphi'})$. Then $f \circ g \in \mathcal{M}[\Sigma_\omega]_T$ for any $f \in \mathcal{M}_L[\Sigma_{\omega'}]_{g(T)}$ and*

$$(f \circ g)(T) = f(g(T)).$$

Proof. Let us first assume that $g \equiv c$ is constant. In this case $g(T) = c\mathcal{I}$. Since g is intrinsic, we have $\overline{c} = \overline{g(s)} = g(\overline{s}) = c$ and so $c \in \mathbb{R}$. Since g maps Σ_φ into $\overline{\Sigma_{\varphi'}}$ for suitable $\varphi \in (\omega, \pi)$ and $\varphi' \in (\omega', \pi)$, we further find $c \in \overline{\Sigma_{\varphi'}} \cap \mathbb{R} = [0, +\infty)$. If $c \neq 0$, then $(f \circ g)(p) \equiv f(c)$ and we deduce easily, for instance from Corollary 6.17, that $(f \circ g)(T) = f(c)\mathcal{I} = f(g(T))$. If on the other hand $c = 0$, then Lemma 6.30 implies that $f(0) := \lim_{p \to 0} f(p)$ as p tends to 0 in Σ_ω exists. Hence $f \circ g$ is well defined. It is the constant function $f \circ g \equiv f(0)$ and so $(f \circ g)(T) = f(0)\mathcal{I}$. If f is intrinsic, then Lemma 6.30 implies $f(g(T)) = f(0)\mathcal{I} = (f \circ g)(T)$. If f is not intrinsic, then $f = f_0 + \sum_{\ell=1}^{3} f_\ell e_\ell$ with intrinsic components f_ℓ. Since $\ker g(T) = \ker(0\mathcal{I}) = V$, for any vector $\mathbf{v}$ also the vectors $e_\ell \mathbf{v}$, $\ell = 1, 2, 3$ belong to $\ker g(T)$ and we conclude, again from Lemma 6.30, that

$$\begin{aligned} f(g(T))\mathbf{v} &= f_0(g(T))\mathbf{v} + \sum_{\ell=1}^{3} f_\ell(g(T)) e_\ell \mathbf{v} = f_0(0)\mathbf{v} + \sum_{\ell=1}^{3} f_\ell(0) e_\ell \mathbf{v} \\ &= \left(f_0(0) + \sum_{\ell=1}^{3} f_\ell(0) e_\ell \right) \mathbf{v} = f(0)\mathbf{v} = (f \circ g)(T)\mathbf{v}. \end{aligned}$$

In the following we shall thus assume that g is not constant.

Let φ' and φ be a couple of angles as in the assumptions of the theorem. Since g is intrinsic, $g|_{\mathbb{C}_{\mathsf{i}} \cap \Sigma_\varphi}$ is a non-constant holomorphic function on $\mathbb{C}_{\mathsf{i}} \cap \Sigma_\varphi$. Hence, it maps the open set $g(\Sigma_\varphi \cap \mathbb{C}_{\mathsf{i}})$ to an open set. The set $g(\Sigma_\varphi) = [g(\Sigma_\varphi \cap \mathbb{C}_{\mathsf{i}})]$ is therefore also open and so actually contained in $\Sigma_{\varphi'}$, not only in $cl(\Sigma_{\varphi'})$. In particular, we find that $f \circ g$ is defined and slice hyperholomorphic on Σ_φ.

We assume for the moment that $f \in \mathcal{E}_L(\Sigma_{\varphi'})$ with $\varphi' \in (\omega', \pi)$ and we choose $\varphi \in (\omega, \pi)$ such that $g \in \mathcal{M}(\Sigma_\varphi)$ and $g(\Sigma_\varphi) \subset \Sigma_{\varphi'}$. Since f is bounded on $\Sigma_{\varphi'}$, the function $f \circ g$ is a bounded function in $\mathcal{SH}_L(\Sigma_\varphi)$. If T is injective, then $e(x) = x(1+x)^{-2} \in \mathcal{E}(\Sigma_\varphi)$ such that $e(T)T(\mathcal{I}+T)^{-2}$ is injective. Moreover, the function $x \mapsto e(x)(f \circ g)(x)$ decays regularly at 0 and infinity in Σ_φ and hence belongs to $\mathcal{E}_L(\Sigma_\varphi)$. In other words, e is a regulariser for $f \circ g$ and hence $f \circ g \in \mathcal{M}_L[\Sigma_\omega]_T$. If T is not injective, then g has polynomial limit $g(0)$ at 0 by Lemma 6.30. Since g is intrinsic, it only takes real values on the real line and so $g(0) \in \mathbb{R}$. It furthermore maps Σ_φ to $\Sigma_{\varphi'}$ and so $g(0) \in cl(\Sigma_{\varphi'}) \cap \mathbb{R} = [0, +\infty)$. Therefore f has polynomial limit at $g(0)$: if $g(0) = 0$ this follows from Corollary 6.13, otherwise it follows from the Taylor expansion of f at $g(0) \in (0, \infty)$, cf. Theorem 2.14. As a consequence, $f \circ g$ has polynomial limit at 0. Therefore the function $x \mapsto (1+x)^{-1}(f \circ g)(x)$ belongs to $\mathcal{E}_L(\Sigma_\varphi)$. Since $(\mathcal{I}+T)^{-1}$ is injective because $-1 \in \rho_S(T)$, we find that $(1+x)^{-1}$ is a regulariser for $f \circ g$ and hence $f \circ g \in \mathcal{M}_L[\Sigma_\omega]_T$.

We have $f(x) = \tilde{f}(x) + (1+x)^{-1}a + b$ with $\tilde{f} \in \mathcal{SH}^\infty_{L,0}(\Sigma_{\varphi'})$ and $a, b \in \mathbb{H}$. Because of the additivity of the functional calculus, we can treat each of these pieces separately. The case that $f \equiv b$ has already been considered above. For $f(x) = (1+x)^{-1}a$, the identity $(f \circ g)(T) = (\mathcal{I} + g(T))^{-1}$ follows from (iii) in Lemma 6.31 because $p \mapsto 1 + g(p)$ and $p \mapsto (f \circ g)(p) = (1 + g(p))^{-1}$ do both belong to $\mathcal{M}_L[\Sigma_\omega]_T$. Hence let us assume that $f = \tilde{f} \in \mathcal{SH}^\infty_{L,0}(\Sigma_{\varphi'})$ with $\varphi' \in (\omega', \pi)$.

We choose $\theta' \in (\omega', \varphi')$ and $\mathsf{i} \in \mathbb{S}$ and set $\Gamma_p = \partial(\Sigma_{\theta'} \cap \mathbb{C}_{\mathsf{i}})$. We furthermore choose $\rho' \in (\omega', \theta')$ and by our assumptions on g, we can find $\varphi \in (\omega, \pi)$ such that $g(\Sigma_\varphi) \subset \Sigma_{\rho'} \subsetneq \Sigma_{\theta'}$. We choose $\theta \in (\omega, \varphi)$ and set $\Gamma_s = \partial(\Sigma_\theta \cap \mathbb{C}_{\mathsf{i}})$. The subscripts s and p in Γ_s and Γ_p refer to the corresponding variable of integration in the following computations.

For any $p \in \Gamma_p$, the functions $s \mapsto \mathcal{Q}_p(g(s))^{-1} = (g(s)^2 - 2p_0 g(s) + |p|^2)^{-1}$ and $s \mapsto S_L^{-1}(p, g(s))$ do then belong to $\mathcal{E}_L(\Sigma_\varphi)$ and $\left[\mathcal{Q}_p(g(\cdot))^{-1}\right](T) = \mathcal{Q}_p(g(T))^{-1}$ and $\left[S_L^{-1}(p, g(\cdot))\right](T) = S_L^{-1}(p, g(T))$. Indeed, by (ii) in Lemma 6.31, we have

$$\begin{aligned} \left[\mathcal{Q}_p(g(\cdot))\right](T) =& (g^2 - 2p_0 g + |p|^2)(T) \\ \supset& g(T)^2 - 2p_0 g(T) + |p|^2\mathcal{I} = \mathcal{Q}_p(g(T)). \end{aligned} \tag{6.6}$$

Taking the closed inverses of these operators, we deduce from (iii) in Lemma 6.31 that

$$\left[\mathcal{Q}_p(g(\cdot))^{-1}\right](T) = \left[\mathcal{Q}_p(g(\cdot))\right](T)^{-1} \supset \mathcal{Q}_p(g(T))^{-1}. \tag{6.7}$$

Since $p \in \rho_S(T)$, the $\mathcal{Q}_p(g(T))^{-1}$ is a bounded operator and hence already defined on all of V. Hence, the inclusion $\supset$ in (6.7) and (6.6) is actually an equality and we find $\left[\mathcal{Q}_p(g(\cdot))^{-1}\right](T) = \mathcal{Q}_p(g(T))^{-1}$. From (ii) we further conclude that also

$$\begin{aligned} \left[S_L^{-1}(p, g(\cdot))\right](T) =& \left[\mathcal{Q}_p(g(\cdot))^{-1}\overline{p} - g(\cdot)\mathcal{Q}_p(g(\cdot))^{-1}\right](T) \\ =& \mathcal{Q}_p(g(T))^{-1}\overline{p} - g(T)\mathcal{Q}_p(g(T))^{-1} = S_L^{-1}(p, g(T)). \end{aligned}$$

We hence have

$$f(g(T)) = \frac{1}{2\pi} \int_{\Gamma_p} S_L^{-1}(p, g(T))\, dp_{\mathsf{i}}\, f(p) = \frac{1}{2\pi} \int_{\Gamma_p} \left[S_L^{-1}(p, g(\cdot))\right](T)\, dp_{\mathsf{i}}\, f(p).$$

Let us first assume that T is injective. Since f and in turn also $f \circ g$ are bounded, we can use $e(x) = x(\mathcal{I} + x)^{-2}$ as a regulariser for $f \circ g$. As e decays regularly at 0 and infinity, also the functions $s \mapsto e(s)S_L^{-1}(p, g(s))$ decays regularly at 0 and infinity for any $p \in \Gamma_p$. Hence it belongs to $\mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ and so

$$\begin{aligned} f(g(T)) &= e(T)^{-1} e(T) f(g(T)) \\ =& e(T)^{-1} \frac{1}{2\pi} \int_{\Gamma_p} e(T) S_L^{-1}(p, g(T))\, dp_{\mathsf{i}}\, f(p) \\ =& e(T)^{-1} \frac{1}{2\pi} \int_{\Gamma_p} \left[e(\cdot) S_L^{-1}(p, g(\cdot))\right](T)\, dp_{\mathsf{i}} f(p) \\ =& e(T)^{-1} \frac{1}{(2\pi)^2} \int_{\Gamma_p} \left(\int_{\Gamma_s} S_L^{-1}(s, T)\, ds_{\mathsf{i}}\, s(1+s)^{-2} S_L^{-1}(p, g(s)) \right) dp_{\mathsf{i}} f(p). \end{aligned} \tag{6.8}$$

We can now apply Fubini's theorem in order to exchange the order of integration: estimating the resolvent using (6.1), we find that the integrand in the above integral is

bounded by the function

$$F(s,p) := C_\theta \left|pS_L^{-1}(p,g(s))\right| \frac{1}{|1+s|^2}\frac{|f(p)|}{|p|}. \tag{6.9}$$

Since p, s and $g(s)$ belong to the same complex plane as g is intrinsic, we have due to (2.18) that

$$\left|pS_L^{-1}(p,g(s))\right| \leq \max_{\tilde{s}\in[s]} \frac{|p|}{|p-g(\tilde{s})|} = \max\left\{\frac{1}{|1-p^{-1}g(s)|}, \frac{1}{|1-p^{-1}g(\overline{s})|}\right\}. \tag{6.10}$$

Since $g(\Gamma_s)\subset \Sigma_{\rho'}\cap\mathbb{C}_{\mathsf{i}} \subsetneq \Sigma_{\theta'}\cap\mathbb{C}_{\mathsf{i}}$ and $\Gamma_p = \partial(\Sigma_{\theta'}\cap\mathbb{C}_{\mathsf{i}})$, these expressions are bounded by a constant depending on θ' and ρ' but neither on p nor on s. Hence $\left|pS_L^{-1}(p,g(s))\right|$ is uniformly bounded on $\Gamma_s\times\Gamma_p$ and $F(s,p)$ is in turn integrable on $\Gamma_p\times\Gamma_s$ because f has polynomial limit 0 both at 0 and infinity.

After exchanging the order of integration in (6.8), we deduce from Cauchy's integral formula that

$$\begin{aligned} f(g(T)) =& e(T)^{-1}\frac{1}{(2\pi)^2}\int_{\Gamma_s} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\, s(1+s)^{-2}\left(\int_{\Gamma_p} S_L^{-1}(p,g(s))\,dp_{\mathsf{i}} f(p)\right)\\ =& e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma_s} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\, e(s)f(g(s))\\ =& e(T)^{-1}e(T)(f\circ g)(T) = (f\circ g)(T). \end{aligned}$$

Let us now consider the case that T is not injective. By Lemma 6.30, the function g has then finite polynomial limit $g(0)\in\mathbb{R}$ in Σ_φ and hence the function $\tilde{g}(p) = g(p)-g(0)\in\mathcal{M}(\Sigma_\varphi)_T$ has finite polynomial limit 0 in at 0. Let us choose a regulariser e for $\tilde{g}$ with polynomial limit 0 at infinity. (This is always possible: if $\tilde{e}$ is an arbitrary regulariser for $\tilde{g}$, we can choose for instance $e(s) = (1+s)^{-1}\tilde{e}(s)$.) We have then $e\tilde{g}\in\mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$. Since $g(0)$ is real, we have $S_L^{-1}(p,g(0)) = (p-g(0))^{-1}$. Moreover $g(s)$ and $\mathcal{Q}_p(g(s))^{-1}$ commute for any $s\in\Gamma_s$. For $p\notin cl(\Sigma_{\rho'})$ we find thus

$$\begin{aligned} & e(s)S_L^{-1}(p,g(s)) - e(s)S_L^{-1}(p,g(0))\\ =& e(s)\mathcal{Q}_p(g(s))^{-1}\left[(\overline{p}-g(s))(p-g(0)) - \mathcal{Q}_p(g(s))\right](p-g(0))^{-1}\\ =& e(s)\mathcal{Q}_p(g(s))^{-1}\Big[(\overline{p}-g(s))p - g(0)(\overline{p}-g(s))\\ & \qquad\qquad + g(s)(\overline{p}-g(s)) - (\overline{p}-g(s))p\Big](p-g(0))^{-1}\\ =& e(s)(g(s)-g(0))S_L^{-1}(p,g(s))(p-g(0))^{-1}\\ =& e(s)\tilde{g}(s)S_L^{-1}(p,g(s))S_L^{-1}(p,g(0)). \end{aligned} \tag{6.11}$$

Hence, e regularises also the function $s\mapsto S_L^{-1}(p,g(s)) - S_L^{-1}(p,g(0))$ and the function

$e(\cdot)\left(S_L^{-1}(p,g(\cdot)) - S_L^{-1}(p,g(0))\right)$ does even belong to $\mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$. We thus have

$$\begin{aligned}
&f(g(T)) = e(T)^{-1}e(T)f(g(T))\\
=&e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma_p} e(T)S_L^{-1}(p,g(T))\,dp_{\mathsf{i}}\,f(p)\\
=&e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma_p} \left[e(\cdot)S_L^{-1}(p,g(\cdot))\right](T)\,dp_{\mathsf{i}}\,f(p)\\
=&e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma_p} \left[e(\cdot)\tilde{g}(\cdot)S_L^{-1}(p,g(\cdot))S_L^{-1}(p,g(0))\right](T)\,dp_{\mathsf{i}}f(p)\\
&+e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma p} e(T)S_L^{-1}(p,g(0))\,dp_{\mathsf{i}}f(p).
\end{aligned}$$

For the second integral, Cauchy's integral formula yields

$$e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma p} e(T)S_L^{-1}(p,g(0))\,dp_{\mathsf{i}}f(p) = e(T)^{-1}e(T)f(g(0)) = f(g(0))\mathcal{I} \quad (6.12)$$

as f decays regularly at infinity in Σ_θ. For the first integral, we have

$$\begin{aligned}
&e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma_p} \left[e(\cdot)\tilde{g}(\cdot)S_L^{-1}(p,g(\cdot))S_L^{-1}(p,g(0))\right](T)\,dp_{\mathsf{i}}f(p)\\
=&e(T)^{-1}\frac{1}{(2\pi)^2}\int_{\Gamma_p}\left(\int_{\Gamma_s} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,e(s)\tilde{g}(s)S_L^{-1}(p,g(s))S_L^{-1}(p,g(0))\right)dp_{\mathsf{i}}f(p)\\
\overset{(A)}{=}&e(T)^{-1}\frac{1}{(2\pi)^2}\int_{\Gamma_s} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\left(\int_{\Gamma_p} e(s)\tilde{g}(s)S_L^{-1}(p,g(s))S_L^{-1}(p,g(0))\,dp_{\mathsf{i}}f(p)\right)\\
\overset{(B)}{=}&e(T)^{-1}\frac{1}{(2\pi)^2}\int_{\Gamma_s} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,e(s)\left(\int_{\Gamma_p} S_L^{-1}(p,g(s)) - S_L^{-1}(p,g(0))\,dp_{\mathsf{i}}f(p)\right)\\
\overset{(C)}{=}&e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma_s} S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,(e(s)f(g(s)) - f(g(0)))\\
=&e(T)^{-1}(e(T)f\circ g(T) - e(T)f(g(0))\mathcal{I}) = f\circ g(T) - f(g(0))\mathcal{I}, \qquad (6.13)
\end{aligned}$$

where the identity (A) follows from Fubini's theorem, the identity (B) follows from (6.11) and the identity (C) finally follows from Cauchy's integral formula. Altogether, we have

$$f(g(T)) = f\circ g(T) - f(g(0))\mathcal{I} + f(g(0))\mathcal{I} = f\circ g(T).$$

In order to justify the application of Fubini's theorem in (A), we observe that the integrand is bounded the function by

$$F(s,p) = C_\theta\left|pS_L^{-1}(p,g(s))\right|\frac{|e(s)\tilde{g}(s)|}{|s|}\frac{|f(p)|}{|p|}\frac{1}{|p-g(0)|},$$

where we used (6.1) in order to estimate the S-resolvent $S_L^{-1}(s,T)$.

If $g(0) \neq 0$ then $|p-g(0)|^{-1}$ is uniformly bounded in p. Just as before, also $\left|pS_L^{-1}(p,g(s))\right|$ is uniformly bounded on $\Gamma_s\times\Gamma_p$. Since $\tilde{g}$ decays regularly at 0, since

e decays regularly at infinity and since f decays regularly both at 0 and infinity, the function F is hence integrable on $\Gamma_s \times \Gamma_p$ and we can apply Fubini's theorem.

If on the other hand $g(0) = 0$, then $g = \tilde{g}$ and we can write

$$\begin{aligned} F(s,p) =& C_\theta \left| S_L^{-1}(p, g(s)) \right| \frac{|e(s)\tilde{g}(s)|}{|s|} \frac{|f(p)|}{|p|} \\ =& C_\theta \left| p^\alpha S_L^{-1}(p, g(s)) g(s)^{1-\alpha} \right| \frac{|e(s)g(s)^\alpha|}{|s|} \frac{|f(p)|}{|p|^{1+\alpha}}, \end{aligned} \tag{6.14}$$

with $\alpha \in (0,1)$ such that $|f(p)|/|p|^{1+\alpha}$ is integrable. This is possible because f decays regularly at 0. Just as in (6.10), we can estimate the first factor in (6.14) by

$$\begin{aligned} &\left| p^\alpha S_L^{-1}(p, g(s)) g(s)^{1-\alpha} \right| \leq \\ &\leq \max \left\{ \frac{|g(s)|^{1-\alpha}}{|p|^{1-\alpha}} \frac{1}{|1 - p^{-1} g(s)|}, \frac{|g(\overline{s})|^{1-\alpha}}{|p|^{1-\alpha}} \frac{1}{|1 - p^{-1} g(\overline{s})|} \right\}, \end{aligned}$$

where we used that $|g(s)| = \left| \overline{g(\overline{s})} \right| = |g(\overline{s})|$ because g is intrinsic. This expression is as before uniformly bounded on $\Gamma_s \times \Gamma_p$ because $g(\Gamma_s) \subset \Sigma_{\rho'} \cap \mathbb{C}_{\mathsf{i}}$. Hence, F is again integrable and it is actually possible to apply Fubini's theorem.

Altogether, we have so far shown that $f(g(T)) = (f \circ g)(T)$ for any $f \in \mathcal{E}_L[\Sigma_{\omega'}]$. Finally, we consider now a general function $f \in \mathcal{M}_L[\Sigma_{\omega'}]_{g(T)}$ that does not necessarily belong to $\mathcal{E}_L[\Sigma_{\omega'}]$. If e is a regulariser for f, then e and ef both belong to $\mathcal{E}_L[\omega']$. By what we have just shown, we hence have $e_g := e \circ g \in \mathcal{M}[\Sigma_\omega]_T$ and $(ef)_g := (ef) \circ g \in \mathcal{M}_L[\Sigma_\omega]_T$ with $e_g(T) = e(g(T))$ and $(ef)_g(T) = (ef)(g(T))$.

Let τ_1 and τ_2 be regularisers for e_g and $(ef)_g$. Then $\tau = \tau_1 \tau_2$ regularises both of them and hence

$$e_g(T) = \tau^{-1}(T)(\tau e_g)(T).$$

Since $e_g(T) = (e \circ g)(T) = e(g(T))$ is injective because e is a regulariser for f, the operator $(\tau e_g)(T)$ is injective too. Moreover, for $f_g := f \circ g$, we find $(\tau e_g) f_g = \tau(e_g f_g) = \tau(ef)_g \in \mathcal{E}_L[\omega]$ because τ was chosen to regularise both e_g and $(ef)_g$. Therefore τe_g is a regulariser for f_g and hence $f_g \in \mathcal{M}_L[\Sigma_\omega]_T$. Finally, we deduce from Lemma 6.31 that

$$\begin{aligned} f(g(T)) =& e(g(T))^{-1}(ef)(g(T)) = (e_g)(T)^{-1}((ef)_g)(T) \\ =& (e_g)(T)^{-1}\tau(T)^{-1}\tau(T)((ef)_g)(T) \\ =& (\tau e_g)(T)^{-1}((\tau e)_g f_g)(T) = f_g(T) = (f \circ g)(T). \end{aligned}$$

□

Corollary 6.34. *Let $T \in \mathrm{Sect}(\omega)$ be injective and let $f \in \mathcal{M}_L[\Sigma_\omega]$. Then we have $f \in \mathcal{M}_L[\Sigma_\omega]_T$ if and only if $p \mapsto f(p^{-1}) \in \mathcal{M}_L[\Sigma_\omega]_{T^{-1}}$ and in this case*

$$f(T) = f(p^{-1})(T^{-1}).$$

Proof. Since T is injective, the function p^{-1} belongs to $\mathcal{M}[\Sigma_\omega]_T$ and the statement follows from Theorem 6.33.

□

6.4 Extensions According to Spectral Conditions

As in the complex case, cf. [59, Section 2.5], one can extend the H^∞-functional calculus for sectorial operators to a larger class of functions if the operator satisfies additional spectral conditions. We shall mention the following three cases, which are relevant in the proof the spectral mapping theorem in Section 6.5. In order to explain them, we introduce the notation

$$\Sigma_{\varphi,r,R} = (\Sigma_\varphi \cap B_R(0)) \setminus B_r(0)$$

for $0 \leq r < R \leq \infty$. (We set $B_\infty(0) = \mathbb{H}$ for $R = \infty$.)

(i) If the operator $T \in \mathrm{Sect}(\omega)$ has a bounded inverse, then $B_\varepsilon(0) \subset \rho_S(T)$ for sufficiently small $\varepsilon > 0$. We can thus define the class

$$\mathcal{E}_L{}^\infty(\Sigma_\varphi) = \{f = \tilde{f} + a \in \mathcal{SH}_L(\Sigma_\varphi) : a \in \mathbb{H}, \tilde{f} \in \mathcal{SH}_L(\Sigma_\varphi) \text{ dec. reg. at } \infty\},$$

and $\mathcal{E}^\infty(\Sigma_\varphi)$ as the set of all intrinsic functions in $\mathcal{E}_L{}^\infty(\Sigma_\varphi)$, where dec. reg. is short for decays regularly. For any function $f \in \mathcal{E}_L{}^\infty(\Sigma_\varphi)$ with $\varphi > 0$, we can define $f(T)$ as

$$f(T) = \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi,r,\infty} \cap \mathbb{C}_\mathsf{i})} S_L^{-1}(s,T)\, ds_\mathsf{i}\, f(s) + a\mathcal{I},$$

with $0 < r < \varepsilon$ arbitrary. It follows as in Lemma 6.16 from Cauchy's integral theorem that this approach is consistent with the usual one if $f \in \mathcal{E}_L(\Sigma_\varphi)$, but the class of admissible functions $\mathcal{E}_L{}^\infty(\Sigma_\varphi)$ is now larger. We can further extend this functional calculus by calling a function $e \in \mathcal{E}_L{}^\infty(\Sigma_\varphi)$ a regulariser for a function $f \in \mathcal{M}_L(\Sigma_\varphi)$, if $e(T)$ is injective and $ef \in \mathcal{E}_L{}^\infty(\Sigma_\varphi)$. In this case, we define $f(T) = e(T)^{-1}(ef)(T)$.

Obviously all the results shown so far still hold for this extended functional calculus since the respective proofs can be carried out in this setting with marginal and obvious modifications. Only in the case of the composition rule we have to consider several cases, just as in the complex case, namely the combinations

a) T is sectorial and $g(T)$ is invertible and sectorial
b) T is invertible and sectorial and $g(T)$ is sectorial
c) T and $g(T)$ are both invertible and sectorial.

In the cases a) and c) one needs the additional assumption $0 \notin cl(g(\Sigma_\omega))$ on the function g.

(ii) If the operator $T \in \mathrm{Sect}(\omega)$ is bounded, then $\mathbb{H} \setminus B_\rho(0) \subset \rho_S(T)$ for sufficiently large $\rho > 0$. We can thus define the class

$$\mathcal{E}_L{}^0(\Sigma_\varphi) = \{f = \tilde{f} + a \in \mathcal{SH}_L(\Sigma_\varphi) : a \in \mathbb{H}, \tilde{f} \in \mathcal{SH}_L(\Sigma_\varphi) \text{ dec. reg. at } 0\}$$

and $\mathcal{E}^0(\Sigma_\varphi)$ as the set of all intrinsic functions in $\mathcal{E}_L{}^0(\Sigma_\varphi)$. For any function $f \in \mathcal{E}_L{}^0(\Sigma_\varphi)$ with $\varphi > 0$, we can define $f(T)$ as

$$f(T) = \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi,0,R} \cap \mathbb{C}_\mathsf{i})} S_L^{-1}(s,T)\, ds_\mathsf{i}\, f(s) + a\mathcal{I},$$

with $0 < \rho < R$ arbitrary. As before this approach is consistent with the usual one if $f \in \mathcal{E}_L(\Sigma_\varphi)$, but the class of admissible functions $\mathcal{E}_L{}^0(\Sigma_\varphi)$ is again larger than $\mathcal{E}_L(\Sigma_\varphi)$. We can further extend this functional calculus by calling $e \in \mathcal{E}_L{}^0(\Sigma_\varphi)$ a regulariser for $f \in \mathcal{M}_L(\Sigma_\varphi)$, if $e(T)$ is injective and $ef \in \mathcal{E}_L{}^0(\Sigma_\varphi)$ and define again $f(T) = e(T)^{-1}(ef)(T)$ for such f.

As before, all results shown so far hold also for this extended functional calculus because the respective proofs can be carried out in this setting with marginal and obvious modifications. For showing the composition rule, we have to consider again several cases and distinguish the following situations:

a) T is sectorial and $g(T)$ is bounded and sectorial

b) T is invertible and sectorial and $g(T)$ is bounded and sectorial

c) T and $g(T)$ are both bounded and sectorial

d) T is bounded and sectorial and $g(T)$ is sectorial

e) T is bounded and sectorial and $g(T)$ is invertible and sectorial.

In the cases a), b) and c) one needs the additional assumption $\infty \notin cl(g(\Sigma_\omega))^{\mathbb{H}_\infty}$ and in the case one needs the additional assumption e) $0 \notin cl(g(\Sigma_\omega))$ on the function g.

(iii) If finally $T \in \operatorname{Sect}(\omega)$ is bounded and has a bounded inverse, then we can set $\mathcal{E}_L{}^{0,\infty}(\Sigma_\varphi) = \mathcal{SH}_L(\Sigma_\varphi)$ and $\mathcal{E}^{0,\infty}(\Sigma_\varphi)$ and define for such functions

$$f(T) = \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi,r,R} \cap \mathbb{C}_{\mathsf{i}})} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, f(s)$$

for sufficiently small r and sufficiently large R. Choosing regularisers in $\mathcal{E}^{0,\infty}(\Sigma_\varphi)$ gives again an extension of the H^∞-functional calculus and of the two extended functional calculi presented in (i) and (ii). All the results presented so far still hold for this extended functional calculs, where the composition rule can be shown again under suitable conditions on the function g.

6.5 The Spectral Mapping Theorem

Finally, let us now show the spectral mapping theorem for the H^∞-functional calculus. We point out that a substantial technical difficulty appears here that does not occur in the classical situation: the proof of the spectral mapping theorem in the complex setting makes use of the fact that $f\left(T|_{V_\sigma}\right) = f(T)|_{V_\sigma}$ if σ is a spectral set and V_σ is the invariant subspace associated with σ, i.e. the range of the spectral projection $\chi_\sigma(T)$ defined in Lemma 2.66. However, subspaces that are invariant under right linear operators are in general only right linear subspaces, but not necessarily left linear subspaces. Hence, they are not two-sided Banach spaces and we cannot define $f\left(T|_{V_\sigma}\right)$ because the S-functional calculus as introduced in Definition 2.65, Definition 2.68 resp. Definition 4.6, and Definition 6.5 requires the operator to act on a two-sided Banach space. The S-resolvents can otherwise not be defined. Instead of using the properties of the S-functional calculus for $T|_{V_\sigma}$ we thus have to find a workaround and prove several steps directly, which is essentially done in Lemma 6.39.

We start with two technical lemmas that are necessary in order to show the spectral inclusion theorem.

Lemma 6.35. *Let $T \in \mathrm{Sect}(\omega)$ and let $s \in \mathbb{H}$. If $\mathcal{Q}_s(T)$ is injective and there exist $e \in \mathcal{M}[\Sigma_\omega]_T$ and $c \in \mathbb{H}$, $c \neq 0$ such that*

$$f(x) := \mathcal{Q}_c(e(x))\,\mathcal{Q}_s(x)^{-1} \in \mathcal{M}[\Sigma_\omega]_T$$

and such that $e(T)$ and $f(T)$ are bounded, then $e(T)\mathcal{Q}_s(T)^{-1} = \mathcal{Q}_s(T)^{-1}e(T)$.

Proof. By assumption the operator $\mathcal{Q}_s(T)$ is injective and hence (iii) in Lemma 6.31 implies that $\mathcal{Q}_s^{-1} \in \mathcal{M}[\omega]_T$. Since $e(T)$ is bounded, it commutes with T and so also with $\mathcal{Q}_s(T)^{-1}$. We thus have

$$e(T)\mathcal{Q}_s(T)^{-1} \subset \mathcal{Q}_s(T)^{-1}e(T).$$

In order to show that this relation is actually an equality, it is sufficient to show that $\mathbf{v} \in \mathrm{dom}\left(\mathcal{Q}_s(T)^{-1}\right)$ for any $\mathbf{v} \in V$ with $e(T)\mathbf{v} \in \mathrm{dom}(\mathcal{Q}_s(T)^{-1})$. This is indeed the case: if $e(T)\mathbf{v}$ belongs to $\mathrm{dom}(\mathcal{Q}_s(T)^{-1})$, then there exists $\mathbf{u} \in \mathrm{dom}(\mathcal{Q}_s(T))$ with $e(T)\mathbf{v} = \mathcal{Q}_s(T)\mathbf{u}$. Hence

$$\begin{aligned} \mathcal{Q}_c(e(T))\mathbf{v} &= e(T)^2\mathbf{v} - 2c_0e(T)\mathbf{v} + |c|^2\mathbf{v} \\ = e(T)\mathcal{Q}_s(T)\mathbf{u} - 2c_0\mathcal{Q}_s(T)\mathbf{u} + |c|^2\mathbf{v} &= \mathcal{Q}_s(T)(e(T)\mathbf{u} - 2c_0\mathbf{u}) + |c|^2\mathbf{v}, \end{aligned} \tag{6.15}$$

where the last identity follows again from (i) in Lemma 6.31 because $e(T)$ is bounded and commutes with T and in turn also with $\mathcal{Q}_s(T)$. Since $f(T) \in \mathcal{B}(V)$, we conclude on the other hand from (ii) of Lemma 6.31 that

$$\mathcal{Q}_c(e(T)) = \mathcal{Q}_s(T)\left[\mathcal{Q}_c(e(\cdot))\mathcal{Q}_s(\cdot)^{-1}\right](T) = \mathcal{Q}_s(T)f(T).$$

Due to (6.15), we then find

$$\begin{aligned} \mathbf{v} &= \frac{1}{|c|^2}\left(\mathcal{Q}_c(e(T))\mathbf{v} - \mathcal{Q}_s(T)(e(T)\mathbf{u} - 2c_0\mathbf{u})\right) \\ &= \mathcal{Q}_s(T)\frac{1}{|c|^2}\left(f(T)\mathbf{v} - e(T)\mathbf{u} + 2c_0\mathbf{u}\right). \end{aligned}$$

Hence, $\mathbf{v}$ belongs to $\mathrm{dom}(\mathcal{Q}_s(T)^{-1})$ and the statement follows. □

Lemma 6.36. *Let $T \in \mathrm{Sect}(\omega)$ and let $f \in \mathcal{M}_L[\Sigma_\omega]_T$. For any $s \in cl(\Sigma_\omega) \setminus \{0\}$ there exists a regulariser e for f with $e(s) \neq 0$.*

Proof. Let $\tilde{e}$ be an arbitrary regulariser of f such that $\tilde{e} \in \mathcal{E}[\Sigma_\omega]$, $\tilde{e}f \in \mathcal{E}_L[\Sigma_\omega]$ and $\tilde{e}(T)$ is injective. If $\tilde{e}(s) \neq 0$, then we can set $e = \tilde{e}$ and we are done. Otherwise recall that $[s]$ is a spherical zero of $\tilde{e}$ and that its order is a finite number $n \in \mathbb{N}$ since $e \not\equiv 0$ as $e(T)$ is injective. We define now $e(x) := \mathcal{Q}_s^{-n}(x)e(x)$ with $\mathcal{Q}_s(x) = x^2 - 2s_0x + |s|^2$. Then $e \in \mathcal{E}[\Sigma_\omega]$ with $e(s) \neq 0$ and $ef = \mathcal{Q}_s^{-n}\tilde{e}f \in \mathcal{E}_L[\Sigma_\omega]$. Furthermore, by (ii) in Lemma 6.31, we have $\tilde{e}(T) = \mathcal{Q}_s(T)e(T)$. Since $\tilde{e}(T)$ is injective, we deduce that also $e(T)$ is injective. Hence e is a regulariser for f with $e(s) \neq 0$. □

Lemma 6.37. *Let $T \in \mathrm{Sect}(\omega)$ and let $s \in cl(\Sigma_\omega)$ with $s \neq 0$. If $f(T)$ has a bounded inverse for some $f \in \mathcal{M}[\Sigma_\omega]_T$ with $f(s) = 0$, then $s \in \rho_S(T)$.*

Proof. Let f be as above and let us first show that $\mathcal{Q}_s(T) = T^2 - 2s_0T + |s|^2\mathcal{I}$ is injective and hence invertible as a closed operator. By Lemma 6.36, there exists a regulariser e for f with $c := e(s) \neq 0$. We have $ef \in \mathcal{E}[\Sigma_\omega]$ with $(ef)(s) = 0$. Since all zeros of intrinsic functions are spherical zeros, we find that also $h = ef\mathcal{Q}_s^{-1} = \mathcal{Q}_s^{-1}ef \in \mathcal{E}[\Sigma_\omega]$. The product rule (ii) in Lemma 6.31 implies therefore

$$h(T)\mathcal{Q}_s(T) \subset (h\mathcal{Q}_s)(T) = (ef)(T) = (fe)(T) = f(T)e(T),$$

where $ef = fe$ because both functions are intrinsic. Since $e(T)$ and $f(T)$ are both injective, we find that also $\mathcal{Q}_s(T)$ is injective. Moreover, e is also a regulariser for $\mathcal{Q}_s^{-1}f$.

Now observe that the function

$$g(x) := \mathcal{Q}_c(e(x))\mathcal{Q}_s(x)^{-1} = (e(x)^2 - 2c_0e(x) + |c|^2)(x^2 - 2s_0x + |s|^2)^{-1}$$

belongs to $\mathcal{E}[\Sigma_\omega]$. Indeed, by Corollary 6.12, the space $\mathcal{E}[\Sigma_\omega]$ is a real algebra such that $\mathcal{Q}_c(e(x)) = e(x)^2 - 2c_0e(x) + |c|^2$ belongs to it as e does. The function $\mathcal{Q}_c(e(x))$ however has a spherical zero at s because $e(s) = c$ such that $g(x) = \mathcal{Q}_c(e(x))\mathcal{Q}_s^{-1}(x)$ is bounded and hence belongs to $\mathcal{E}[\Sigma_\omega]$ by Corollary 6.13. In particular this implies that $g(T)$ is bounded.

We deduce from Lemma 6.35 that $e(T)\mathcal{Q}_s(T)^{-1} = \mathcal{Q}_s(T)^{-1}e(T)$ and inverting both sides of this equation yields $\mathcal{Q}_s(T)e(T)^{-1} = e(T)^{-1}\mathcal{Q}_s(T)$. The product rule in (ii) of Lemma 6.31, the boundedness of $h(T) = (e\mathcal{Q}_s^{-1}f)(T)$ and the fact that $\mathcal{Q}_s^{-1}$ and e commute because both are intrinsic functions imply

$$\begin{aligned} f(T) = e(T)^{-1}(ef)(T) &= e(T)^{-1}\left(\mathcal{Q}_s e\mathcal{Q}_s^{-1}f\right)(T) = e(T)^{-1}\mathcal{Q}_s(T)\left(e\mathcal{Q}_s^{-1}f\right)(T) \\ &= \mathcal{Q}_s(T)e(T)^{-1}(e\mathcal{Q}_s^{-1}f)(T) = \mathcal{Q}_s(T)(\mathcal{Q}_s^{-1}f)(T). \end{aligned}$$

Since $f(T)$ is surjective, we find that $\mathcal{Q}_s(T)$ is surjective too. Hence $\mathcal{Q}_s(T)^{-1}$ is an everywhere defined closed operator and thus bounded by the closed graph theorem. Consequently $s \in \rho_S(T)$.

□

Proposition 6.38. *If $T \in \mathrm{Sect}(\omega)$ and $f \in \mathcal{M}[\Sigma_\omega]_T$, then*

$$f(\sigma_S(T) \setminus \{0\}) \subset \sigma_{SX}(f(T)).$$

Proof. Let $s \in \sigma_S(T) \setminus \{0\}$ and set $c := f(s)$. If $c \neq \infty$, then Lemma 6.37 implies that $\mathcal{Q}_c(f(T))^2 = f(T)^2 - 2c_0f(T) + |c|^2\mathcal{I}$ does not have a bounded inverse because $g = f^2 - 2c_0f + |c|^2$ belongs to $\mathcal{M}[\Sigma_\omega]_T$ and satisfies $g(c) = 0$. Hence $c = f(s)$ belongs to $\sigma_S(f(T))$ for $s \in \sigma_S(T) \setminus \{0\}$ with $f(s) \neq \infty$.

If on the other hand $c = \infty$, then suppose that $c \notin \sigma_{SX}(f(T))$, i.e. that $f(T)$ is bounded. In this case there exists $p \in \mathbb{H}$ such that $\mathcal{Q}_p(f(T))$ has a bounded inverse. By (iii) in Lemma 6.31, this implies $g(x) = \mathcal{Q}_p(f(x))^{-1} \in \mathcal{M}[\Sigma_\omega]_T$. The operator $g(T)$ is invertible as $g(T)^{-1} = \mathcal{Q}_p(f(T))$ belongs to $\mathcal{B}(V)$ because $f(T)$ is bounded. Since moreover $g(s) = 0$ as $f(s) = \infty$, another application of Lemma 6.37 yields $s \in \rho_S(T)$. But this contradicts our assumption $s \in \sigma_S(T) \setminus \{0\}$. Hence, we must have $c \in \sigma_{SX}(f(T))$.

□

We have so far shown the spectral inclusion theorem for spectral values not equal to 0 or ∞. These two values need a special treatment. They also need additional assumptions on the function f for a spectral inclusion theorem to hold as we shall see in the following. (The assumptions presented here might however not be the most general ones that are possible, cf. [58].)

First we however have to show a technical lemma. We recall that if $\sigma \subset \sigma_{SX}(T)$ is a spectral set, then $E_\sigma := \chi_\sigma(T)$ is by Theorem 4.33 a projection that commutes with T, i.e. it is a projection onto a right-linear subspace of V that is invariant under T. If $\infty \notin \sigma$, then we can choose a bounded slice Cauchy domain $U_\sigma \subset \mathbb{H}$ such that $\sigma \subset U_\sigma$ and such that $(\sigma_S(T) \setminus \sigma) \cap U_\sigma = \emptyset$. The projection E_σ is then given by

$$E_\sigma = \frac{1}{2\pi} \int_{\partial(U_\sigma \cap \mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\, S_R^{-1}(s,T) = \frac{1}{2\pi} \int_{\partial(U_\sigma \cap \mathbb{C}_{\mathsf{i}})} S_L^{-1}(p,T)\, dp_{\mathsf{i}}. \tag{6.16}$$

If on the other hand $\infty \in \sigma$, then we can choose an unbounded slice Cauchy domain $U_\sigma \subset \mathbb{H}$ such that $\sigma \subset U_\sigma$ and such that $(\sigma_S(T) \setminus \sigma) \cap U_\sigma = \emptyset$. The projection E_σ is then given by

$$E_\sigma = \mathcal{I} + \frac{1}{2\pi} \int_{\partial(U_\sigma \cap \mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\, S_R^{-1}(s,T) = \mathcal{I} + \frac{1}{2\pi} \int_{\partial(U_\sigma \cap \mathbb{C}_{\mathsf{i}})} S_L^{-1}(p,T)\, dp_{\mathsf{i}}.$$

Lemma 6.39. *Let $T \in \operatorname{Sect}(\omega)$ be unbounded and assume that $\sigma_S(T)$ is bounded. Furthermore let E_∞ be the spectral projection onto the invariant subspace associated with ∞. If $f \in \mathcal{M}[\Sigma_\omega]_T$ has polynomial limit 0 at infinity, then $\{f(T)\}_\infty := f(T)E_\infty$ is a bounded operator that is given by the slice hyperholomorphic Cauchy integral*

$$\{f(T)\}_\infty = \int_{\partial(\Sigma_\varphi \setminus B_r(0)) \cap \mathbb{C}_{\mathsf{i}}} f(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T), \tag{6.17}$$

where $B_r(0)$ is the ball centered at 0 with $r > 0$ sufficiently large such that it contains $\sigma_S(T)$ and any singularity of f. Moreover, for two such functions, we have

$$\{f(T)\}_\infty \{g(T)\}_\infty = \{(fg)(T)\}_\infty. \tag{6.18}$$

Proof. Let us first assume that $f \in \mathcal{E}[\Sigma_\omega]$, i.e. $f \in \mathcal{E}(\Sigma_\varphi)$ with $\omega < \varphi < \pi$. Since f decays regularly at infinity, it is of the form $f(s) = \tilde{f}(s) + a(1+s)^{-1}$ with $a \in \mathbb{R}$ and $f \in \mathcal{SH}_0^\infty(\Sigma_\varphi)$. The operator $\tilde{f}(T)$ is given by the slice hyperholomorphic Cauchy integral

$$\tilde{f}(T) = \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi'} \cap \mathbb{C}_{\mathsf{i}})} \tilde{f}(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \tag{6.19}$$

with $\mathsf{i} \in \mathbb{S}$ and $\varphi' \in (\omega, \varphi)$. Let now $r_1 < r_2$ be such that $\sigma_S(T) \subset B_r(0)$. Cauchy's integral theorem allows us to replace the path of integration in (6.19) by the union of $\Gamma_{s,1} = \partial(\Sigma_{\varphi'} \cap B_{r_1}(0)) \cap \mathbb{C}_{\mathsf{i}}$ and $\Gamma_{s,2} = \partial(\Sigma_{\varphi'} \setminus B_{r_2}(0)) \cap \mathbb{C}_{\mathsf{i}}$ such that

$$\tilde{f}(T) = \frac{1}{2\pi} \int_{\Gamma_{s,1}} \tilde{f}(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) + \frac{1}{2\pi} \int_{\Gamma_{s,2}} \tilde{f}(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T). \tag{6.20}$$

Let us choose $R \in (r_1, r_2)$. Since $\sigma_{SX}(T) = \sigma_S(T) \cup \{\infty\}$, we have $E_\infty = \mathcal{I} - E_{\sigma_S(T)}$ and the spectral projection $E_{\sigma_S(T)}$ is given by the slice hyperholomorphic

Cauchy integral (6.16) along $\Gamma_p = \partial(B_R(0) \cap \mathbb{C}_{\mathsf{i}})$. The subscripts s and p in $\Gamma_{s,1}$, $\Gamma_{s,2}$ and Γ_p are chosen in order to indicate the corresponding variable of integration in the following computation.

If we write the operators $\tilde{f}(T)$ and $E_{\sigma_S(T)}$ in terms of the slice hyperholomorphic Cauchy integrals defined above, we find that

$$\begin{aligned}\tilde{f}(T)E_{\sigma_s(T)} =& \frac{1}{2\pi}\int_{\Gamma_{s,1}} \tilde{f}(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)\frac{1}{2\pi}\int_{\Gamma_p} S_L^{-1}(p,T)\, dp_{\mathsf{i}} \\ &+ \frac{1}{2\pi}\int_{\Gamma_{s,2}} \tilde{f}(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)\frac{1}{2\pi}\int_{\Gamma_p} S_L^{-1}(p,T)\, dp_{\mathsf{i}}.\end{aligned} \tag{6.21}$$

If we apply the S-resolvent equation (2.30) in the first integral, which we denote by Ψ_1 for neatness, we find

$$\begin{aligned}\Psi_1 =& \frac{1}{(2\pi)^2}\int_{\Gamma_{s,1}} \tilde{f}(s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)\int_{\Gamma_p} p\left(p^2 - 2s_0p + |s|^2\right)^{-1}\, dp_{\mathsf{i}} \\ &- \frac{1}{(2\pi)^2}\int_{\Gamma_{s,1}} \tilde{f}(s)\, ds_{\mathsf{i}}\, \overline{s}S_R^{-1}(s,T)\int_{\Gamma_p} \left(p^2 - 2s_0p + |s|^2\right)^{-1}\, dp_{\mathsf{i}} \\ &- \frac{1}{(2\pi)^2}\int_{\Gamma_p}\Bigg(\int_{\Gamma_{s,1}} \tilde{f}(s)\, ds_{\mathsf{i}}\, (S_L^{-1}(p,T)p \\ &\qquad\qquad - \overline{s}S_L^{-1}(p,T))\left(p^2 - 2s_0p + |s|^2\right)^{-1}\Bigg)\, dp_{\mathsf{i}}.\end{aligned} \tag{6.22}$$

For $s \in \Gamma_s$, the the functions $p \mapsto \left(p^2 - 2s_0p + |s|^2\right)^{-1}$ and $p \mapsto p\left(p^2 - 2s_0p + |s|^2\right)^{-1}$ are rational functions on $\mathbb{C}_{\mathsf{i}}$ that have two singularities, namely $s = s_0 + \mathsf{i}s_1$ and $\overline{s} = s_0 - \mathsf{i}s_1$. Since we chose $r_1 < R$, these singularities lie inside of $B_R(0)$ for any $s \in \Gamma_s$. As $\Gamma_p = \partial(B_R(0) \cap \mathbb{C}_{\mathsf{i}})$, the residue theorem yields

$$\frac{1}{2\pi}\int_{\Gamma_p} p\left(p^2 - 2s_0p + |s|^2\right)^{-1} dp_{\mathsf{i}} = \lim_{\mathbb{C}_{\mathsf{i}}\ni p\to s} p(p-\overline{s})^{-1} + \lim_{\mathbb{C}_{\mathsf{i}}\ni p\to \overline{s}} p(p-s)^{-1} = 1$$

and

$$\frac{1}{2\pi}\int_{\Gamma_p} \left(p^2 - 2s_0p + |s|^2\right)^{-1} dp_{\mathsf{i}} = \lim_{\mathbb{C}_{\mathsf{i}}\ni p\to s} (p-\overline{s})^{-1} + \lim_{\mathbb{C}_{\mathsf{i}}\ni p\to \overline{s}} (p-\overline{s})^{-1} = 0,$$

where $\lim_{\mathbb{C}_{\mathsf{i}}\ni p\to s} \tilde{f}(p)$ denotes the limit of $\tilde{f}(p)$ as p tends to s in $\mathbb{C}_{\mathsf{i}}$. If we apply the identity (2.31) with $B = S_L^{-1}(p,T)$ in the third integral in (6.22), it turns into

$$\begin{aligned}&\frac{1}{(2\pi)^2}\int_{\Gamma_p}\left(\int_{\Gamma_{s,1}} \tilde{f}(s)\, ds_{\mathsf{i}}\, \left(s^2 - 2p_0s + |p|^2\right)^{-1} sS_L^{-1}(p,T)\right) dp_{\mathsf{i}} \\ &- \frac{1}{(2\pi)^2}\int_{\Gamma_p}\left(\int_{\Gamma_{s,1}} \tilde{f}(s)\, ds_{\mathsf{i}}\, \left(s^2 - 2p_0s + |p|^2\right)^{-1} S_L^{-1}(p,T)\overline{p}\right) dp_{\mathsf{i}} = 0.\end{aligned}$$

The last identity follows from Cauchy's integral theorem because $\tilde{f}(s)$ is right slice hyperholomorphic and the functions $s \mapsto (s^2 - 2p_0s + |p|^2)^{-1}S_L^{-1}(p,T)$ and $s \mapsto$

$s(s^2-2p_0s+|p|^2)^{-1}S_L^{-1}(p,T)$ are left slice hyperholomorphic on $\Sigma_{\varphi'}\cap B_{r_1}(0)$ for any $p\in\Gamma_p$ as we chose $R>r_1$. Hence, we find

$$\Psi_1=\frac{1}{2\pi}\int_{\Gamma_{s,1}}\tilde{f}(s)\,ds_{\mathsf{i}}\,S_R^{-1}(p,T).$$

The second integral in (6.21), which we denote by Ψ_2 for neatness, turns after an application of the S-resolvent equation (2.30) into

$$\begin{aligned}\Psi_2=&\frac{1}{(2\pi)^2}\int_{\Gamma_{s,2}}\tilde{f}(s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)\int_{\Gamma_p}p\left(p^2-2s_0p+|s|^2\right)^{-1}\,dp_{\mathsf{i}}\\&-\frac{1}{(2\pi)^2}\int_{\Gamma_{s,2}}\tilde{f}(s)\,ds_{\mathsf{i}}\,\overline{s}S_R^{-1}(s,T)\int_{\Gamma_p}\left(p^2-2s_0p+|s|^2\right)^{-1}\,dp_{\mathsf{i}}\\&-\frac{1}{(2\pi)^2}\int_{\Gamma_{s,2}}\left(\int_{\Gamma_p}\tilde{f}(s)\,ds_{\mathsf{i}}\,(S_L^{-1}(p,T)p-\right.\\&\qquad\left.-\overline{s}S_L^{-1}(p,T))\left(p^2-2s_0p+|s|^2\right)^{-1}\right)dp_{\mathsf{i}}.\end{aligned}\tag{6.23}$$

Since we chose $R<r_2$ the singularities of $p\mapsto\left(p^2-2s_0p+|s|^2\right)^{-1}$ and $p\mapsto p\left(p^2-2s_0p+|s|^2\right)^{-1}$ lie outside of $cl(B_R(0))$ for any $s\in\Gamma_{s,2}$. Hence, these functions are right slice hyperholomorphic on $cl(B_R(0))$ and so Cauchy's integral theorem implies that the first two integrals in (6.23) vanish. Since $\tilde{f}$ decays regularly at infinity, since (6.1) holds true and since Γ_p is a path of finite length, we can apply Fubini's theorem and exchange the order of integration in the third integral of (6.23). After applying the identity (2.31), we find

$$\begin{aligned}\Psi_2=\frac{1}{(2\pi)^2}\int_{\Gamma_p}&\left(\int_{\Gamma_{s,2}}\tilde{f}(s)ds_{\mathsf{i}}\,\left(s^2-2p_0s+|p|^2\right)^{-1}\right.\\&\qquad\left.\cdot\left(sS_L^{-1}(p,T)-S_L^{-1}(p,T)\overline{p}\right)\right)dp_{\mathsf{i}}.\end{aligned}$$

However, also this integral vanishes: as f decays regularly at infinity, the integrand decays sufficiently fast so that we can use Cauchy's integral theorem to transform the path of integration and write

$$\begin{aligned}&\int_{\Gamma_{s,2}}\tilde{f}(s)ds_{\mathsf{i}}\,\left(s^2-2p_0s+|p|^2\right)^{-1}\left(sS_L^{-1}(p,T)-S_L^{-1}(p,T)\overline{p}\right)\\&=\lim_{\rho\to+\infty}\int_{\partial(U_\rho\cap\mathbb{C}_{\mathsf{i}})}\tilde{f}(s)\,ds_{\mathsf{i}}\,\left(s^2-2p_0s+|p|^2\right)^{-1}\left(sS_L^{-1}(p,T)-S_L^{-1}(p,T)\overline{p}\right)=0\end{aligned}$$

where $U_\rho=(\Sigma_\varphi\setminus U_{r_2})\cap U_\rho$. The last identity follows again from Cauchy's integral theorem because the singularities p and $\overline{p}$ of the functions $s\mapsto(s^2-2p_0s+|p|^2)^{-1}$ and $s\mapsto(s^2-2p_0s+|p|^2)^{-1}s$ lie outside of $cl(U_\rho)$ because we chose $R<r_2$.

Putting these pieces together, we find that

$$\tilde{f}(T)E_{\sigma_S(T)}=\frac{1}{2\pi}\int_{\Gamma_{s,1}}\tilde{f}(p)\,dp_{\mathsf{i}}\,S_R^{-1}(p,T).\tag{6.24}$$

We therefore deduce from (6.20) and $E_\infty = \mathcal{I} - E_{\sigma_S(T)}$ that

$$\tilde{f}(T)E_\infty = \tilde{f}(T) - \tilde{f}(T)E_{\sigma_S(T)} = \frac{1}{2\pi}\int_{\Gamma_{s,2}} \tilde{f}(p)\, dp_{\mathsf{i}}\, S_R^{-1}(p,T). \tag{6.25}$$

Let us now consider the operator $a(\mathcal{I}+T)^{-1}$. Since it is slice hyperholomorphic on $\sigma_S(T)$ and at infinity, it is admissible for the S-fuctional calculus. If we set $\chi_{\{\infty\}}(s) := \chi_{\mathbb{H}\setminus U_R(0)}$—that is $\chi_{\{\infty\}}(s) = 1$ if $s \notin U_R(0)$ and $\chi_{\{\infty\}}(s) = 0$ if $s \in cl(U_R(0))$—then $\chi_{\{\infty\}}(T) = E_\infty$ via the S-functional calculus. The product rule of the S-functional calculus yields $a(\mathcal{I}+T)^{-1}E_\infty = g(T)$ with $g(s) = a(1+s)\chi_{\{\infty\}}(s)$. If we set

$$U_{\rho,1} := (\Sigma_\varphi \setminus B_{r_2}(0)) \cup (\mathbb{H}\setminus B_\rho(0)) \quad \text{and} \quad U_{\rho,2} = (\Sigma_\varphi \cap B_{r_1}(0)) \cup B_\varepsilon(0)$$

with $0 < \varepsilon < 1$ sufficiently small, then $U_\rho = U_{\rho,1} \cup U_{\rho,2}$ is an unbounded slice Cauchy domain that contains $\sigma_S(T)$ and such that g is slice hyperholomorphic on $cl(U_\rho)$. Hence,

$$\begin{aligned} a(\mathcal{I}+T)^{-1}E_\infty =& g(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U_\rho\cap\mathbb{C}_{\mathsf{i}})} dp_{\mathsf{i}}\, S_R^{-1}(p,T)\\ =& \frac{1}{2\pi}\int_{\partial(U_{\rho,1}\cap\mathbb{C}_{\mathsf{i}})} a(1+s)\, dp_{\mathsf{i}}\, S_R^{-1}(p,T) \end{aligned}$$

and letting ρ tend to infinity, we finally find

$$a(\mathcal{I}+T)^{-1}E_\infty = \frac{1}{2\pi}\int_{\Gamma_{s,2}} a(1+s)\, dp_{\mathsf{i}}\, S_R^{-1}(p,T). \tag{6.26}$$

Adding (6.25) and (6.26), we find that (6.17) holds true for $f \in \mathcal{E}[\Sigma_\omega]$

Now let f be an arbitrary function in $\mathcal{M}[\Sigma_\omega]_T$ that decays regularly at infinity and let e be a regulariser for f. We can assume that e decays regularly at infinity—otherwise, we can replace e by $s \mapsto (1+s)^{-1}e(s)$, which is a regulariser for f with this property. We expect that

$$\begin{aligned} f(T)E_\infty =& e^{-1}(T)(ef)(T)E_\infty = e^{-1}(T)\{(ef)(T)\}_\infty\\ \overset{(*)}{=}& e^{-1}(T)\{e(T)\}_\infty\{f(T)\}_\infty = e^{-1}(T)e(T)E_\infty\{f(T)\}_\infty\\ =& E_\infty\{f(T)\}_\infty \overset{(**)}{=} \{f(T)\}_\infty \end{aligned} \tag{6.27}$$

such that (6.17) holds true. The boundedness of $f(T)E_\infty$ follows then also from the boundedness of the integral $\{f(T)\}_\infty$. The second and the fourth of the above equalities follow from the above arguments since ef and e both belong to $\mathcal{E}[\Sigma_\varphi]$ and decay regularly at infinity. The equalities marked with $(*)$ and $(**)$ however remain to be shown.

Let $\omega < \varphi_2 < \varphi_1 < \varphi$ be such that $e, f \in \mathcal{E}(\Sigma_\varphi)$ and let $r_1 < r_2$ be such that $B_{r_1}(0)$ contains $\sigma_S(T)$ and any singularity of f. We set $U_s = \Sigma_{\varphi_1} \setminus B_{r_1}(0)$ and $U_p = \Sigma_{\varphi_2} \setminus B_{r_2}(0)$, where the subscripts s and p indicate again the respective variable of integration in the following computation. An application of the S-resolvent equation

(2.30) shows then that

$$\begin{aligned}
&\{e(T)\}_\infty\{f(T)\}_\infty = \\
=&\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_{\mathsf{i}})} e(s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)\frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(p,T)\,dp_{\mathsf{i}}\,f(p)\\
=&\frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_{\mathsf{i}})} e(s)\,ds_{\mathsf{i}}S_R^{-1}(s,T)\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})} p(p^2-2s_0p+|s|^2)^{-1}\,dp_{\mathsf{i}}f(p)\\
&+\frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_{\mathsf{i}})} e(s)\,ds_{\mathsf{i}}S_R^{-1}(s,T)\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})} (p^2-2s_0p+|s|^2)^{-1}\,dp_{\mathsf{i}}f(p)\\
&+\frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_{\mathsf{i}})} e(s)\,ds_{\mathsf{i}}\left(\overline{s}S_L^{-1}(p,T)-S_L^{-1}(p,T)p\right)\\
&\qquad\qquad\qquad\cdot\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})}(p^2-2s_0p+|s|^2)^{-1}\,dp_{\mathsf{i}}f(p).
\end{aligned}$$

Because of our choice of U_s and U_p, the singularities of $p\mapsto(p^2-s_0p+|s|^2)^{-1}$ lie outside U_p for any $s\in\partial(U_s\cap\mathbb{C}_{\mathsf{i}})$ such that $p\mapsto(p^2-2s_0p+|s|)^{-1}$ and $p\mapsto p(p^2-2s_0p+|s|)^{-1}$ are right slice hyperholomorphic on $cl(U_p)$ for any such s. Since f also decays regularly in U_p at infinity, Cauchy's integral theorem implies that the first two of the above integrals equal zero. The fact that e and f decay polynomially at infinity allows us to exchange the order of integration in the third integral, such that

$$\begin{aligned}
\{e(T)\}_\infty\{f(T)\}_\infty =& \frac{1}{(2\pi)^2}\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})}\left[\int_{\partial(U_s\cap\mathbb{C}_{\mathsf{i}})} e(s)\,ds_{\mathsf{i}}\right.\\
&\left.\cdot\left(\overline{s}S_L^{-1}(p,T)-S_L^{-1}(p,T)p\right)(p^2-2s_0p+|s|^2)^{-1}\right]dp_{\mathsf{i}}f(p).
\end{aligned}$$

If $p\in\partial(U_p\cap\mathbb{C}_{\mathsf{i}})$, then p lies for sufficiently large ρ in the bounded axially symmetric Cauchy domain $U_{s,\rho}=U_s\cap B_\rho(0)$. Since f is an intrinsic function on $cl(U_{s,\rho})$, Lemma 4.18 implies

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_{\mathsf{i}})} e(s)\,ds_{\mathsf{i}}\left(\overline{s}S_L^{-1}(p,T)-S_L^{-1}(p,T)p\right)(p^2-2s_0p+|s|^2)^{-1}\\
=&\lim_{\rho\to\infty}\frac{1}{2\pi}\int_{\partial(U_s\cap B_\rho(0)\cap\mathbb{C}_{\mathsf{i}})} e(s)\,ds_{\mathsf{i}}\left(\overline{s}S_L^{-1}(p,T)-S_L^{-1}(p,T)p\right)(p^2-2s_0p+|s|^2)^{-1}\\
=&S_L^{-1}(p,T)e(p).
\end{aligned}$$

Recalling the equivalence of right and left slice hyperholomorphic Cauchy integrals for intrinsic functions, c.f. Remark 4.14, we finally find that

$$\{e(T)\}_\infty\{f(T)\}_\infty=\frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(p,T)\,dp_{\mathsf{i}}e(p)f(p)=\{(ef)(T)\}_\infty,$$

Hence, the identity $(*)$ in (6.27) is true.

Similar arguments show that also $(**)$ holds true. We choose $0<R<r$ such that $B_R(0)$ contains $\sigma_S(T)$ and all singularities of $f(T)$ and we choose $\omega<\varphi'<\varphi$ such

that $f \in \mathcal{E}(\Sigma_{\varphi'})$ and set $U_p := \Sigma_{\varphi'} \setminus B_r(0)$. An application of the S-resolvent equation (2.30) shows that

$$\begin{aligned}
&E_{\sigma_s(T)}\{f(T)\}_\infty = \\
=&\frac{1}{2\pi}\int_{\partial(B_R(0)\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\, S_R^{-1}(s,T)\frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, f(p)\\
=&\frac{1}{(2\pi)^2}\int_{\partial(B_R(0)\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\, S_R^{-1}(s,T)\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})} p(p^2-2s_0p+|s|^2)^{-1}\, dp_{\mathsf{i}}\, f(p)\\
&-\frac{1}{(2\pi)^2}\int_{\partial(B_R(0)\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\, \overline{s}S_R^{-1}(s,T)\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})} (p^2-2s_0p+|s|^2)^{-1}\, dp_{\mathsf{i}}\, f(p)\\
&+\frac{1}{(2\pi)^2}\int_{\partial(B_R(0)\cap\mathbb{C}_{\mathsf{i}})} \left[\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\, \left(\overline{s}S_L^{-1}(p,T) - S_L^{-1}(p,T)p\right)\right.\\
&\qquad\qquad\qquad\qquad\left.\cdot\,(p^2-2s_0p+|s|^2)^{-1}\right] dp_{\mathsf{i}}\, f(p).
\end{aligned}$$

Again, the first two integrals vanish as a consequence of Cauchy's integral theorem because the poles of the function $p \mapsto (p^2-2s_0p+|s|^2)^{-1}$ lie outside of $cl(U_p)$ for any $s \in \partial(B_R(0)\cap\mathbb{C}_{\mathsf{i}})$ and f decays regularly at infinity. Because of (6.1) and the regular decay of f at infinity, we can however apply Fubini's theorem to exchange the order of integration in the third integral and find

$$\begin{aligned}
E_{\sigma_S(T)}\{f(T)\}_\infty = \frac{1}{(2\pi)^2}\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{i}})}&\left[\int_{\partial(B_R(0)\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\right.\\
&\left.(s^2-2p_0s+|p|^2)^{-1}\left(sS_L^{-1}(p,T) - S_L^{-1}(p,T)\overline{p}\right)\right] dp_{\mathsf{i}}\, f(p).
\end{aligned}$$

As the functions $s \mapsto (s^2-2p_0s+|p|^2)^{-1}$ and $s \mapsto (s^2-2p_0s+|p|^2)^{-1}s$ are right slice hyperholomorphic on $\overline{B_R(0)}$ for any $p \in \partial(U_p\cap\mathbb{C}_{\mathsf{i}})$, also this integral vanishes due to Cauchy's integral theorem. Consequently, the identity $(**)$ in (6.27) holds also true as

$$E_\infty\{f(T)\}_\infty = \{f(T)\}_\infty - E_\sigma\{f(T)\}_\infty = \{f(T)\}_\infty.$$

Finally, we point out that the above computations, which proved that $\{(ef)(T)\}_\infty = \{e(T)\}_\infty\{f(T)\}_\infty$ did not require that $e \in \mathcal{E}[\Sigma_\omega]$. They also work if e belongs to $\mathcal{M}[\Sigma_\omega]_T$ and decays regularly at infinity. Hence the same calculations show that (6.18) holds true.

□

Theorem 6.40. *Let $T \in \mathrm{Sect}(\omega)$ and $s \in \{0,\infty\}$. If $f \in \mathcal{M}[\Sigma_\omega]_T$ has polynomial limit c at s and $s \in \sigma_{SX}(T)$, then $c \in \sigma_{SX}(f(T))$.*

Proof. If $c \neq \infty$, then $c \in \mathbb{R}$ because, as an intrinsic function, f takes only real values on the real line. We can hence consider the function $f-c$ instead of f because $\sigma_{SX}(f(T)) = \sigma_{SX}(f(T)-c\mathcal{I})+c$ so that it is sufficient to consider the cases $c=0$ or $c=\infty$.

Let us start with the case $c=0$ and $s=\infty$. If $\infty \in \overline{\sigma_S(T)\setminus\{0\}}^{\mathbb{H}_\infty}$, then

$$0 \in \overline{f(\sigma_S(T)\setminus\{0\})}^{\mathbb{H}_\infty} \subset \sigma_{SX}(f(T))$$

because $f(\sigma_S(T) \setminus \{0\}) \subset \sigma_{SX}(f(T))$ by Proposition 6.38 and the latter is a closed subset of $\mathbb{H}_\infty$. In case $\infty \notin cl(\sigma_S(T))^{\mathbb{H}_\infty}$, we show that $0 \notin \sigma_{SX}(f(T))$ implies that T is bounded, i.e. that even $\infty \notin \sigma_{SX}(T)$. Let us hence assume that $\infty \notin cl(\sigma_S(T))^{\mathbb{H}_\infty}$ and that $0 \notin \sigma_{SX}(f(T))$. In this case, there exists $R > 0$ such that $\sigma_S(T)$ is contained in the open ball $B_R(0)$ of radius R centered at zero. The integal

$$E_{\sigma_S(T)} := \frac{1}{2\pi} \int_{\partial(B_R(0)\cap\mathbb{C}_{\mathsf{i}})} ds_{\mathsf{i}}\, S_R^{-1}(s,T)$$

defines then a bounded projection that commutes with T, namely the spectral projection associated with the spectral set $\sigma_S(T) \subset \sigma_{SX}(T)$ that is obtained from the S-functional calculus, cf. Lemma 2.66. The compatibility of the S-functional calculus with polynomials in T moreover implies

$$TE_{\sigma_S(T)} = (s\chi_{\sigma_S(T)})(T) = \frac{1}{2\pi} \int_{\partial(B_R(0)\cap\mathbb{C}_{\mathsf{i}})} s\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \in \mathcal{B}(V),$$

where $\chi_{\sigma_S(T)}(s)$ denotes the characteristic function of an arbitrary axially symmetric bounded set that contains $cl(B_R(0))$.

Set $E_\infty := \mathcal{I} - E_{\sigma_S(T)}$ and let $V_\infty := E_\infty V$ be the range of E_∞. Since T commutes with E_∞, the operator $T_\infty := T|_{V_\infty}$ is a closed operator on V_∞ with domain $\operatorname{dom}(T_\infty) = \operatorname{dom}(T) \cap V_\infty$. Moreover, we conclude from Lemma 2.66 that

$$\sigma_{SX}(T_\infty) = \sigma_{SX}(T) \setminus \sigma_S(T) \subset \{\infty\}$$

and so in particular

$$\sigma_S(T_\infty) = \sigma_{SX}(T_\infty) \setminus \{\infty\} = \emptyset. \tag{6.28}$$

Now observe that $f(T)$ commutes with E_∞ because of (i) in Lemma 6.31. Hence, $f(T)$ leaves V_∞ invariant and $f(T)_\infty := f(T)|_{V_\infty}$ defines a closed operator on V_∞ with domain $\operatorname{dom}(f(T)_\infty) = \operatorname{dom}(f(T)) \cap V_\infty$. (Note that $f(T)_\infty$ intuitively corresponds to $f(T_\infty)$. The S-functional calculus is however only defined on two-sided Banach spaces. As V_∞ is only a right-linear subspace of V and hence not a two-sided Banach space, we can not define the operator $f(T_\infty)$, cf. the remark at the beginning of Section 6.5.) Since $f(T)$ is invertible because we assumed $0 \notin \sigma_{SX} f(T)$, the operator $f(T)_\infty$ is invertible too and its inverse is $f(T)^{-1}|_{V_\infty} \in \mathcal{B}(V_\infty)$.

Our goal is now to show that T_∞ is bounded. Any bounded operator on a nontrivial Banach space has non-empty S-spectrum and hence we can then conclude from (6.28) that $V_\infty = \{0\}$. Since f decays regularly at infinity, there exists $n \in \mathbb{N}$ such that $sf^n(s) \in \mathcal{M}[\omega]_T$ decays regularly at infinity too. Because of Lemma 6.31, the operators $Tf^n(T)$ and $(sf^n)(T)$ both commute with E_∞. Hence, they leave V_∞ invariant and we find, again because of Lemma 6.31, that

$$Tf^n(T)|_{V_\infty} \subset (sf^n)(T)|_{V_\infty} \in \mathcal{B}(V_\infty)$$

with

$$\begin{aligned} &\operatorname{dom}\left(Tf^n(T)|_{V_\infty}\right) = \operatorname{dom}\left(Tf^n(T)\right) \cap V_\infty \\ &= \operatorname{dom}\left((sf^n)(T)\right) \cap \operatorname{dom}\left(f^n(T)\right) \cap V_\infty \\ &= \operatorname{dom}\left((sf^n)(T)|_{V_\infty}\right) \cap \operatorname{dom}\left(f^n(T)|_{V_\infty}\right). \end{aligned}$$

But since sf^n and f^n both decay regularly at infinity in Σ_φ, Lemma 6.39 implies that $f^n(T)|_{V_\infty}$ and $(sf^n)(T)|_{V_\infty}$ are both bounded linear operators on V_∞. Hence their domain is the entire space V_∞ and we find that

$$Tf^n(T)|_{V_\infty} = (sf^n)(T)|_{V_\infty} \in \mathcal{B}(V_\infty).$$

Finally, observe that Lemma 6.39 also implies that $f^n(T)|_{V_\infty} = (f(T)|_{V_\infty})^n$. As $f(T)|_{V_\infty}$ has a bounded inverse on V_∞, namely $f(T)^{-1}|_{V_\infty}$, we find that $T_\infty \in \mathcal{B}(V_\infty)$ too. As pointed out above, this implies $V_\infty = \{0\}$.

Altogether we find that $V = V_{\sigma_S(T)} := E_{\sigma_S(T)}V$ such that $T = T|_{V_{\sigma_S(T)}}$ belongs to $\mathcal{B}(V_{\sigma_S(T)}) = \mathcal{B}(V)$ and in turn $\infty \notin \sigma_{SX}(T)$ if $0 = f(\infty) \notin \sigma_{SX}(f(T))$.

Now let us consider the case that $s = 0$ and $c = 0$, that is $f(0) = 0$. If 0 does not belong to $\sigma_{SX}(f(T))$, then $f(T)$ has a bounded inverse. Let e be a regulariser for f such that $ef \in \mathcal{E}[\Sigma_\omega]$. Since $f(T) = e(T)^{-1}(ef)(T)$ is injective, the operator $(ef)(T)$ must be injective too. As the function ef has polynomial limit 0 at 0, we conclude from Lemma 6.30 that even T is injective. If we define $\tilde{f}(x) := f(x^{-1})$, then $\tilde{f}$ has polynomial limit 0 at ∞ and $\tilde{f}(T^{-1})$ is invertible as $\tilde{f}(T^{-1}) = f(T)$ by Corollary 6.34. Hence, $0 = \tilde{f}(\infty) \notin \sigma_{SX}(\tilde{f}(T^{-1}))$ and arguments as the ones above show that $\infty \notin \sigma_{SX}(T^{-1})$ such that $T^{-1} \in \mathcal{B}(V)$. Thus, T has a bounded inverse and in turn $0 \notin \sigma_{SX}(T)$ if $0 = f(0) \notin \sigma_{SX}(f(T))$.

Finally, let us consider the case $c = f(s) = \infty$ with $s = 0$ or $s = \infty$ and let us assume that $\infty \notin \sigma_{SX}(f(T))$, that is that $f(T)$ is bounded. If we choose $a \in \mathbb{R}$ with $|a| > \|f(T)\|$, then $a \in \rho_S(f(T))$ and hence $a\mathcal{I} - f(T)$ has a bounded inverse. By (iii) in Lemma 6.31, the function $g(x) := (a - f(x))^{-1}$ belongs to $\mathcal{M}[\Sigma_\omega]_T$. Moreover, $g(T)$ is invertible and $g(T)$ has polynomial limit 0 at s. As we have shown above, this implies $s \notin \sigma_{SX}(T)$, which concludes the proof.

□

Combining Proposition 6.38 and Theorem 6.40, we arrive at the following theorem.

Theorem 6.41. *Let $T \in \mathrm{Sect}(\omega)$. If $f \in \mathcal{M}[\Sigma_\omega]_T$ and f has polynomial limits at $\sigma_{SX}(T) \cap \{0, \infty\}$, then*

$$f(\sigma_{SX}(T)) \subset \sigma_{SX}(f(T)).$$

Let us now consider the inverse inclusion. We start with the following auxiliary lemma.

Lemma 6.42. *Let $T \in \mathrm{Sect}(\omega)$ and let $f \in \mathcal{M}[\Sigma_\omega]_T$ have finite polynomial limits at $\{0, \infty\} \cap \sigma_{SX}(T)$ in Σ_φ for some $\varphi \in (\omega, \pi)$. Furthermore assume that all poles of f are contained in $\rho_S(T)$.*

(i) If $\{0, \infty\} \subset \sigma_{SX}(T)$, then $f(T)$ is defined by the H^∞-functional calculus for sectorial operators.

(ii) If $0 \in \sigma_{SX}(T)$ but $\infty \notin \sigma_{SX}(T)$, then $f(T)$ is defined by the extended H^∞-functional calculus for bounded sectorial operators.

(iii) If $\infty \in \sigma_{SX}(T)$ but $0 \notin \sigma_{SX}(T)$, then $f(T)$ is defined by the extended H^∞-functional calculus for invertible sectorial operators.

(iv) If $0, \infty \notin \sigma_{SX}(T)$, then $f(T)$ is defined by the H^∞-functional calculus for bounded and invertible sectorial operators.

In all of these cases $f(T) \in \mathcal{B}(V)$.

Proof. Let us first consider the case (i), i.e. we assume that $\{0, \infty\} \subset \sigma_{SX}(T)$. Since f has polynomial limits at 0 and ∞ in Σ_ω, the function f has only finitely many poles $[s_1], \dots, [s_n]$ in $cl(\Sigma_\omega)$. Moreover, for suitably large $m_1 \in \mathbb{N}$, the function $f_1(x) = (1+x)^{-2m_1}\mathcal{Q}_{s_1}(x)^{m_1} f(x)$ has also polynomial limits at 0 and ∞ and poles at $[s_2], \dots, [s_n]$ but it does not have a pole at $[s_1]$. Moreover, if we set $r_1(x) = (1+x)^{-2m_1}\mathcal{Q}_{s_1}(x)^{m_1}$, then $r_1(T)$ is bounded and injective because $[s_1] \subset [\rho_S(T))]$. We can now repeat this argument and find inductively $m_2, \dots, m_n$ such that, after setting $r_\ell(x) = (1+x)^{-2m_\ell}\mathcal{Q}_{s_\ell}(x)^{m_\ell}$ for $\ell = 2, \dots, n$ and $r := r_n \cdots r_1$, the function $\tilde{f} = rf$ belongs to $\mathcal{M}[\Sigma_\omega]_T$, has polynomial limits at 0 and ∞ and does not have any poles in $cl(\Sigma_\omega)$. Hence it belongs to $\mathcal{E}[\Sigma_\omega]$. Moreover, r belongs to $\mathcal{E}[\Sigma_\omega]$ too and since $r(T) = r_n(T) \cdots r_1(T)$ is the product of invertible operators, it is invertible itself. Hence r regularises f such that $f(T)$ is defined in terms of the H^∞-functional calculus. Moreover, $f(T) = r(T)^{-1}\tilde{f}(T)$ is bounded as it is the product of two bounded operators.

Similar arguments show the other cases: in (ii) for example, the function f has polynomial limit at 0 but not at ∞, such that the poles of f may accumulate at ∞. However, we integrate along the boundary of $\Sigma_{\omega,0,R} = \Sigma_\omega \cap B_R(0)$ in $\mathbb{C}_i$ for sufficiently large R when we define the H^∞-functional calculus for bounded sectorial operators. Hence, only finitely many poles are contained in $\Sigma_{\omega,0,R}$ and hence relevant. Therefore we can apply the above strategy again in order to show that f is regularised by a rational intrinsic function and that $f(T)$ is hence defined and a bounded operator. Similar, we can argue for (iii) and (iv), where the poles may of f accumulate at 0 resp. at 0 and ∞, but only finitely many of them are relevant.

□

Proposition 6.43. *Let $T \in \mathrm{Sect}(\omega)$. If $f \in \mathcal{M}[\Sigma_\omega]_T$ has polynomial limits at any point in $\sigma_S(T) \cap \{0, \infty\}$, then*

$$f(\sigma_{SX}(T)) \supset \sigma_{SX}(f(T)).$$

Proof. Let $s \in \mathbb{H}$ with $s \notin f(\sigma_{SX}(T))$. The function $x \mapsto \mathcal{Q}_s(f(x))^{-1}$ belongs then to $\mathcal{M}[\Sigma_\omega]_T$ and has finite polynomial limits at $\sigma_{SX}(T) \cap \{0, \infty\}$. Moreover the set of poles of $\mathcal{Q}_s(f(\cdot))$ as an element of $\mathcal{M}[\Sigma_\omega]$, which consists of those spheres $[x]$ in $cl(\Sigma_\omega) \setminus \{0\}$ for which $f([x]) = [f(x)] = [s]$, is contained in the S-resolvent set of T as we chose $s \notin f(\sigma_{SX}(T))$. From Lemma 6.42 we therefore deduce that $\mathcal{Q}_s(f(T))^{-1}$ is defined and belongs to $\mathcal{B}(V)$. Hence $\mathcal{Q}_s(f(T))$ has a bounded inverse and so $s \in \sigma_{SX}(f(T))$.

If finally $s = \infty \notin f(\sigma_{SX}(T))$, then the poles of f are contained in the S-resolvent set of T. Hence, Lemma 6.42 implies that $f(T)$ is a bounded operator and in turn $s = \infty \notin \sigma_{SX}(f(T))$.

□

Combining Theorem 6.41 and Proposition 6.43, we obtain the following spectral mapping theorem

Theorem 6.44 (Spectral Mapping Theorem)**.** *Let* $T \in \mathrm{Sect}(\omega)$ *and let* $f \in \mathcal{M}[\Sigma_\omega]_T$ *have polynomial limits at* $\{0, \infty\} \cap \sigma_{SX}(T)$*. Then*

$$f(\sigma_{SX}(T)) = \sigma_{SX}(f(T)).$$

CHAPTER 7

Fractional Powers of Quaternionic Linear Operators

Fractional powers of operators arose the interest of mathematicians in the 1960s in [14, 62, 63, 65, 88, 89] and have been extensively studied since then. They have applications in the theory of semi-groups [57], allow to define interpolation spaces [39] and provide the theoretical background for defining fractional evolution equations [18]. In this chapter, we generalise three classical approaches for defining fractional powers of operators to the quaternionic setting.

Definition 7.1. The slice hyperholomorphic logarithm on $\mathbb{H}$ is defined as

$$\log x := \ln|x| + \mathsf{i}_x \arg(x) = \ln|x| + \mathsf{i}_x \arccos(x_0/|x|) \tag{7.1}$$

for $x \in \mathbb{H} \setminus (-\infty, 0]$.

Note that for $x = x_0 \in [0, \infty)$ we have $\arccos(x_0/|x|) = 0$ and so $\log x = \ln x$. Therefore, $\log x$ is well defined also on the positive real axis and does not depend on the choice of the imaginary unit i_x. It is

$$e^{\log x} = x \quad \text{for } x \in \mathbb{H}$$

and

$$\log e^x = x \quad \text{for } x \in \mathbb{H} \quad \text{with } |\underline{x}| < \pi.$$

The logarithm is real differentiable on $\mathbb{H} \setminus (-\infty, 0]$. Moreover, for any $\mathsf{i} \in \mathbb{S}$, the restriction of $\log x$ to the complex plane $\mathbb{C}_{\mathsf{i}}$ coincides with the principle branch of the complex logarithm on $\mathbb{C}_{\mathsf{i}}$ and is therefore holomorphic on $\mathbb{C}_{\mathsf{i}} \setminus (-\infty, 0]$. Since $\log \overline{x} = \overline{\log x}$, the function $\log x$ is intrinsic slice hyperholomorphic on $\mathbb{H} \setminus (-\infty, 0]$.

Remark 7.2. Observe that there exist other definitions of the quaternionic logarithm in the literature. In [56], the logarithm of a quaternion is for instance defined as

$$\log_{k,\ell} x := \begin{cases} \ln|x| + \mathbf{i}_x\left(\arccos\frac{x_0}{|x|} + 2k\pi\right), & |\underline{x}| \neq 0 \text{ or } |\underline{x}| = 0, x_0 > 0 \\ \ln|x| + e_\ell\pi, & |\underline{x}| = 0, x_0 < 0 \end{cases}$$

where $k \in \mathbb{Z}$ and e_ℓ is one of the generating units of $\mathbb{H}$. This logarithm is however not continuous (and therefore in particular not slice hyperholomorphic) at the real line, unless $k = 0$. But for $k = 0$ this definition of the logarithm coincides with the one given in Definition 7.1. Even more, the identity principle implies that (7.1) defines the maximal slice hyperholomorphic extension of the natural logarithm on $(0, +\infty)$ to a subset of the quaternions.

Definition 7.3. For $\alpha \in \mathbb{R}$, we define the fractional power x^α of $x \in \mathbb{H} \setminus (-\infty, 0]$ as

$$x^\alpha := e^{\alpha \log x} = e^{\alpha(\ln|x| + \mathbf{i}_x \arg(x))}. \tag{7.2}$$

Remark 7.4. The function $x \mapsto x^\alpha$ is by Corollary 2.7 intrinsic slice hyperholomorphic on its domain $\mathbb{H} \setminus (-\infty, 0]$ as it is the composition of two intrinsic slice hyperholomorphic functions. Observe however that this is only true for real α: if $\alpha \notin \mathbb{R}$, then the inner function $x \mapsto \alpha \log x$ is not intrinsic. Since the composition of two slice hyperholomorphic functions in in general only slice hyperholomorphic if the inner function is intrinsic, defining x^α as in (7.2) does in this case not yield a slice hyperholomorphic function.

Since x^α is a slice hyperholomorphic function for $\alpha \in \mathbb{R}$, our setting provides suitable techniques for defining fractional powers of quaternionic linear operators. In the complex setting, several approaches for defining fractional powers of sectorial operators are known and they make different assumptions on the respective operator. In this chapter, we generalise three of these approaches to the quaternionic setting.

1) In Section 7.1 we follow the approach of [39] and define fractional powers of invertible sectorial operators directly via a slice hyperholomorphic Cauchy integral. This approach allows to define interpolation spaces, but the theory of interpolation spaces in the quaternionic setting does not show any significant difference to the complex one. The results presented in this section can be found in [21].

2) In Section 7.2, we follow [59] and develop the most general approach to fractional powers in the quaternionic setting. We use the H^∞-functional calculus introduced in Chapter 6 in order to introduce fractional powers of arbitrary sectorial operators and show several of their properties. The results in this section are part of [19].

3) In Section 7.3, we finally follow [63] and introduce fractional powers of exponent $\alpha \in (0, 1)$ indirectly. We first find integral representations for operators that should correspond to the S-resolvents of T^α. Then we show that there exists actually an operator such that these integral coincide with its resolvent. In particular, this implies that the famous Kato-formula for the resolvent of the fractional power of an operator has an analogue in the quaternionic setting. The results in this section can again be found in [21].

7.1 A Direct Approach to Fractional Powers of Invertible Sectorial Operators with Negative Exponent

In the following we assume that T is a densely defined closed quaternionic right linear operator such that $(-\infty, 0] \subset \rho_S(T)$ and such that there exists a positive constant $M > 0$ such that

$$\|S_R^{-1}(s,T)\| \leq \frac{M}{1+|s|} \qquad \text{for } s \in (-\infty, 0]. \tag{7.3}$$

Obviously this is the case if T is a sectorial operator in the sense of Definition 6.1 and has a bounded inverse. In order to show that also the converse is true we recall the notation

$$\Sigma_\varphi = \{s \in \mathbb{H} : \arg(s) < \varphi\}$$

for the sector of angle $\varphi \in (0, \pi)$ around the positive real axis.

Lemma 7.5. *The estimate* (7.3) *implies the existence of constants $a > 0$, $\varphi \in (0, \pi)$ and $M_n > 0$ for $n \in \mathbb{N}$ such that $\sigma_S(T)$ is contained in the translated sector*

$$\Sigma_\varphi + a := \{s \in \mathbb{H} : \arg(s - a) < \varphi\}$$

and such that, for any $s \notin \Sigma_\varphi + a$ and any $n \in \mathbb{N}$, we have

$$\|S_R^{-n}(s,T)\| \leq \frac{M_n}{(1+|s|)^n} \qquad \textit{and } \|S_L^{-n}(s,T)\| \leq \frac{M_n}{(1+|s|)^n}. \tag{7.4}$$

Proof. By Proposition 4.42, we have

$$\partial_S{}^k S_R^{-1}(s,T) = (-1)^k k!\, S_R^{-(k+1)}(s,T) \quad \text{for } \ k \in \mathbb{N}. \tag{7.5}$$

Hence, by Lemma 2.61 and Remark 2.13, the map $s \mapsto S_R^{-n}(s,T)$ is a left slice hyperholomorphic function on $\rho_S(T)$ with values in $\mathcal{B}(V)$ for any $n \in \mathbb{N}$. From the identity (7.5) we deduce

$$\begin{aligned}\partial_S{}^k S_R^{-n}(s,T) =& \partial_S{}^{k+n-1} \frac{(-1)^{n-1}}{(n-1)!} S_R^{-1}(s,T) \\ =& \frac{(-1)^k (k+n-1)!}{(n-1)!}\, S_R^{-(k+n)}(s,T).\end{aligned}$$

When we apply Theorem 2.14 in order to expand $S_R^{-n}(s,T)$ into a Taylor series at a real point $\alpha \in \rho_S(T)$, we therefore get

$$\begin{aligned} S_R^{-n}(s,T) &= \sum_{k=0}^{+\infty} \frac{1}{k!}(s-\alpha)^k \partial_S{}^k S_R^{-n}(\alpha,T) \\ &= \sum_{k=0}^{+\infty} \frac{1}{k!}(s-\alpha)^k \frac{(-1)^k (k+n-1)!}{(n-1)!}\, S_R^{-(k+n)}(\alpha,T) \\ &= \sum_{k=0}^{+\infty} (-1)^k \binom{n+k-1}{k} (s-\alpha)^k S_R^{-(n+k)}(\alpha,T) \end{aligned} \tag{7.6}$$

on any ball $B_r(\alpha)$ contained in $\rho_S(T)$. Since α is real, $S_R^{-n}(\alpha,T) = \left(S_R^{-1}(\alpha,T)\right)^n$ and thus $\|S_R^{-n}(\alpha,T)\| \leq \|S_R^{-1}(\alpha,T)\|^n$. The ratio test and the estimate (7.3) therefore imply that this series converges on the ball with radius $(1+|\alpha|)/M$ centered at α for $\alpha \in (-\infty, 0]$. In particular, considering the case $n=1$, we deduce from Theorem 3.12 that any such ball is contained in $\rho_S(T)$. Otherwise the above series would give a nontrivial slice hyperholomorphic continuation of $S_R^{-1}(s,T)$, which cannot exist by Theorem 3.12.

Set $a = \min\left\{\frac{1}{4M}, 1\right\}$. Then the closed ball $cl(B_a(0))$ is contained in $\rho_S(T)$ and for any $s \in cl(B_a(0))$, we have the estimate

$$\begin{aligned}
\left\|S_R^{-n}(s,T)\right\| &\leq \sum_{k=0}^{+\infty} \binom{n+k-1}{k} |s|^k \|S_R^{-(n+k)}(0,T)\| \\
&\leq \sum_{k=0}^{+\infty} \binom{n+k-1}{k} \frac{1}{(4M)^k} M^{n+k} \frac{(1+|s|)^{n+k}}{(1+|s|)^{n+k}} \\
&= \frac{2^n M^n}{(1+|s|)^n} \sum_{k=0}^{+\infty} \binom{n+k-1}{k} \frac{1}{2^k} = \frac{4^n M^n}{(1+|s|)^n},
\end{aligned}$$

where the last equation follows from the Taylor series expansion

$$(1-z)^{-n} = \sum_{k=0}^{+\infty} \binom{n+k-1}{k} z^k \qquad \text{for } |z| < 1.$$

Now set $\varphi_0 = \pi - \arctan(\frac{1}{2M})$ and consider the sector Σ_{φ_0}. If $s = s_0 + \mathbf{i}_s s_1 \notin \Sigma_{\varphi_0}$, we have $0 \leq s_1 \leq |s_0|/(2M)$ and from the power series expansion (7.6) of $S_R^{-n}(s,T)$ at s_0, we conclude

$$\begin{aligned}
\left\|S_R^{-n}(s,T)\right\| &\leq \sum_{k=0}^{+\infty} \binom{n+k-1}{k} |s_1|^k \left\|S_R^{-1}(s_0,T)\right\|^{n+k} \\
&\leq \sum_{k=0}^{+\infty} \binom{n+k-1}{k} \left(\frac{|s_0|}{2M}\right)^k \left(\frac{M}{1+|s_0|}\right)^{n+k} \\
&\leq \left(\frac{M}{1+|s_0|}\right)^n \sum_{k=0}^{+\infty} \binom{n+k-1}{k} \frac{1}{2^k} = \frac{2^n M^n}{(1+|s_0|)^n}.
\end{aligned}$$

Since $|s| \leq |s_0| + |s_1| \leq (1 + \frac{1}{2M})|s_0|$, we get

$$\|S_R^{-n}(s,T)\| \leq \frac{2^n M^n}{\left(1 + \left(1+\frac{1}{2M}\right)^{-1} |s|\right)^n} \leq \frac{\left(1+\frac{1}{2M}\right)^n 2^n M^n}{(1+|s|)^n}.$$

Hence, the estimate

$$\left\|S_R^{-n}(s,T)\right\| \leq \frac{M_n}{(1+|s|)^n} \tag{7.7}$$

with

$$M_n := \left(1 + \frac{1}{2M}\right)^n 4^n M^n$$

holds true for any $s \notin \Omega := \Sigma_(\varphi_0) \setminus B_a(0)$. Now observe that the sector $\Sigma_\varphi + a$ with

$$\varphi := \arctan\left(\frac{a\sin\varphi_0}{a(-1+\cos\varphi_0)}\right)$$

is contains the set V. Hence, if $s \notin \Sigma_\varphi + a$, then $s \notin \Omega$ and so $s \in \rho_S(T)$ and (7.7) holds true.

Since $S_L^{-1}(s,T) = S_R^{-1}(s,T)$ for $s \in (-\infty, 0]$, the estimate (7.3) applies also to the left S-resolvent. Thus we can use analogous arguments to prove that the the estimate for the left S-resolvent in (7.4) also holds true with these constants.

□

Definition 7.6. Let $\mathsf{i} \in \mathbb{S}$ and let $\Sigma_\varphi + a$ be the sector obtained from Lemma 7.5. Let $\theta \in (\varphi, \pi)$ and choose a piecewise smooth path Γ in $\mathbb{H} \setminus (\Sigma_\varphi + a)$ that goes in $\mathbb{C}_\mathsf{i}$ from $\infty e^{\mathsf{i}\theta}$ to $\infty e^{-\mathsf{i}\theta}$ and avoid the negative real axis $(-\infty, 0]$. For $\alpha > 0$, we define

$$T^{-\alpha} := \frac{1}{2\pi}\int_\Gamma s^{-\alpha}\, ds_\mathsf{i}\, S_R^{-1}(s,T). \tag{7.8}$$

Theorem 7.7. *For any $\alpha > 0$, the operator $T^{-\alpha}$ is bounded and independent of the choice of $\mathsf{i} \in \mathbb{S}$, of $\theta \in (\theta_0, \pi)$ and of the concrete path Γ in $\mathbb{C}_\mathsf{i}$ and therefore well-defined.*

Proof. The estimate (7.4) assures that the integral in (7.8) exists and that it defines a bounded right-linear operator. Since $s \mapsto s^{-\alpha}$ is right slice hyperholomorphic and $s \mapsto S_R^{-1}(s,T)$ is left slice hyperholomorphic, the independence of the choice of θ and the independence of the choice of the path Γ in the complex plane $\mathbb{C}_\mathsf{i}$ follow from Cauchy's integral theorem.

In order to show that $T^{-\alpha}$ is independent of the choice of the imaginary unit $\mathsf{i} \in \mathbb{S}$, we consider an arbitrary imaginary unit $\mathsf{j} \in \mathbb{S}$ with $\mathsf{j} \neq \mathsf{i}$. If $\Sigma_\varphi + a$ is the sector obtained from Lemma 7.5, then let $\varphi < \theta_s < \theta_p < \pi$ and set $U_s := \Sigma_{\theta_s} \setminus B_{a/2}(0)$ and $U_p := \Sigma_{\theta_p} \setminus B_{a/3}(0)$. (The indices s and p are chosen in order to indicate the variable of integration over the boundary of the respective set in the following calculation.) Then U_p and U_s are slice domains that contain $\sigma_S(T)$ and $\partial(U_s \cap \mathbb{C}_\mathsf{i})$ and $\partial(U_p \cap \mathbb{C}_\mathsf{j})$ are paths that are admissible in Definition 7.6.

Observe that $s \mapsto s^{-\alpha}$ is right slice hyperholomorphic on $cl(U_p)$ and that, by our choices of U_p and U_s, we have $s \in U_p$ for any $s \in \partial(U_s \cap \mathbb{C}_\mathsf{i})$. If we choose $r > 0$ large enough, then $s \in U_p \cap B_r(0)$ and we obtain from Theorem 2.30 that

$$\begin{aligned} s^\alpha &= \lim_{r\to+\infty} \frac{1}{2\pi}\int_{\partial(U_p \cap B_r(0) \cap \mathbb{C}_\mathsf{i})} p^{-\alpha}\, dp_\mathsf{j}\, S_R^{-1}(p,s) \\ &= \frac{1}{2\pi}\int_{\partial(U_p \cap \mathbb{C}_\mathsf{j})} p^{-\alpha}\, dp_\mathsf{j}\, S_R^{-1}(p,s), \end{aligned}$$

where the second equations holds since $p^{-\alpha} \to 0$ uniformly as $p \to \infty$ in U_p. For the

operator $T^{-\alpha}$, we thus obtain

$$\begin{aligned} T^{-\alpha} &= \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_{\mathsf{i}})} s^{-\alpha}\,ds_{\mathsf{i}}\,S_R^{-1}(s,T) \\ &= \frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_{\mathsf{i}})}\left(\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{j}})} p^{-\alpha}\,dp_{\mathsf{j}}\,S_R^{-1}(p,s)\right) ds_{\mathsf{i}}\,S_R^{-1}(s,T). \end{aligned} \tag{7.9}$$

We now apply Fubini's theorem, but we postpone the estimate that justifies this to the end of the proof as is very technical and quite long. By exchanging the order of integration, we get

$$\begin{aligned} T^{-\alpha} &= \frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{j}})} p^{-\alpha}\,dp_{\mathsf{j}}\left(\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_{\mathsf{i}})} S_R^{-1}(p,s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)\right) \\ &= \frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_{\mathsf{j}})} p^{-\alpha}\,dp_{\mathsf{j}}\,S_R^{-1}(p,T), \end{aligned}$$

where the last equation follows as an application of the S-functional calculus since $S_R^{-1}(p,\infty) = \lim_{s\to\infty} S_R^{-1}(p,s) = 0$. Hence, the operator $T^{-\alpha}$ is also independent of the choice of the imaginary unit $\mathsf{i}\in\mathbb{S}$ provided that it is actually possible to apply Fubini's theorem in (7.9). In order to show that the integrand in (7.9) is absolutely integrable, we consider the parametrisations Γ_s and Γ_p of $\partial(U_s\cap\mathbb{C}_{\mathsf{i}})$ and $\partial(U_p\cap\mathbb{C}_{\mathsf{j}})$ that are given by

$$\Gamma_s(r) = \begin{cases} \Gamma_s^+(r) := re^{-\theta_s\mathsf{i}}, & r\in[a/2,+\infty) \\ \Gamma_s^0(r) := \frac{a}{2}e^{-\frac{2\theta_s}{a}\mathsf{i}r} & r\in(-a/2,a/2) \\ \Gamma_s^-(r) := re^{\theta_s\mathsf{i}}, & r\in(-\infty,-a/2] \end{cases}$$

and

$$\Gamma_p(t) = \begin{cases} \Gamma_p^+(t) := te^{-\theta_p\mathsf{j}}, & t\in[a/3,+\infty) \\ \Gamma_p^0(t) := \frac{a}{3}e^{-\frac{3\theta_p}{a}\mathsf{j}t} & t\in(-a/3,a/3) \\ \Gamma_p^-(t) := te^{\theta_p\mathsf{j}}, & t\in(-\infty,-a/3] \end{cases}.$$

Then

$$\begin{aligned} &\frac{1}{(2\pi)^2}\int_{\Gamma_s}\int_{\Gamma_p}\left\|p^{-\alpha}\,dp_{\mathsf{j}}\,S_R^{-1}(p,s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)\right\| \\ &= \sum_{\tau,\nu\in\{-,0,+\}}\int_{\Gamma_s^\tau}\int_{\Gamma_p^\nu}\left\|p^{-\alpha}\,dp_{\mathsf{j}}\,S_R^{-1}(p,s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)\right\| \end{aligned} \tag{7.10}$$

and it is sufficient to estimate each of the terms in the sum separately. Applying Theorem 2.9 allows us to estimate

$$|S_R^{-1}(p,s)| \leq \frac{1}{2}|1-\mathsf{ij}|\frac{1}{|p_{\mathsf{i}}-s|} + \frac{1}{2}|1+\mathsf{ij}|\frac{1}{|\overline{p_{\mathsf{i}}}-s|} \leq \frac{2}{|p_{\mathsf{i}}-s_{\mathsf{i}}|}, \tag{7.11}$$

where $p_{\mathsf{i}} = p_0 + \mathsf{i}p_1$ for $p = p_0 + \mathsf{i}_p p_1$. By applying (7.4), we find for any $\tau,\nu\in\{+,-\}$

that

$$
\begin{aligned}
&\int_{\Gamma_s^\tau}\int_{\Gamma_p^\nu}\left\|p^{-\alpha}\,dp_{\mathsf{j}}\,S_R^{-1}(p,s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)\right\|\\
&=\int_{\frac{a}{2}}^{+\infty}\int_{\frac{a}{3}}^{+\infty} t^{-\alpha}\frac{2}{|te^{\theta_p\mathsf{i}}-re^{\theta_s\mathsf{i}}|}\frac{M_1}{1+r}\,dt\,dr\\
&=\int_{\frac{a}{2}}^{+\infty}\int_{\frac{a}{3}}^{+\infty} \frac{t^{-\alpha}}{|t-re^{(\theta_s-\theta_p)\mathsf{i}}|}\frac{2M_1}{1+r}\,dt\,dr\\
&=\int_{\frac{a}{2}}^{+\infty}\int_{\frac{a}{3}}^{+\infty} \frac{\left(\frac{t}{r}\right)^{-\alpha}}{\left|\frac{t}{r}-e^{(\theta_s-\theta_p)\mathsf{i}}\right|}\,dt\,\frac{1}{r^{1+\alpha}}\frac{2M_1}{1+r}\,dr\\
&=\int_{\frac{a}{2}}^{+\infty}\int_{\frac{a}{3r}}^{+\infty} \frac{\mu^{-\alpha}}{|\mu-e^{(\theta_s-\theta_p)\mathsf{i}}|}\,d\mu\,\frac{1}{r^{\alpha}}\frac{2M_1}{1+r}\,dr.
\end{aligned}
$$

The modulus of $\mu - e^{(\theta_s-\theta_p)\mathsf{i}}$ can be estimated from below by the absolute value of its real part or by the absolute value of its imaginary part, and therefore,

$$
\begin{aligned}
&\int_{\Gamma_s^\tau}\int_{\Gamma_p^\nu}\left\|p^{-\alpha}\,dp_{\mathsf{j}}\,S_R^{-1}(p,s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)\right\|\\
&\le\int_{\frac{a}{2}}^{+\infty}\int_{2}^{+\infty} \frac{\mu^{-\alpha}}{\mu-\cos(\theta_s-\theta_p)}\,d\mu\,\frac{1}{r^{\alpha}}\frac{2M_1}{1+r}\,dr\\
&\quad+\int_{\frac{a}{2}}^{+\infty}\int_{\frac{a}{3r}}^{2} \frac{\mu^{-\alpha}}{\sin(\theta_p-\theta_s)}\,d\mu\,\frac{1}{r^{\alpha}}\frac{2M_1}{1+r}\,dr\\
&=\underbrace{\int_{2}^{+\infty} \frac{\mu^{-\alpha}}{\mu-\cos(\theta_s-\theta_p)}\,d\mu}_{=:C_1<+\infty}\underbrace{\int_{\frac{a}{2}}^{+\infty}\frac{1}{r^{\alpha}}\frac{2M_1}{1+r}\,dr}_{=:C_2<+\infty}\\
&\quad+\int_{\frac{a}{2}}^{+\infty}\frac{1}{\sin(\theta_p-\theta_s)}\left(\frac{2^{1-\alpha}}{1-\alpha}-\frac{a^{1-\alpha}}{(1-\alpha)3^{1-\alpha}}\frac{1}{r^{1-\alpha}}\right)\frac{1}{r^{\alpha}}\frac{2M_1}{1+r}\,dr\\
&=C_1C_2+\frac{2M_1}{\sin(\theta_p-\theta_s)}\frac{2^{1-\alpha}}{1-\alpha}\int_{\frac{a}{2}}^{+\infty}\frac{r^{-\alpha}}{1+r}\,dr\\
&\quad+\frac{2M_1}{\sin(\theta_p-\theta_s)}\frac{a^{1-\alpha}}{(1-\alpha)3^{1-\alpha}}\int_{\frac{a}{2}}^{+\infty}\frac{1}{r(1+r)}\,d\mu\,dr
\end{aligned}
$$

where each of these integrals is finite.

For $\tau = 0$ and $\nu = +$, we can again use (7.11) to estimate

$$
\begin{aligned}
&\int_{\Gamma_s^0}\int_{\Gamma_p^+}\left\|p^{-\alpha}\,dp_{\mathsf{j}}\,S_R^{-1}(p,s)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)\right\|\\
&\le\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{\frac{a}{3}}^{+\infty} t^{-\alpha}\left|S_R^{-1}\left(te^{\mathsf{j}\theta_p},\frac{a}{2}e^{-\frac{2\theta_s}{a}r\mathsf{i}}\right)\right|\frac{\theta_sM_1}{1+\frac{a}{2}}\,dt\,dr
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{\frac{a}{3}}^{+\infty} t^{-\alpha} \frac{2}{\left|te^{\mathsf{i}\theta_p} - \frac{a}{2} e^{\frac{2\theta_s}{a} r \mathsf{i}}\right|} \frac{\theta_s M_1}{1+\frac{a}{2}}\, dt\, dr \\
&= \frac{2\theta_s M_1}{1+\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{\frac{a}{3}}^{+\infty} \frac{t^{-\alpha}}{\left|t - \frac{a}{2} e^{\left(\frac{2\theta_s}{a} r - \theta_p\right)\mathsf{i}}\right|}\, dt\, dr.
\end{aligned}
$$

Since $0 < \theta_s < \theta_p < \pi$, the distance δ of the set

$$
\left\{ \frac{a}{2} e^{\left(\frac{2\theta_s}{a} r - \theta_p\right)\mathsf{i}} : -\frac{a}{2} < r < \frac{a}{2} \right\}
$$

to the positive real axis is greater than zero, and hence,

$$
\begin{aligned}
&\int_{\Gamma_s^0} \int_{\Gamma_p^+} \left\| p^{-\alpha}\, dp_{\mathsf{j}}\, S_R^{-1}(p,s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \right\| \\
\leq& \frac{2\theta_s M_1}{1+\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{\frac{a}{3}}^{a} \frac{t^{-\alpha}}{\delta}\, dt\, dr + \frac{2\theta_s M_1}{1+\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{a}^{+\infty} t^{-\alpha} \frac{1}{t - \frac{a}{2}}\, dt\, dr \\
=& \frac{2\theta_s a M_1}{\delta\left(1+\frac{a}{2}\right)} \int_{\frac{a}{3}}^{a} t^{-\alpha}\, dt + \frac{2\theta_s a M_1}{1+\frac{a}{2}} \int_{a}^{+\infty} \frac{t^{-\alpha}}{t-\frac{a}{2}}\, dt,
\end{aligned}
$$

where again these integrals are finite. A similar computation can be done for the case $\tau = 0$ and $\nu = -$.

For $\tau = +$ and $\nu = 0$, we apply once more (7.11) and obtain

$$
\begin{aligned}
&\int_{\Gamma_s^+} \int_{\Gamma_p^0} \left\| p^{-\alpha}\, dp_{\mathsf{j}}\, S_R^{-1}(p,s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \right\| \\
\leq& \int_{\frac{a}{2}}^{+\infty} \int_{-\frac{a}{3}}^{\frac{a}{3}} \left(\frac{a}{3}\right)^{-\alpha} \theta_p \left| S_R^{-1}\left(\frac{a}{3} e^{\frac{-3\theta_p}{a}\mathsf{j}t}, re^{-\theta_s \mathsf{i}} \right) \right| \frac{M_1}{1+r}\, dt\, dr \\
\leq& 2\left(\frac{a}{3}\right)^{-\alpha} \theta_p M_1 \int_{\frac{a}{2}}^{+\infty} \int_{-\frac{a}{3}}^{\frac{a}{3}} \frac{1}{\left| \frac{a}{3} e^{\frac{3\theta_p}{a}\mathsf{i}t} - re^{\theta_s \mathsf{i}} \right|} \frac{1}{1+r}\, dt\, dr \\
=& 2\left(\frac{a}{3}\right)^{-\alpha} \theta_p M_1 \int_{\frac{a}{2}}^{+\infty} \int_{-\frac{a}{3}}^{\frac{a}{3}} \frac{1}{\left| r - \frac{a}{3} e^{\left(\frac{3\theta_p}{a} t - \theta_s\right)\mathsf{i}} \right|} \frac{1}{1+r}\, dt\, dr.
\end{aligned}
$$

Estimating the modulus of the denominator from below with the modulus of its real part, we obtain

$$
\begin{aligned}
&\int_{\Gamma_s^+} \int_{\Gamma_p^0} \left\| p^{-\alpha}\, dp_{\mathsf{j}}\, S_R^{-1}(p,s)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \right\| \\
\leq& 2\left(\frac{a}{3}\right)^{-\alpha} \theta_p M_1 \int_{\frac{a}{2}}^{+\infty} \int_{-\frac{a}{3}}^{\frac{a}{3}} \frac{1}{\left| r - \frac{a}{3}\cos\left(\frac{3\theta_p}{a} t - \theta_s\right) \right|} \frac{1}{1+r}\, dt\, dr \\
\leq& 2\left(\frac{a}{3}\right)^{-\alpha} \theta_p M_1 \int_{\frac{a}{2}}^{+\infty} \int_{-\frac{a}{3}}^{\frac{a}{3}} \frac{1}{r-\frac{a}{3}} \frac{1}{1+r}\, dt\, dr \\
=& \frac{4a}{3}\left(\frac{a}{3}\right)^{-\alpha} \theta_p M_1 \int_{\frac{a}{2}}^{+\infty} \frac{1}{r-\frac{a}{3}} \frac{1}{1+r}\, dr,
\end{aligned}
$$

and this last integral is finite. The estimate of the case $\tau = -$ and $\nu = 0$ can be done in a similar way.

Finally, the summand for $\tau = 0$ and $\nu = 0$ consists of the integral of a continuous function over a bounded domain and is therefore finite.

Putting these pieces together, we obtain that the integrand in (7.9) is absolutely integrable, which allows us to apply Fubini's theorem in order to exchange the order of integration.

□

If $\alpha \in \mathbb{N}$, then $s^{-\alpha}$ is right slice hyperholomorphic at infinity. The following Corollary then immediately follows as an application of the S-functional calculus.

Corollary 7.8. *If $\alpha \in \mathbb{N}$, then the operator $T^{-\alpha}$ defined in* (7.8) *coincides with the α-th inverse power of T.*

If we follow the arguments of the proof of Theorem 5.27 in [39, Chapter II], we obtain an integral representation of $T^{-\alpha}$ that is almost identical to the one derived for the complex case: the only difference is the different constant in front of the integral. This is due to the different choice of the branch of the logarithm that is used in [39] in order to define the fractional powers. As pointed out in Remark 7.2, it is not possible to define different branches of the logarithm in a quaternionic slice hyperholomorphic setting. In Corollary 7.11 we however obtain an integral representation that is clearly different from any integral representation known from the classical complex setting.

Theorem 7.9. *Let $n \in \mathbb{N}$. For $\alpha \in (0, n+1)$ with $\alpha \notin \mathbb{N}$, the operator $T^{-\alpha}$ defined in* (7.8) *has the representation*

$$T^{-\alpha} = (-1)^{n+1} \frac{\sin(\alpha\pi)}{\pi} \frac{n!}{(n-\alpha)\cdots(1-\alpha)} \int_0^{+\infty} t^{n-\alpha} S_R^{-(n+1)}(-t, T)\, dt. \tag{7.12}$$

Proof. Let a and φ be the constants obtained from Lemma 7.5. For $b \in (0, a)$ and $\theta \in (\varphi, \pi)$, we can choose $U = \Sigma_\theta + b$ and integrate over the boundary $\partial(U \cap \mathbb{C}_{\mathsf{i}})$ of U in $\mathbb{C}_{\mathsf{i}}$ for some $\mathsf{i} \in \mathbb{S}$ in the integral representation of $T^{-\alpha}$. The boundary consists of the path

$$\gamma(t) = \begin{cases} b - te^{\mathsf{i}\theta}, & t \in (-\infty, 0] \\ b + te^{-\mathsf{i}\theta}, & t \in (0, \infty) \end{cases},$$

and hence

$$\begin{aligned} T^{-\alpha} =& \frac{1}{2\pi} \int_{-\infty}^{0} (b - te^{\mathsf{i}\theta})^{-\alpha}(-\mathsf{i})(-e^{\mathsf{i}\theta}) S_R^{-1}(b - te^{\mathsf{i}\theta}, T)\, dt \\ &+ \frac{1}{2\pi} \int_0^{+\infty} (b + te^{-\mathsf{i}\theta})^{-\alpha}(-\mathsf{i}) e^{-\mathsf{i}\theta} S_R^{-1}(b + te^{-\mathsf{i}\theta}, T)\, dt \\ =& \frac{\mathsf{i}}{2\pi} \int_0^{+\infty} (b + te^{\mathsf{i}\theta})^{-\alpha} e^{\mathsf{i}\theta} S_R^{-1}(b + te^{\mathsf{i}\theta}, T)\, dt \\ &- \frac{\mathsf{i}}{2\pi} \int_0^{+\infty} (b + te^{-\mathsf{i}\theta})^{-\alpha} e^{-\mathsf{i}\theta} S_R^{-1}(b + te^{-\mathsf{i}\theta}, T)\, dt. \end{aligned}$$

Integrating n times by parts yields

$$\begin{aligned} T^{-\alpha} =& \frac{n!}{(n-\alpha)\cdots(1-\alpha)} \frac{\mathsf{i}}{2\pi} \int_0^{+\infty} (b+te^{\mathsf{i}\theta})^{n-\alpha} e^{\mathsf{i}\theta} S_R^{-(n+1)}(b+te^{\mathsf{i}\theta},T)\,dt \\ & - \frac{n!}{(n-\alpha)\cdots(1-\alpha)} \frac{\mathsf{i}}{2\pi} \int_0^{+\infty} (b+te^{-\mathsf{i}\theta})^{n-\alpha} e^{-\mathsf{i}\theta} S_R^{-(n+1)}(b+te^{-\mathsf{i}\theta},T)\,dt. \end{aligned}$$

Because of the estimate (7.4), we can apply Lebesgue's dominated convergence theorem with dominating function

$$f(t) = \begin{cases} C(1+t^{n-\alpha}) & \text{if } t \leq 1 \\ Ct^{-\alpha-1} & \text{if } t > 1, \end{cases}$$

where $C > 0$ is a sufficiently large constant. Taking the limit $b \to 0$, we obtain

$$\begin{aligned} T^{-\alpha} =& \frac{n!}{(n-\alpha)\cdots(1-\alpha)} \frac{\mathsf{i}}{2\pi} \int_0^{+\infty} t^{n-\alpha} e^{\mathsf{i}\theta(n-\alpha)} e^{\mathsf{i}\theta} S_R^{-(n+1)}(te^{\mathsf{i}\theta},T)\,dt \\ & - \frac{n!}{(n-\alpha)\cdots(1-\alpha)} \frac{\mathsf{i}}{2\pi} \int_0^{+\infty} t^{n-\alpha} e^{-\mathsf{i}\theta(n-\alpha)} e^{-\mathsf{i}\theta} S_R^{-(n+1)}(te^{-\mathsf{i}\theta},T)\,dt \end{aligned} \tag{7.13}$$

and then, taking the limit $\theta \to \pi$, we get

$$\begin{aligned} T^{-\alpha} =& - \frac{n!}{(n-\alpha)\cdots(1-\alpha)} \frac{\mathsf{i}}{2\pi} \int_0^{+\infty} t^{n-\alpha} e^{\mathsf{i}\pi(n-\alpha)} S_R^{-(n+1)}(-t,T)\,dt \\ & + \frac{n!}{(n-\alpha)\cdots(1-\alpha)} \frac{\mathsf{i}}{2\pi} \int_0^{+\infty} t^{n-\alpha} e^{-\mathsf{i}\pi(n-\alpha)} S_R^{-(n+1)}(-t,T)\,dt \\ =& (-1)^{n+1} \frac{\sin(\alpha\pi)}{\pi} \frac{n!}{(n-\alpha)\cdots(1-\alpha)} \int_0^{+\infty} t^{n-\alpha} S_R^{-(n+1)}(-t,T)\,dt, \end{aligned}$$

where the last equation follows from the identity

$$-\mathsf{i}e^{\mathsf{i}\pi(n-\alpha)} + \mathsf{i}e^{-\mathsf{i}\pi(n-\alpha)} = \sin((n-\alpha))\pi = (-1)^{n+1}\sin(\alpha\pi).$$

□

Corollary 7.10. *It is $\mathcal{I}^{-\alpha} = \mathcal{I}$ for $\alpha \geq 0$, where $\mathcal{I}$ denotes the identity operator on V.*

Proof. If $\alpha \in \mathbb{N}$, this follows immediately from Corollary 7.8. For $\alpha \notin \mathbb{N}$, we consider $n \in \mathbb{N}$ with $\alpha \in (0, n+1)$. Since $S_R^{-1}(s,\mathcal{I}) = (s-1)^{-1}\mathcal{I}$, we then have

$$\begin{aligned} \mathcal{I}^{-\alpha} &= (-1)^{n+1} \frac{\sin(\alpha\pi)}{\pi} \frac{n!}{(n-\alpha)\cdots(1-\alpha)} \int_0^{+\infty} \frac{t^{n-\alpha}}{(-t-1)^{n+1}}\,dt\,\mathcal{I} \\ &= \frac{\sin(\alpha\pi)}{\pi} \frac{n!}{(n-\alpha)\cdots(1-\alpha)} \int_0^{+\infty} \frac{t^{n-\alpha}}{(t+1)^{n+1}}\,dt\,\mathcal{I}. \end{aligned}$$

By [55, p. 3.194], we have

$$\int_0^{+\infty} \frac{t^{n-\alpha}}{(t+1)^{n+1}}\,dt = B(n-\alpha+1,\alpha) = \frac{(n-\alpha)\cdots(1-\alpha)}{n!} \frac{\pi}{\sin(\pi\alpha)}, \tag{7.14}$$

where $B(x,y)$ denotes the Beta function, and hence $\mathcal{I}^{-\alpha} = \mathcal{I}$.

□

Corollary 7.11. *Let $\alpha \in (0,1)$. Then*

$$T^{-\alpha} = -\frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} t^{-\alpha} S_R^{-1}(-t,T)\,dt. \tag{7.15}$$

Corollary 7.12. *For $\alpha \in (0, n+1)$, the operators $T^{-\alpha}$ are uniformly bounded by the constant M_{n+1} obtained from Lemma 7.5.*

Proof. From (7.12), Lemma 7.5 and (7.14), we obtain the estimate

$$\|T^{-\alpha}\| \leq \frac{\sin(\alpha\pi)}{\pi} \frac{n!}{(n-\alpha)\cdots(1-\alpha)} \int_0^{+\infty} t^{n-\alpha} \frac{M_{n+1}}{(1+t)^{n+1}}\,dt = M_{n+1}.$$

□

Corollary 7.13. *Assume that $\sigma_S(T) \subset \{s \in \mathbb{H} : \operatorname{Re}(s) > 0\}$ and that φ in Lemma 7.5 can be chosen lower or equal to $\pi/2$. For $\alpha \in (0,1)$, we then have*

$$T^{-\alpha} = \frac{1}{\pi} \int_0^{+\infty} \tau^{-\alpha} \left(\cos\left(\frac{\alpha\pi}{2}\right) T + \sin\left(\frac{\alpha\pi}{2}\right) \tau\mathcal{I} \right) (T^2 + \tau^2)^{-1}\,d\tau.$$

Proof. By our assumptions, we can choose $n = 0$ and $\theta = \pi/2$ in (7.13). Since $e^{\mathsf{i}\frac{\pi}{2}} = \mathsf{i}$ and $e^{-\mathsf{i}\frac{\pi}{2}} = -\mathsf{i}$, we then have

$$T^{-\alpha} = \frac{\mathsf{i}}{2\pi} \int_0^{+\infty} t^{-\alpha} e^{-\mathsf{i}\frac{\alpha-1}{2}\pi} S_R^{-1}(\mathsf{i}t,T)\,dt - \frac{\mathsf{i}}{2\pi} \int_0^{+\infty} t^{-\alpha} e^{\mathsf{i}\frac{\alpha-1}{2}\pi} S_R^{-1}(-\mathsf{i}t,T)\,dt.$$

We observe that

$$S_R^{-1}(\pm t\mathsf{i}, T) = -(T \pm t\mathsf{i}\mathcal{I})(T^2 + t^2)^{-1},$$

and so

$$T^{-\alpha} = \frac{\mathsf{i}}{2\pi} \int_0^{+\infty} t^{-\alpha} \left(-e^{-\mathsf{i}\frac{\alpha-1}{2}\pi}(T + t\mathsf{i}\mathcal{I}) + e^{\mathsf{i}\frac{\alpha-1}{2}\pi}(T - t\mathsf{i}\mathcal{I}) \right) (T^2 + t^2)^{-1}\,dt.$$

Some easy simplifications show

$$-e^{-\mathsf{i}\frac{\alpha-1}{2}\pi}(T + t\mathsf{i}\mathcal{I}) + e^{\mathsf{i}\frac{\alpha-1}{2}\pi}(T - t\mathsf{i}\mathcal{I}) = -2\mathsf{i}\left[\cos\left(\frac{\alpha\pi}{2}\right) T + 2\sin\left(\frac{\alpha\pi}{2}\right) t\mathcal{I}\right],$$

and in turn

$$T^{-\alpha} = \frac{1}{\pi} \int_0^{+\infty} t^{-\alpha} \left(\cos\left(\frac{\alpha\pi}{2}\right) T + \sin\left(\frac{\alpha\pi}{2}\right) t\mathcal{I} \right) (T^2 + t^2)^{-1}\,dt.$$

□

Observe that $s \mapsto s^{-\alpha}$ is intrinsic slice hyperholomorphic. Hence, we could also use the left S-resolvent operator to define fractional powers of T. Indeed, this yields exactly the same operator.

Proposition 7.14. *Let $\alpha > 0$ and let Γ be an admissible path as in Definition 7.6. The operator $T^{-\alpha}$ satisfies*

$$T^{-\alpha} = \frac{1}{2\pi} \int_\Gamma S_L^{-1}(s,T)\,ds_{\mathsf{i}}\, s^{-\alpha}. \tag{7.16}$$

Proof. We can proof this statement in two different ways. We can either perform computations as (4.12) in order to show that the slice hyperholomorphic Cauchy integral in (7.16) equals the one in (7.8).

As an alternative approach, we can perform computations analogue to those in the proof of Theorem 7.9 to show that, for $n \in \mathbb{N}$ and $\alpha \in (0, n+1)$ with $\alpha \notin \mathbb{N}$, one has

$$\begin{aligned}
&\frac{1}{2\pi} \int_{\Gamma} S_L^{-1}(s,T)\, ds_{\mathsf{i}}\, s^{-\alpha} \\
=&(-1)^{n+1} \frac{\sin(\alpha\pi)}{\pi} \frac{n!}{(n-\alpha)\cdots(1-\alpha)} \int_0^{+\infty} S_L^{-(n+1)}(-t,T) t^{n-\alpha}\, dt.
\end{aligned}$$

But for real t one has $S_L^{-1}(-t,T) = (-t-T)^{-1} = S_R^{-1}(-t,T)$, and in turn this integral equals

$$(-1)^{n+1} \frac{\sin(\alpha\pi)}{\pi} \frac{n!}{(n-\alpha)\cdots(1-\alpha)} \int_0^{+\infty} t^{n-\alpha} S_R^{-(n+1)}(-t,T)\, dt = T^{-\alpha},$$

where the last equation follows from Theorem 7.9. If $\alpha \in \mathbb{N}$, the statement follows immediately from the S-functional calculus and Corollary 7.8 because $s^{-\alpha}$ is intrinsic slice hyperholomorphic at infinity.

□

Theorem 7.15. *The family $\{T^{-\alpha}\}_{\alpha>0}$ has the semigroup property $T^{-\alpha}T^{-\beta} = T^{-(\alpha+\beta)}$.*

Proof. Let φ and a be the constants obtained from Lemma 7.5. We choose θ_p and θ_s such that $\max\{\varphi, \pi/2\} < \theta_p < \theta_s < \pi$ and we choose a_p and a_s with $0 < a_s < a_p < a$ and a_p sufficiently small such that $cl(B_{a_p}(0)) \cap cl(\Sigma_\varphi) + a = \emptyset$. Then the sets

$$G_p = \Sigma_{\theta_p} \setminus cl(B_{a_p}(0)) \quad \text{and} \quad G_s = \Sigma_{\theta_s} \setminus cl(B_{a_s}(0))$$

satisfy $\sigma_S(T) \subset G_p$ and $cl(G_p) \subset G_s$ and for $\mathsf{i} \in \mathbb{S}$ their boundaries $\partial(G_p \cap \mathbb{C}_{\mathsf{i}})$ and $\partial(G_s \cap \mathbb{C}_{\mathsf{i}})$ are admissible paths as in Definition 7.6. The subscripts p and s refer again to the respective variables of integration in the following calculation.

The S-resolvent equation (2.30) and Proposition 7.14 imply

$$\begin{aligned}
T^{-\alpha}T^{-\beta} =& \frac{1}{(2\pi)^2} \int_{\partial(G_s\cap\mathbb{C}_{\mathsf{i}})} s^{-\alpha}\, ds_{\mathsf{i}}\, S_R^{-1}(s,T) \int_{\partial(G_p\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(p,T)\, dp_{\mathsf{i}}\, p^{-\beta} \\
=&\frac{1}{(2\pi)^2} \int_{\partial(G_s\cap\mathbb{C}_{\mathsf{i}})} s^{-\alpha}\, ds_{\mathsf{i}} \int_{\partial(G_p\cap\mathbb{C}_{\mathsf{i}})} S_R^{-1}(s,T) p (p^2 - 2s_0 p + |s|^2)^{-1} dp_{\mathsf{i}}\, p^{-\beta} \\
&- \frac{1}{(2\pi)^2} \int_{\partial(G_s\cap\mathbb{C}_{\mathsf{i}})} s^{-\alpha}\, ds_{\mathsf{i}} \int_{\partial(G_p\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(p,T) p (p^2 - 2s_0 p + |s|^2)^{-1} dp_{\mathsf{i}}\, p^{-\beta} \\
&- \frac{1}{(2\pi)^2} \int_{\partial(G_s\cap\mathbb{C}_{\mathsf{i}})} s^{-\alpha}\, ds_{\mathsf{i}} \int_{\partial(G_p\cap\mathbb{C}_{\mathsf{i}})} \overline{s} S_R^{-1}(s,T) (p^2 - 2s_0 p + |s|^2)^{-1} dp_{\mathsf{i}}\, p^{-\beta} \\
&+ \frac{1}{(2\pi)^2} \int_{\partial(G_s\cap\mathbb{C}_{\mathsf{i}})} s^{-\alpha}\, ds_{\mathsf{i}} \int_{\partial(G_p\cap\mathbb{C}_{\mathsf{i}})} \overline{s} S_L^{-1}(p,T) (p^2 - 2s_0 p + |s|^2)^{-1} dp_{\mathsf{i}}\, p^{-\beta}.
\end{aligned}$$

But since the functions $p \mapsto p(p^2 - 2s_0 p + |s|^2)^{-1} p^{-\beta}$ and $p \mapsto (p^2 - 2s_0 p + |s|^2)^{-1} p^{-\beta}$ are holomorphic on an open set that contains $cl(G_p \cap \mathbb{C}_{\mathsf{i}})$ and since they tend uniformly

to zeros as $p \to \infty$ in G_p, Cauchy's integral theorem implies

$$\frac{1}{(2\pi)^2} \int_{\partial(G_s \cap \mathbb{C}_{\mathsf{i}})} s^{-\alpha}\, ds_{\mathsf{i}} \int_{\partial(G_p \cap \mathbb{C}_{\mathsf{i}})} S_R^{-1}(s,T) p (p^2 - 2s_0 p + |s|^2)^{-1} dp_{\mathsf{i}}\, p^{-\beta} = 0$$

and

$$-\frac{1}{(2\pi)^2} \int_{\partial(G_s \cap \mathbb{C}_{\mathsf{i}})} s^{-\alpha}\, ds_{\mathsf{i}} \int_{\partial(G_p \cap \mathbb{C}_{\mathsf{i}})} \overline{s} S_R^{-1}(s,T) (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_{\mathsf{i}}\, p^{-\beta} = 0.$$

It follows that

$$\begin{aligned} &T^{-\alpha}\, T^{-\beta} \\ &= -\frac{1}{(2\pi)^2} \int_{\partial(G_s \cap \mathbb{C}_{\mathsf{i}})} s^{-\alpha}\, ds_{\mathsf{i}} \int_{\partial(G_p \cap \mathbb{C}_{\mathsf{i}})} S_L^{-1}(p,T) p (p^2 - 2s_0 p + |s|^2)^{-1} dp_{\mathsf{i}}\, p^{-\beta} \\ &\quad + \frac{1}{(2\pi)^2} \int_{\partial(G_s \cap \mathbb{C}_{\mathsf{i}})} s^{-\alpha}\, ds_{\mathsf{i}} \int_{\partial(G_p \cap \mathbb{C}_{\mathsf{i}})} \overline{s} S_L^{-1}(p,T) (p^2 - 2s_0 p + |s|^2)^{-1} dp_{\mathsf{i}}\, p^{-\beta}. \end{aligned} \tag{7.17}$$

Let us assume that we can apply Fubini's theorem in order to exchange the order of integration. We then find that

$$\begin{aligned} T^{-\alpha} T^{-\beta} &= \frac{1}{(2\pi)^2} \int_{\partial(G_s \cap \mathbb{C}_{\mathsf{i}})} s^{-\alpha}\, ds_{\mathsf{i}} \\ &\quad \cdot \int_{\partial(G_p \cap \mathbb{C}_{\mathsf{i}})} [\overline{s} S_L^{-1}(p,T) - S_L^{-1}(p,T) p] (p^2 - 2s_0 p + |s|^2)^{-1} dp_{\mathsf{i}}\, p^{-\beta}. \end{aligned}$$

Applying Lemma 4.18 with $B = S_L^{-1}(p,T)$, we obtain

$$\begin{aligned} T^{-\alpha}\, T^{-\beta} &= \frac{1}{2\pi} \int_{\partial(G_p \cap \mathbb{C}_{\mathsf{i}})} S_L^{-1}(p,T) dp_{\mathsf{i}}\, p^{-\alpha}\, p^{-\beta} \\ &= \frac{1}{2\pi} \int_{\partial(G_p \cap \mathbb{C}_{\mathsf{i}})} S_L^{-1}(p,T) dp_{\mathsf{i}}\, p^{-(\alpha+\beta)} = T^{-(\alpha+\beta)}. \end{aligned}$$

What remains to show is that we can actually apply Fubini's theorem in (7.17). For $\tau \in \{s, p\}$ we therefore decompose $\partial(G_\tau \cap \mathbb{C}_{\mathsf{i}}) = \Gamma_\tau^- \cup \Gamma\tau^- \cup \Gamma_\tau^+$ with

$$\begin{aligned} \Gamma_\tau^- &= \left\{ -re^{\mathsf{i}\theta_\tau}, r \in (-\infty, -a_\tau] \right\} \\ \Gamma_\tau^0 &= \left\{ a_\tau e^{-\mathsf{i}\theta}, \theta \in (-\theta_\tau, \theta_\tau) \right\} \\ \Gamma_\tau^+ &= \left\{ re^{-\mathsf{i}\theta_\tau}, r \in [a_\tau, +\infty) \right\} \end{aligned}$$

such that

$$\begin{aligned} &T^{-\alpha}\, T^{-\beta} \\ &= \sum_{u,v \in \{+,0,-\}} -\frac{1}{(2\pi)^2} \int_{\Gamma_s^u} s^{-\alpha}\, ds_{\mathsf{i}} \int_{\Gamma_p^v} S_L^{-1}(p,T) p (p^2 - 2s_0 p + |s|^2)^{-1} dp_{\mathsf{i}}\, p^{-\beta} \\ &\quad + \sum_{u,v \in \{+,0,-\}} \frac{1}{(2\pi)^2} \int_{\Gamma_s^u} s^{-\alpha}\, ds_{\mathsf{i}} \int_{\Gamma_p^v} \overline{s} S_L^{-1}(p,T) (p^2 - 2s_0 p + |s|^2)^{-1} dp_{\mathsf{i}}\, p^{-\beta}. \end{aligned} \tag{7.18}$$

Since p and s commute, we have $(p^2-2s_0p+|s|^2)^{-1}=(p-s)^{-1}(p-\overline{s})^{-1}$ and thus for $u=+$ and $v=+$

$$\begin{aligned}
&\int_{\Gamma_s^+} s^{-\alpha}\,ds_{\mathsf{i}}\int_{\Gamma_p^+} S_L^{-1}(p,T)p(p^2-2s_0p+|s|^2)^{-1}dp_{\mathsf{i}}\,p^{-\beta}\\
=&\int_{a_s}^{+\infty} r^{-\alpha}e^{\mathsf{i}\alpha\theta_s}e^{-\mathsf{i}\theta_s}(-\mathsf{i})\int_{a_p}^{+\infty} S_L^{-1}(te^{-\mathsf{i}\theta_p},T)\cdot\\
&\qquad\cdot te^{-\mathsf{i}\theta_p}\left(te^{-\mathsf{i}\theta_p}-re^{-\mathsf{i}\theta_s}\right)^{-1}\left(te^{-\mathsf{i}\theta_p}-re^{\mathsf{i}\theta_s}\right)^{-1}e^{-\mathsf{i}\theta_p}(-\mathsf{i})t^{-\beta}e^{\mathsf{i}\beta\theta_p}\,dt\,dr.
\end{aligned}$$

Using the estimate

$$\|S_L^{-1}(s,T)\|\le\frac{M_1}{1+|s|}\tag{7.19}$$

obtained from Lemma 7.5 and setting

$$C:=\sup_{t\in[0,+\infty)}\frac{M_1t}{1+t}<+\infty,\tag{7.20}$$

we find that the integral of the norm of the integrand is lower or equal to

$$\begin{aligned}
&\int_{a_s}^{+\infty}\int_{a_p}^{+\infty} r^{-\alpha}\frac{M_1t}{1+t}\frac{1}{|t-re^{\mathsf{i}(\theta_p-\theta_s)}|}\frac{1}{|t-re^{\mathsf{i}(\theta_p+\theta_s)}|}t^{-\beta}\,dt\,dr\\
\le&C\int_{a_s}^{+\infty}\int_{a_p}^{+\infty}\frac{r^{-(1+\alpha)}}{\left|\frac{t}{r}-e^{-\mathsf{i}(\theta_s-\theta_p)}\right|}\frac{t^{-(1+\beta)}}{\left|\frac{r}{t}-e^{-\mathsf{i}(\theta_s+\theta_p)}\right|}\,dt\,dr.
\end{aligned}\tag{7.21}$$

Since $\pi/2<\theta_p<\theta_s<\pi$, we have $0<\theta_s-\theta_p<\pi$ and $\pi<\theta_s+\theta_p<2\pi$. Therefore

$$\inf_{\xi\in\mathbb{R}}|\xi-e^{-\mathsf{i}(\theta_s\pm\theta_p)}|<\inf_{\xi\in\mathbb{R}}\left|\operatorname{Im}\left(\xi-e^{-\mathsf{i}(\theta_s\pm\theta_p)}\right)\right|=\inf_{\xi\in\mathbb{R}}|\sin(\theta_s\pm\theta_p)|\ \neq 0$$

and so

$$\begin{aligned}
K_1&:=\sup_{\xi\in[0,+\infty)}\frac{1}{|\xi-e^{-\mathsf{i}(\theta_s-\theta_p)}|}<+\infty\\
K_2&:=\sup_{\xi\in[0,+\infty)}\frac{1}{\left|\frac{r}{t}-e^{-\mathsf{i}(\theta_s+\theta_p)}\right|}<+\infty.
\end{aligned}\tag{7.22}$$

As $\alpha,\beta>0$ and $a_s,a_p>0$, we conclude from (7.21) that

$$\begin{aligned}
&\int_{a_s}^{+\infty}\int_{a_p}^{+\infty} r^{-\alpha}\frac{M_1t}{1+t}\frac{1}{|t-re^{\mathsf{i}(\theta_p-\theta_s)}|}\frac{1}{|t-re^{\mathsf{i}(\theta_p+\theta_s)}|}t^{-\beta}\,dt\,dr\\
\le&CK_1K_2\int_{a_s}^{+\infty} r^{-(1+\alpha)}\,dr\int_{a_p}^{+\infty} t^{-(1+\beta)}\,dt<+\infty.
\end{aligned}\tag{7.23}$$

The second integral in (7.18) with $u=+$ and $v=+$ is

$$\begin{aligned}
&\int_{\Gamma_s^+} s^{-\alpha}\,ds_{\mathsf{i}}\int_{\Gamma_p^+}\overline{s}S_L^{-1}(p,T)(p^2-2s_0p+|s|^2)^{-1}\,dp_{\mathsf{i}}\,p^{-\beta}\\
=&\int_{a_s}^{+\infty} r^{-\alpha}e^{\mathsf{i}\alpha\theta_s}e^{-\mathsf{i}\theta_s}(-\mathsf{i})\int_{a_p}^{+\infty} re^{\mathsf{i}\theta_s}S_L^{-1}(te^{\mathsf{i}\theta_p},T)\cdot\\
&\qquad\cdot\left(te^{-\mathsf{i}\theta_p}-re^{-\mathsf{i}\theta_s}\right)^{-1}\left(te^{-\mathsf{i}\theta_p}-re^{\mathsf{i}\theta_s}\right)^{-1}e^{-\mathsf{i}\theta_p}(-\mathsf{i})t^{-\beta}e^{\mathsf{i}\beta\theta_p}\,dt\,dr.
\end{aligned}$$

With (7.19) and (7.22), we can estimate the integral of the norm of the integrand by

$$
\begin{aligned}
&\int_{a_s}^{+\infty}\int_{a_p}^{+\infty} r^{-\alpha+1}\frac{M_1}{1+t}\frac{1}{|te^{-\mathsf{i}(\theta_p-\theta_s)}-r|}\frac{1}{|te^{-\mathsf{i}(\theta_p+\theta_s)}-r|}t^{-\beta}\,dt\,dr\\
\le& M_1\int_{a_s}^{+\infty} r^{-(1+\alpha)}\int_{a_p}^{+\infty}\frac{t^{-\beta}}{1+t}\frac{1}{\left|\frac{t}{r}-e^{-\mathsf{i}(\theta_s-\theta_p)}\right|}\frac{1}{\left|\frac{t}{r}-e^{\mathsf{i}(\theta_p+\theta_s)}\right|}\,dt\,dr\\
\le& M_1K_1K_2\int_{a_s}^{+\infty} r^{-(1+\alpha)}\,dr\int_{a_p}^{+\infty}\frac{t^{-\beta}}{1+t}\,dt<+\infty.
\end{aligned}
$$

For $u=+$ and $v=0$, we have

$$
\begin{aligned}
&\int_{\Gamma_s^+} s^{-\alpha}\,ds_{\mathsf{i}}\int_{\Gamma_p^0} S_L^{-1}(p,T)p(p^2-2s_0p+|s|^2)^{-1}\,dp_{\mathsf{i}}\,p^{-\beta}\\
=&\int_{a_s}^{+\infty} r^{-\alpha}e^{\mathsf{i}\alpha\theta_s}e^{-\mathsf{i}\theta_s}(-\mathsf{i})\int_{-\theta_p}^{\theta_p} S_L^{-1}(a_pe^{-\mathsf{i}\theta},T)a_pe^{-\mathsf{i}\theta}\cdot\\
&\qquad\cdot\left(a_pe^{-\mathsf{i}\theta}-re^{-\mathsf{i}\theta_s}\right)^{-1}\left(a_pe^{-\mathsf{i}\theta}-re^{\mathsf{i}\theta_s}\right)^{-1}a_pe^{-\mathsf{i}\theta}(-\mathsf{i})^2a_p^{-\beta}e^{\mathsf{i}\beta\theta}\,d\theta\,dr
\end{aligned}
$$

and, again using (7.19), we find that the integral of the absolute value of the integrand is lower or equal to

$$
\int_{a_s}^{+\infty} r^{-\alpha}\int_{-\theta_p}^{\theta_p}\frac{M_1a_p^{2-\beta}}{1+a_p}\frac{1}{|r-a_pe^{-\mathsf{i}(\theta-\theta_s)}|}\frac{1}{|r-a_pe^{-\mathsf{i}(\theta+\theta_s)}|}\,d\theta\,dr.
$$

Since $\pi/2<\theta_p<\theta_s<\pi$, the distance δ between the set

$$
\left\{a_pe^{-\mathsf{i}(\theta+\theta_s)},\theta\in[-\theta_p,\theta_p]\right\}\cup\left\{a_pe^{-\mathsf{i}(\theta-\theta_s)},\theta\in[-\theta_p,\theta_p]\right\}\tag{7.24}
$$

and the positive real axis is greater than zero and hence the above integral can be estimated by

$$
\begin{aligned}
&\int_{a_s}^{a_s+2a_p} r^{-\alpha}\int_{-\theta_p}^{\theta_p}\frac{M_1a_p^{2-\beta}}{1+a_p}\frac{1}{\delta^2}\,dr\,d\theta\\
&+\int_{a_s+2a_p}^{+\infty}\frac{r^{-\alpha}}{(r-a_p)^2}\int_{-\theta_p}^{\theta_p}\frac{M_1a_p^{2-\beta}}{1+a_p}\,d\theta\,dr<+\infty.
\end{aligned}
$$

For the second integral in (7.18) with $u=+$ and $v=0$, we have

$$
\begin{aligned}
&\int_{\Gamma_s^+} s^{-\alpha}\,ds_{\mathsf{i}}\int_{\Gamma_p^0}\overline{s}S_L^{-1}(p,T)(p^2-2s_0p+|s|^2)^{-1}\,dp_{\mathsf{i}}\,p^{-\beta}\\
=&\int_{a_s}^{+\infty} r^{-\alpha}e^{\mathsf{i}\alpha\theta_s}e^{-\mathsf{i}\theta_s}(-\mathsf{i})\int_{-\theta_p}^{\theta_p} re^{\mathsf{i}\theta_s}S_L^{-1}(a_pe^{-\mathsf{i}\theta},T)\cdot\\
&\qquad\cdot\left(a_pe^{-\mathsf{i}\theta}-re^{-\mathsf{i}\theta_s}\right)^{-1}\left(a_pe^{-\mathsf{i}\theta}-re^{\mathsf{i}\theta_s}\right)^{-1}a_pe^{-\mathsf{i}\theta}(-\mathsf{i})^2a_p^{-\beta}e^{\mathsf{i}\beta\theta}\,d\theta\,dr.
\end{aligned}
$$

Using (7.19), we can estimate the integral of the absolute value of the integrand by

$$\begin{aligned}
&\int_{a_s}^{+\infty} r^{1-\alpha} \int_{-\theta_p}^{\theta_p} \frac{a_p^{1-\beta} M_1}{1+a_p} \frac{1}{|a_p e^{-\mathsf{i}(\theta-\theta_s)} - r|} \frac{1}{|a_p e^{-\mathsf{i}(\theta+\theta_s)} - r|}\, d\theta\, dr \\
\leq &\int_{a_s}^{a_s+2a_p} r^{1-\alpha} \int_{-\theta_p}^{\theta_p} \frac{a_p^{1-\beta} M_1}{1+a_p} \frac{1}{\delta^2}\, d\theta\, dr \\
&+ \int_{a_s+2a_p}^{+\infty} \frac{r^{1-\alpha}}{(r-a_p)^2} \int_{-\theta_p}^{\theta_p} \frac{a_p^{1-\beta} M_1}{1+a_p}\, d\theta\, dr < +\infty,
\end{aligned}$$

where $\delta > 0$ is again the distance between the set in (7.24) and the positive real axis. Similar estimates hold true if $u = -$ and $v = 0$.

If $u = 0$ and $v = +$, then

$$\begin{aligned}
&\int_{\Gamma_s^0} s^{-\alpha}\, ds_{\mathsf{i}} \int_{\Gamma_p^+} S_L^{-1}(p,T) p (p^2 - 2s_0 p + |s|^2)^{-1} dp_{\mathsf{i}}\, p^{-\beta} \\
= &\int_{-\theta_s}^{\theta_s} a_s^{-\alpha} e^{\mathsf{i}\alpha\theta} a_s e^{-\mathsf{i}\theta} (-\mathsf{i})^2 \int_{a_p}^{+\infty} S_L^{-1}(t e^{-\mathsf{i}\theta_p}, T) t e^{-\mathsf{i}\theta_s} \cdot \\
&\qquad \cdot \left(t e^{-\mathsf{i}\theta_p} - a_s e^{-\mathsf{i}\theta}\right)^{-1} \left(t e^{-\mathsf{i}\theta_p} - a_s e^{\mathsf{i}\theta}\right)^{-1} e^{-\mathsf{i}\theta_s} (-\mathsf{i}) t^{-\beta} e^{\mathsf{i}\beta\theta_p}\, dt\, d\theta.
\end{aligned}$$

Once more (7.19) allows us to estimate the integral of the absolute value of the integrand by

$$\begin{aligned}
&\int_{-\theta_s}^{\theta_s} a_s^{1-\alpha} \int_{a_p}^{+\infty} \frac{M_1 t}{1+t} \frac{1}{|t - a_s e^{\mathsf{i}(\theta_p - \theta)}|} \frac{1}{|t - a_s e^{\mathsf{i}(\theta_p + \theta)}|} t^{-\beta}\, dt\, d\theta \\
\leq &C \int_{-\theta_s}^{\theta_s} a_s^{1-\alpha} \int_{a_p}^{+\infty} \frac{t^{-\beta}}{(t-a_s)^2}\, dt\, d\theta < +\infty,
\end{aligned}$$

where C is again as in (7.20), and the second inequality follows because $a_s < a_p$. For the second integral in (7.18) with $u = 0$ and $v = +$, we similarly have

$$\begin{aligned}
&\int_{\Gamma_s^0} s^{-\alpha}\, ds_{\mathsf{i}} \int_{\Gamma_p^+} \overline{s} S_L^{-1}(p,T) (p^2 - 2s_0 p + |s|^2)^{-1} dp_{\mathsf{i}}\, p^{-\beta} \\
= &\int_{-\theta_s}^{\theta_s} a_s^{-\alpha} e^{\mathsf{i}\alpha\theta} a_s e^{-\mathsf{i}\theta} (-\mathsf{i})^2 \int_{a_p}^{+\infty} a_s e^{\mathsf{i}\theta_s} S_L^{-1}(t e^{-\mathsf{i}\theta_p}, T) \cdot \\
&\qquad \cdot \left(t e^{-\mathsf{i}\theta_p} - a_s e^{-\mathsf{i}\theta}\right)^{-1} \left(t e^{-\mathsf{i}\theta_p} - a_s e^{\mathsf{i}\theta}\right)^{-1} e^{-\mathsf{i}\theta_s} (-\mathsf{i}) t^{-\beta} e^{\mathsf{i}\beta\theta_p}\, dt\, d\theta.
\end{aligned}$$

As above, we can estimate the integral of the absolute value of the integrand by

$$\begin{aligned}
&\int_{-\theta_s}^{\theta_s} a_s^{2-\alpha} \int_{a_p}^{+\infty} \frac{M_1}{1+t} \frac{1}{|t - a_s e^{\mathsf{i}(\theta_p - \theta)}|} \frac{1}{|t - a_s e^{\mathsf{i}(\theta_p + \theta)}|} t^{-\beta}\, dt\, d\theta \\
\leq &C \int_{-\theta_s}^{\theta_s} a_s^{2-\alpha} \int_{a_p}^{+\infty} \frac{t^{-(1+\beta)}}{(t-a_s)^2}\, dt\, d\theta < +\infty,
\end{aligned}$$

where the last inequality follows again because $a_s < a_p$. Similar estimates hold for the case $u = 0$ and $v = -$.

Finally, the integrals in (7.18) with $u = 0$ and $v = 0$ are absolutely convergent since, in this case, we integrate a continuous and hence bounded function over a bounded domain.

Putting these pieces together, we obtain that we can actually apply Fubini's theorem in (7.17) in order to exchange the order of integration.

□

Lemma 7.16. *Since* $\mathrm{dom}(T)$ *is dense in* V*, the semigroup* $(T^{-\alpha})_{\alpha\geq 0}$ *is strongly continuous.*

Proof. We first consider $\mathbf{v} \in \mathrm{dom}(T)$. For $\alpha \in (0,1)$, we have

$$S_R^{-1}(t,T)\mathbf{v} - S_R^{-1}(t,\mathcal{I})\mathbf{v} = S_R^{-1}(t,T)S_R^{-1}(t,\mathcal{I})(T\mathbf{v} - \mathcal{I}\mathbf{v})$$

if $t \in \mathbb{R}$. Hence, we deduce from Corollary 7.11 that

$$\begin{aligned} &T^{-\alpha}\mathbf{v} - \mathcal{I}^{-\alpha}\mathbf{v} = \\ &= -\frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty} t^{-\alpha}S_R^{-1}(-t,T)\mathbf{v}\,dt + \frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty} t^{-\alpha}S_R^{-1}(-t,\mathcal{I})\mathbf{v}\,dt \\ &= -\frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty} t^{-\alpha}S_R^{-1}(t,T)S_R^{-1}(t,\mathcal{I})(T\mathbf{v} - \mathcal{I}\mathbf{v})\,dt \overset{\alpha\to 0}{\longrightarrow} 0 \end{aligned}$$

because $\sin(\alpha\pi) \to 0$ as $\alpha \to 0$ and the integral is uniformly bounded for $\alpha \in [0, 1/2]$ due to (7.3). Since $\mathcal{I}^{-\alpha} = \mathcal{I}$ by Corollary 7.10, we get $T^{-\alpha}\mathbf{v} \to \mathbf{v}$ as $\alpha \to 0$ for any $\mathbf{v} \in \mathrm{dom}(T)$.

For arbitrary $\mathbf{v} \in V$ and $\varepsilon > 0$, there exists $\mathbf{v}_\varepsilon \in \mathrm{dom}(T)$ with $\|\mathbf{v} - \mathbf{v}_\varepsilon\| < \varepsilon$ because $\mathrm{dom}(T)$ is dense in V. Corollary 7.12 therefore implies

$$\begin{aligned} \lim_{\alpha\to 0}\|T\mathbf{v} - \mathbf{v}\| &\leq \lim_{\alpha\to 0}\|T\mathbf{v} - T^{-\alpha}\mathbf{v}_\varepsilon\| + \|T^{-\alpha}\mathbf{v}_\varepsilon - \mathbf{v}_\varepsilon\| + \|\mathbf{v}_\varepsilon - \mathbf{v}\| \\ &\leq (M_1 + 1)\|\mathbf{v} - \mathbf{v}_\varepsilon\| \leq (M_1+1)\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we deduce that $T^{-\alpha}\mathbf{v} \to \mathbf{v}$ as $\alpha \to 0$ even for arbitrary $\mathbf{v} \in V$. This is equivalent to the strong continuity of the semigroup $(T^{-\alpha})_{\alpha\geq 0}$.

□

Proposition 7.17. *The operator* $T^{-\alpha}$ *is injective for any* $\alpha > 0$.

Proof. For $\alpha > 0$ choose $\beta > 0$ with $n = \alpha + \beta \in \mathbb{N}$. Then $T^{-\beta}T^{-\alpha} = T^{-n}$ and in turn $T^nT^{-\beta}T^{-\alpha} = \mathcal{I}$, which implies the injectivity of $T^{-\alpha}$.

□

The previous proposition allows us to define powers of T also for $\alpha > 0$.

Definition 7.18. For $\alpha > 0$ we define the operator T^α as the inverse of the operator $T^{-\alpha}$, which is defined on $\mathrm{dom}(T^\alpha) = \mathrm{ran}(T^{-\alpha})$.

Corollary 7.19. *Let* $\alpha, \beta \in \mathbb{R}$. *Then the operators* $T^\alpha T^\beta$ *and* $T^{\alpha+\beta}$ *agree on* $\mathrm{dom}(T^\gamma)$ *with* $\gamma = \max\{\alpha, \beta, \alpha+\beta\}$.

Proof. If $\alpha, \beta \geq 0$ and $\mathbf{v} \in \operatorname{dom}(T^{\alpha+\beta})$ then, since $T^{-(\alpha+\beta)} = T^{-\beta}T^{-\alpha}$ by Theorem 7.15, we have

$$T^{\alpha}T^{\beta}\mathbf{v} = T^{\alpha}T^{\beta}(T^{-\beta}T^{-\alpha}T^{\alpha+\beta})\mathbf{v} = (T^{\alpha}T^{\beta}T^{-\beta}T^{-\alpha})T^{\alpha+\beta}\mathbf{v} = T^{\alpha+\beta}\mathbf{v}.$$

The other cases follow in a similar way.

□

With these definitions it is possible to establish a theory of interpolation spaces for strongly continuous quaternionic semigroups analogue to the one for complex operator semigroups. Since the proofs follow the lines of this classical case and the quaternionic theory does not show any significant difference to the complex one, we refer to Chapter II in [39] for an overview on these results.

7.2 Fractional Powers via the H^∞-Functional Calculus

The most general strategy for introducing fractional powers of complex linear operators is via the H^∞-functional calculus. It applies to arbitrary sectorial operators and does not require further assumptions on the operator. Following [59], we show now that this is also possible in the quaternionic setting.

Let $T \in \operatorname{Sect}(\omega)$ and let $\alpha \in (0, +\infty)$. The function $s \mapsto s^\alpha$ does then obviously belong to $\mathcal{M}[\Sigma_\omega]_T$ and we can define T^α using the quaternionic H^∞-functional calculus introduced in Chapter 6. Precisely, we can choose $n \in \mathbb{N}$ with $n > \alpha$ and find

$$T^{\alpha} := s^{\alpha}(T) = (\mathcal{I} + T)^{n}\left(s^{\alpha}(1+s)^{-n}\right)(T), \tag{7.25}$$

where $\left(s^{\alpha}(1+s)^{-n}\right)(T)$ is defined via a slice hyperholomorphic Cauchy integral as in (6.2) or (6.3).

Definition 7.20. Let $T \in \operatorname{Sect}(\omega)$ and $\alpha > 0$. We call the operator defined in (7.25) the fractional power with exponent α of T.

The following properties are immediate consequences of the properties of the H^∞-functional calculus.

Lemma 7.21. *Let $T \in \operatorname{Sect}(\omega)$ and let $\alpha \in (0, +\infty)$.*

(i) If T is injective, then $(T^{-1})^{\alpha} = (T^{\alpha})^{-1}$. Thus $0 \in \rho_S(T)$ if and only if $0 \in \rho_S(T^\alpha)$.

(ii) Any bounded operator that commutes with T commutes also with T^α.

(iii) The spectral mapping theorem holds, namely

$$\sigma_S(T^\alpha) = \{s^\alpha : s \in \sigma_S(T)\}.$$

Another important property is analyticity in the exponent. Observe that, although in the complex case the mapping $\alpha \mapsto T^\alpha$ is holomorphic in α, we cannot expect slice hyperholomorphicity here because the fractional powers are only defined for real exponents, cf. Remark 7.4.

Proposition 7.22. *If $T \in \mathrm{Sect}(\omega)$, then the following statements hold true.*

(i) If T is bounded, then T^α is bounded too and the mapping $\Lambda : \alpha \to T^\alpha$ is analytic on $(0, +\infty)$ and has a left and a right slice hyperholomorphic extension to $\mathbb{H}^+ = \{s \in \mathbb{H} : \mathrm{Re}(s) > 0\}$. In particular, for any $\alpha_0 \in (0, +\infty)$ the Taylor series expansion of f_α at α_0 converges on $(0, 2\alpha_0)$.

(ii) If $n \in \mathbb{N}$ and $0 < \alpha < n$, then $\mathrm{dom}(T^n) \subset \mathrm{dom}(T^\alpha)$. For each $\mathbf{v} \in \mathrm{dom}(T^n)$, the mapping $\Lambda_{\mathbf{v}} : \alpha \mapsto A^\alpha \mathbf{v}$ is analytic on $(0, n)$ and the power series expansion of $\Lambda_{\mathbf{v}}$ at $\alpha_0 \in (0, n)$ converges on $(-r_{\alpha_0} + \alpha, \alpha + r_{\alpha_0})$ with $r_{\alpha_0} = \min\{\alpha_0, n - \alpha_0\}$. Hence, $\Lambda_{\mathbf{v}}$ has a left and a right slice hyperholomorphic expansion to the set $\bigcup_{\alpha_0 \in (0,n)} B_{r_{\alpha_0}}(\alpha_0)$.

Proof. Let us first show (ii). If $n \in \mathbb{N}$ and $\alpha \in (0, n)$, then

$$T^\alpha = (\mathcal{I} + T)^n \left(s^\alpha (1+s)^{-n}\right)(T).$$

If $\mathbf{v} \in \mathrm{dom}\,(T^n)$, then T^n and $(s^\alpha(1+s)^{-n})(T)$ commute because of Lemma 6.31(i) such that $T^\alpha \mathbf{v} = (s^\alpha(1+s)^{-n})(T)(\mathcal{I}+T)^n \mathbf{v}$ and hence $\mathbf{v} \in \mathrm{dom}(T^\alpha)$.

Let now $\alpha_0 \in (0, n)$ and set $r := r_{\alpha_0} = \min\{\alpha_0, n - \alpha_0\}$. The Taylor series expansion of $\alpha \mapsto s^\alpha$ at α_0 is $s^\alpha = \sum_{n=0}^{+\infty} \frac{(\alpha-\alpha_0)^k}{k!} s^{\alpha_0} \log(s)^k$ and converges on $(0, 2\alpha_0)$. If $\varepsilon \in (0,1)$ and $\alpha \in (0,n)$ with $|\alpha - \alpha_0| < (1-\varepsilon)r$, then we have after choosing $\varphi \in (\omega, \pi)$ that

$$\begin{aligned} T^\alpha \mathbf{v} &= \left(s^\alpha(1+s)^{-n}\right)(T)(\mathcal{I}+T)^n \mathbf{v} \\ &= \frac{1}{2\pi} \int_{\partial(\Sigma_\varphi \cap \mathbb{C}_{\mathbf{i}})} s^\alpha (1+s)^{-n}\, ds_{\mathbf{i}}\, S_R^{-1}(s,T)(\mathcal{I}+T)^n \mathbf{v} \\ &= \frac{1}{2\pi} \int_{\partial(\Sigma_\varphi \cap \mathbb{C}_{\mathbf{i}})} \sum_{k=0}^{+\infty} \frac{(\alpha-\alpha_0)^k}{k!} s^{\alpha_0} \log(s)^k (1+s)^{-n}\, ds_{\mathbf{i}}\, S_R^{-1}(s,T)(\mathcal{I}+T)^n \mathbf{v}. \end{aligned} \tag{7.26}$$

We want to apply the theorem of dominated convergence in order to exchange the integral and the series. Using (6.1) we find that $\tilde{\Psi}(s) = M \left\|(\mathcal{I}+T)^n \mathbf{v}\right\| \Psi(s)$ with

$$\Psi(s) := \sum_{k=0}^{+\infty} \frac{|\alpha - \alpha_0|^k}{k!} |s|^{\alpha_0 - 1} \frac{|\log(s)|^k}{|1+s|^n}$$

is a dominating function for the integrand in (7.26). In order to show the integrability of $\Psi(s)$ along $\partial(\Sigma_\varphi \cap \mathbb{C}_{\mathbf{i}})$, we choose $C_{est} > 1$ such that $(1-\varepsilon)C_{est} < 1$ and $0 < t_0 < 1$ and $1 < t_1$ such that

$$|\ln(t) + I\theta| < C_{est} |\ln(t)| \quad \forall t \in (0, t_0] \cup [t_1, +\infty). \tag{7.27}$$

We then have

$$\begin{aligned} \frac{1}{2} \int_{\partial(\Sigma_\varphi \cap \mathbb{C}_{\mathbf{i}})} \psi(s)\, d|s_{\mathbf{i}}| &= \int_0^{+\infty} \sum_{k=0}^{+\infty} \frac{|\alpha-\alpha_0|^k}{k!} t^{\alpha_0 - 1} \frac{|\ln(t) + \mathbf{i}\varphi|^k}{|1 + te^{\mathbf{i}\varphi}|^n}\, dt \\ \leq & \sum_{k=0}^{+\infty} \frac{|\alpha-\alpha_0|^k}{k!} \bigg(C_0 C_{est}^k \int_0^{t_0} t^{\alpha_0-1}(-\ln(t))^k\, dt \\ & + C_0 C_1^k \int_{t_0}^{t_1} t^{\alpha_0 - 1}\, dt + C_2 C_{est}^k \int_{t_1}^{+\infty} t^{\alpha_0 - (n+1)} \ln(t)^k\, dt \bigg), \end{aligned}$$

with the constants

$$C_0 := \max_{t\in[0,t_1]} \frac{1}{|1+te^{\mathbf{i}\varphi}|^n}, \qquad C_1 := \max_{t\in[t_0,t_1]} |\ln(t) + \mathbf{i}\varphi|$$

and a constant $C_2 > 0$ such that

$$\frac{1}{|1+te^{\mathbf{i}\varphi}|} < \frac{C_2}{t} \qquad \forall t \in [t_1, +\infty).$$

Since

$$\int_0^{t_1} t^{\alpha_0-1}(-\ln(t))^k\,dt \le \int_{-\infty}^0 e^{\alpha_0\xi}(-\xi)^k\,d\xi = \frac{k!}{\alpha_0^{k+1}}$$

and similarly

$$\int_{t_1}^{+\infty} t^{\alpha_0-(n+1)}\ln(t)^k\,dt \le \int_1^{+\infty} e^{-(n-\alpha_0)\xi}\xi^k\,d\xi = \frac{k!}{(n-\alpha_0)^{k+1}},$$

we can further estimate

$$\begin{aligned}&\frac{1}{2}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_i)}\Psi(s)\,d|s_{\mathbf{i}}| \le \\ &\le \sum_{k=0}^{+\infty}\frac{|\alpha-\alpha_0|^k}{k!}\left(C_0C_{est}^k\frac{k!}{\alpha_0^{k+1}} + C_0C_1^k\left(\frac{t_1^{\alpha_0}-t_0^{\alpha_0}}{\alpha_0}\right) + C_2C_{est}^k\frac{k!}{(n-\alpha_0)^{k+1}}\right).\end{aligned}$$

As $|\alpha-\alpha_0| < (1-\varepsilon)r = (1-\varepsilon)\min\{\alpha_0, n-\alpha_0\}$, we finally find

$$\begin{aligned}\frac{1}{2}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_i)}\Psi(s)\,d|s_{\mathbf{i}}| &\le \frac{C_0}{\alpha_0}\sum_{k=0}^{+\infty}((1-\varepsilon)C_{est})^k \\ &\quad + \frac{C_0(t_1^{\alpha_0}-t_0^{\alpha_0})}{\alpha_0}\sum_{k=0}^{+\infty}\frac{(C_1|\alpha-\alpha_0|)^k}{k!} + \frac{C_2}{\alpha_0}\sum_{k=0}^{+\infty}((1-\varepsilon)C_{est})^k.\end{aligned}$$

Since $(1-\varepsilon)C_1 < 1$ these series are finite and hence $\tilde{\Psi}$ is an integrable majorant of the integrand in (7.26). We can thus exchange the series and the integral in (7.26) such that

$$T^\alpha\mathbf{v} = \sum_{k=0}^{+\infty}\frac{(\alpha-\alpha_0)^k}{k!}\frac{1}{2\pi}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_{\mathbf{i}})} s^{\alpha_0}\log(s)^k(1+s)^{-n}\,ds_{\mathbf{i}}\,S_R^{-1}(s,T)(\mathcal{I}+T)^n\mathbf{v},$$

where this series converges uniformly for $|\alpha-\alpha_0| < (1-\varepsilon)r$. Since $\varepsilon\in(0,1)$ was arbitrary, we obtain the statement.

If T is bounded, then (7.25) is the composition of two bounded operators and hence bounded. With arguments as the ones used above one can show that the power series expansion of Λ at α_0 converges in $\mathcal{B}(V)$ on $(0, 2\alpha_0)$. If we write the scalar variable $(\alpha-\alpha_0)$ in the power series expansion on the left or on the right side of the coefficients and extend α to a quaternionic variable, we find that Λ has a left resp. a right slice hyperholomorphic extension to $B_{\alpha_0}(\alpha_0)$. Finally, any point in $\mathbb{H}^+$ is contained in a ball of this form, and hence we find that we can extend Λ to a left or to a right slice hyperholomorphic function on all of $\mathbb{H}^+$.

□

We show now that the usual computational rules that we expect to hold for fractional powers of an operator hold true with our approach.

Proposition 7.23 (First Law of Exponents)**.** *Let* $T \in \mathrm{Sect}(\omega)$*. For all* $\alpha, \beta > 0$ *the identity* $T^{\alpha+\beta} = T^\alpha T^\beta$ *holds. In particular* $\mathrm{dom}(T^\gamma) \subset \mathrm{dom}(T^\alpha)$ *for* $0 < \alpha < \gamma$*.*

Proof. Because of (ii) in Lemma 6.31, we have $T^\alpha T^\beta \subset T^{\alpha+\beta}$ with $\mathrm{dom}\left(T^\alpha T^\beta\right) = \mathrm{dom}\left(T^{\alpha+\beta}\right) \cap \mathrm{dom}\left(T^\beta\right)$. We choose $n \in \mathbb{N}$ with $\alpha, \beta < n$ and define the bounded operators

$$\Lambda_\alpha := \left(s^\alpha(1+s)^{-n}\right)(T) \qquad \text{and} \qquad \Lambda_\beta := \left(s^\beta(1+s)^{-n}\right)(T).$$

If now $\mathbf{v} \in \mathrm{dom}\left(T^{\alpha+\beta}\right)$, then (ii) in Lemma 6.31 implies

$$\begin{aligned} T^{\alpha+\beta}\mathbf{v} =& (\mathcal{I}+T)^{2n}(\mathcal{I}+T)^{-2n}T^{\alpha+\beta}\mathbf{v} = (\mathcal{I}+T)^{2n}T^{\alpha+\beta}(\mathcal{I}+T)^{-2n}\mathbf{v} \\ =& (\mathcal{I}+T)^{2n}\left(s^{\alpha+\beta}(1+s)^{-2n}\right)(T)\mathbf{v} = (\mathcal{I}+T)^{2n}\Lambda_\alpha\Lambda_\beta\mathbf{v} \end{aligned}$$

and hence $\Lambda_\alpha\Lambda_\beta\mathbf{v} \in \mathrm{dom}\left((\mathcal{I}+T^{2n})\right) = \mathrm{dom}\left(T^{2n}\right)$. Since $\left(s^{n-\alpha}(1+s)^{-n}\right)(T)$ commutes with T^{2n} because of (i) in Lemma 6.31, we thus find

$$T^n(\mathcal{I}+T)^{-2n}\Lambda_\beta\mathbf{v} = \left(s^{n+\beta}(1+s)^{-3n}\right)(T)\mathbf{v} = \left(s^{n-\alpha}(1+s)^{-n}\right)(T)\Lambda_\alpha\Lambda_\beta\mathbf{v}$$

belongs to $\mathrm{dom}\left(T^{2n}\right)$. Since T and $T(\mathcal{I}+T)^{-1}$ commute, we have $T(\mathcal{I}+T)^{-1}T\mathbf{v} = T^2(\mathcal{I}+T)\mathbf{v}$ and hence $\mathbf{v} \in \mathrm{dom}(T)$ implies $T(\mathcal{I}+T)^{-1}\mathbf{v} \in \mathrm{dom}(T)$. If on the other hand $T(\mathcal{I}+T)^{-1}\mathbf{v} \in \mathrm{dom}(T)$, then the identity

$$T(\mathcal{I}+T)^{-1}\mathbf{v} = \mathbf{v} - (\mathcal{I}+T)^{-1}\mathbf{v}$$

implies $\mathbf{v} \in \mathrm{dom}(T)$ and hence $\mathbf{v} \in \mathrm{dom}(T)$ if and only if $T(\mathcal{I}+T)^{-1}\mathbf{v} \in \mathrm{dom}(T)$. By induction, we find that $\mathbf{v} \in \mathrm{dom}(T^m)$ if and only if $T^n(\mathcal{I}+T)^{-n}\mathbf{v} \in \mathrm{dom}(T^m)$. We thus conclude that $(\mathcal{I}+T)^{-n}\Lambda_\beta\mathbf{v} \in \mathrm{dom}(T^{2n})$, which in turn implies that $\Lambda_\beta\mathbf{v} \in \mathrm{dom}(T^n)) = \mathrm{dom}((\mathcal{I}+T)^n)$. Thus, $T^\beta\mathbf{v} = (\mathcal{I}+T)^n\Lambda_\beta\mathbf{v}$ is defined, such that in turn $\mathbf{v} \in \mathrm{dom}\left(T^\beta\right)$ for any $\mathbf{v} \in \mathrm{dom}(T^{\alpha+\beta})$. We conclude that

$$\mathrm{dom}\left(T^\alpha T^\beta\right) = \mathrm{dom}\left(T^{\alpha+\beta}\right) \cap \mathrm{dom}\left(T^\beta\right) = \mathrm{dom}\left(T^{\alpha+\beta}\right)$$

and so $T^\alpha T^\beta = T^{\alpha+\beta}$.

□

Proposition 7.24 (Scaling Property)**.** *Let* $T \in \mathrm{Sect}(\omega)$ *and let* $\Lambda = [\delta_1, \delta_2] \subset (0, \pi/\omega)$ *be a compact interval. Then the family* $(T^\alpha)_{\alpha\in\Lambda}$ *is uniformly sectorial of angle* $\delta_2\omega$*. In particular, for every* $\alpha \in (0, \pi/\omega_T)$*, the operator* T^α *is sectorial with* $\omega_{T^\alpha} = \alpha\omega_T$*.*

Proof. The second statement obviously follows from the first by choosing $\lambda = [\alpha, \alpha]$. Because of (iii) in Lemma 7.21, we know that $\sigma_S(T^\alpha) = (\sigma_S(T))^\alpha \subset \overline{\Sigma_{\alpha\omega}} \subset \overline{\Sigma_{\delta_2\omega}}$ for $\alpha \in \Lambda$. What remains to show are the uniform estimates (6.1) for the S-resolvents.

We choose $\varphi \in (\delta_2\omega, \pi)$. In order to show that $\|S_L^{-1}(s, T^\alpha)s\|$ is uniformly bounded for $s \notin \overline{\Sigma_\varphi}$ and $\alpha \in \Lambda$, we define for $\alpha \in \Lambda$ and $s \notin \Sigma_\varphi$ the function

$$\begin{aligned} \Psi_{s,\alpha}(p) =& S_L^{-1}\left(s, p^\alpha\right)s + S_L^{-1}\left(-|s|^{\frac{1}{\alpha}}, p\right)|s|^{\frac{1}{\alpha}} \\ =& \mathcal{Q}_s\left(p^\alpha\right)^{-1}\left(|s|^{\frac{1}{\alpha}} + p\right)^{-1}\left(p(\overline{s} - p^\alpha)s + p^\alpha(\overline{s} - p^\alpha)|s|^{\frac{1}{\alpha}}\right). \end{aligned} \tag{7.28}$$

This function belongs to $\mathcal{SH}_{L,0}^{\infty}[\Sigma_\omega]$: as $s \notin \Sigma_\varphi$, it is left slice hyperholomorphic on Σ_{θ_0} with $\theta_0 := \min\{\alpha^{-1}\varphi, \pi\} > \omega$. The first line in (7.28) implies that $\Psi_{s,\alpha}$ has polynomial limit 0 at infinity because $S_L^{-1}(s, p^\alpha)$ and $S_L^{-1}\left(|s|^{1/\alpha}, p\right)$ have polynomial limit 0 at infinity and the second line in (7.28) implies that $\Psi_{s,\alpha}$ has polynomial limit 0 at 0 because $\mathcal{Q}_s(p^\alpha)^{-1}$ and $\left(|s|^{1/\alpha} - p\right)^{-1}$ are bounded for p sufficiently close to 0. Since the function $S_L^{-1}\left(|s|^{1/\alpha}, p\right) = \left(|s|^{1/\alpha} - p\right)^{-1}$ belongs to $\mathcal{E}_L[\Sigma_\omega]$, we find that also $S_L^{-1}(s, p^\alpha)s = \Psi_{s,\alpha}(p) + S_L^{-1}(|s|^{1/\alpha}, T)|s|^{1/\alpha}$ belongs to $\mathcal{E}_L[\Sigma_\omega]$ and that

$$S_L^{-1}(s, T^\alpha)s = S_L^{-1}\left(|s|^{\frac{1}{\alpha}}, T\right)|s|^{\frac{1}{\alpha}} + \Psi_{s,\alpha}(T).$$

The function $\Psi_{s,\alpha}$ satisfies the scaling property $\Psi_{t^\alpha s,\alpha}(tp) = \Psi_{s,\alpha}(p)$ and so we have $\Psi_{\frac{s}{|s|},\alpha}\big(|s|^{-\frac{1}{\alpha}}, p\big) = \Psi_{s,\alpha}(p)$. If we choose $\theta \in (\omega, \min\{\pi, \delta_2^{-1}\varphi\})$ and $\mathsf{i} = \mathsf{i}_s$, we therefore find that

$$\begin{aligned}
\|S_L^{-1}(s, T^\alpha)s\| \leq & C_{\theta',T} + \left\|\Psi_{s/|s|,\alpha}\left(|s|^{-\frac{1}{\alpha}}T\right)\right\| \\
\leq & C_{\theta',T} + \frac{1}{2\pi}\left\|\int_{\partial(\Sigma_\theta\cap\mathbb{C}_{\mathsf{i}})} S_L^{-1}(p,T)\, ds_{\mathsf{i}}\, \Psi_{s/|s|,\alpha}\left(|s|^{-\frac{1}{\alpha}}p\right)\right\| \\
\leq & C_{\theta',T} + \frac{C_{\theta',T}}{2\pi}\int_{\partial(\Sigma_\theta\cap\mathbb{C}_{\mathsf{i}})} |p|^{-1} d|p|\, \left|\Psi_{s/|s|,\alpha}(p)\right|,
\end{aligned}$$

where $C_{\theta',T}$ is the respective constant in (6.1) for some $\theta' \in (\omega, \theta)$, which is independent of s and $\alpha \in \Lambda$. Hence, if we are able to show that

$$\sup\left\{\int_{\partial(\Sigma_\theta\cap\mathbb{C}_{\mathsf{i}_s})} |p|^{-1} d|p|\, |\Psi_{s,\alpha}(p)| : |s| = 1, s \notin \Sigma_\varphi, \alpha \in \Lambda\right\} < +\infty, \tag{7.29}$$

then we are done. Since we integrate along a path in the complex plane $\mathbb{C}_{\mathsf{i}_s}$, we find that p and s commute and $\Psi_{s,\alpha}(p)$ simplifies to

$$\Psi_{s,\alpha}(p) = (s - p^\alpha)^{-1}\left(p + |s|^{1/\alpha}\right)^{-1}\left(ps + |s|^{1/\alpha}p^\alpha\right).$$

As $|s| = 1$, we can therefore estimate

$$|\Psi_{s,\alpha}(p)| \leq \frac{|p|^{1-\varepsilon}}{|s - p^\alpha|}\frac{|p|^\varepsilon}{|1+p|} + \frac{|p|^{\alpha-\varepsilon}}{|s - p^\alpha|}\frac{|p|^\varepsilon}{|1+p|} \leq K\frac{|p|^\varepsilon}{|1+p|}$$

with $\varepsilon \in (0, \delta_1)$, because $|p|^{1-\varepsilon}/\,|s - p^\alpha|$ and $|p|^{\alpha-\varepsilon}/\,|s - p^\alpha|$ are uniformly bounded by some constant $K > 0$ for our parameters s, α and p. Thus we have an estimate for the integrand in (7.29) that is independent of the parameters such that (7.29) is actually true.

With analogous arguments using the right slice hyperholomorphic version of the S-functional calculus for sectorial operators, we can show that also $\|sS_R^{-1}(s, T^\alpha)\|$ is uniformly bounded for $s \notin \overline{\Sigma_\varphi}$ and $\alpha \in \Lambda$. Since $\varphi \in (\delta_2\omega, \pi)$ was arbitrary, the proof is finished.

□

As immediate consequences of Proposition 7.24 and the composition rule Theorem 6.33, we obtain the following two results.

Proposition 7.25. *Let $T \in \operatorname{Sect}(\omega)$ for some $\omega \in (0,\pi)$ and let $\alpha \in (0,\pi/\omega)$ and $\varphi \in (\omega,\pi/\alpha)$. If $f \in \mathcal{SH}^\infty_{L,0}(\Sigma_{\alpha\varphi})$ (or $f \in \mathcal{M}_L[\Sigma_{\alpha\omega}]_{T^\alpha}$), then the function $p \mapsto f(p^\alpha)$ belongs to $\mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ (resp. $\mathcal{M}_L[\Sigma_\omega]_T$) and*

$$f(T^\alpha) = (f(p^\alpha))(T).$$

Corollary 7.26 (Second Law of Exponents)**.** *Let $T \in \operatorname{Sect}(\omega)$ with $\omega \in (0,\pi)$ and let $\alpha \in (0,\pi/\omega)$. For all $\beta > 0$, we have*

$$(T^\alpha)^\beta = T^{\alpha\beta}.$$

Corollary 7.27. *Let $T \in \operatorname{Sect}(\omega)$ and $\gamma > 0$. For any $\mathbf{v} \in \operatorname{dom}(T^\gamma)$, the mapping $\Lambda_{\mathbf{v}} : \alpha \mapsto T^\alpha\mathbf{v}$ defined on $(0,\gamma)$ is analytic in α. Moreover, the power series expansion of $\Lambda_{\mathbf{v}}$ at any point $\alpha_0 \in (0,\gamma)$ converges on $(-r+\alpha_0, \alpha_0+r)$ with $r = \min\{\gamma-\alpha_0, \alpha_0\}$.*

Proof. Let $n > \gamma$ and set $A := T^{\gamma/n}$. Because of Corollary 7.26, we have $T^\alpha = A^{\alpha n/\gamma}$. If $\mathbf{v} \in \operatorname{dom}(T^\gamma)$, then $\mathbf{v} \in \operatorname{dom}(A^n)$ and the mapping $\Upsilon(\beta) := A^\beta\mathbf{v}$ is analytic on $(0,n)$ by Proposition 7.22. The radius of convergence of its power series expansion at $\beta_0 \in (0,n)$ is greater than or equal to $r' = \min\{\beta_0, n-\beta_0\}$. Hence, $\Lambda_{\mathbf{v}}(\alpha) = \Upsilon(n\alpha/\gamma)$ is also an analytic function and the radius of convergence of its power series expansion at any point $\alpha_0 \in (0,\gamma)$ is greater than or equal to $\min\{\alpha_0, \gamma-\alpha_0\}$, which is exactly what we wanted to show.

□

We conclude this section with the generalisation of the famous Balakrishnan representation of fractional powers and some of its consequences. This formula was introduced in [14] as one of the first approaches for defining fractional powers of sectorial operators.

Theorem 7.28 (Balakrishnan Representation)**.** *Let $T \in \operatorname{Sect}(\omega)$. For $0 < \alpha < 1$, we have*

$$T^\alpha\mathbf{v} = \frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty} t^{\alpha-1}(t+T)^{-1}T\mathbf{v}\,dt, \qquad \forall \mathbf{v} \in \operatorname{dom}(T). \tag{7.30}$$

More general, for $0 < \alpha < n \le m$, we have

$$T^\alpha\mathbf{v} = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)}\int_0^{+\infty} t^{\alpha-1}[T(t+T)^{-1}]^m\mathbf{v}\,dt, \qquad \forall \mathbf{v} \in \operatorname{dom}(T^n). \tag{7.31}$$

Proof. We first show (7.30) and hence assume that $\alpha \in (0,1)$. For $\mathbf{v} \in \operatorname{dom}(T)$, we have because of (ii) in Lemma 6.31 and with arbitrary $\varphi \in (\omega,\pi)$ and $\varepsilon > 0$ that

$$\begin{aligned} T^\alpha\mathbf{v} &= \left(p^\alpha(p+\varepsilon)^{-1}\right)(T)(T+\varepsilon\mathcal{I})\mathbf{v} \\ &= \left(p^\alpha(p+\varepsilon)^{-1}\right)(T)T\mathbf{v} + \varepsilon\left(p^\alpha(p+\varepsilon)^{-1}(1+p)^{-1}\right)(T)(\mathcal{I}+T)\mathbf{v} \\ &= \frac{1}{2\pi}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_{\mathsf{i}})} s^{\alpha-1}s(s+\varepsilon)^{-1}\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)T\mathbf{v} \\ &\quad + \frac{1}{2\pi}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_{\mathsf{i}})} s^\alpha\varepsilon(s+\varepsilon)^{-1}(1+s)^{-1}\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)(\mathcal{I}+T)\mathbf{v}. \end{aligned}$$

Now observe that there exists a positive constant $K < +\infty$ such that

$$\left|\varepsilon(s+\varepsilon)^{-1}\right| \leq \frac{K}{|s|} \qquad \forall \varepsilon > 0, s \in \partial(\Sigma_\varphi \cap \mathbb{C}_{\mathbf{i}}).$$

Together with the estimate (6.1), this implies that the integrand in the second integral is bounded for all $\varepsilon > 0$ by the functions $s \mapsto K C_{\varphi,T} |s|^{\alpha-1}(|1+s|)^{-1}\|(\mathcal{I}+T)\mathbf{v}\|$, which is integrable along $\partial(\Sigma_\varphi \cap \mathbb{C}_{\mathbf{i}})$ because of the assumption $\alpha \in (0,1)$. Hence, we can apply Lebesgue's theorem in order to exchange the integral with the limit and find that the second integral vanishes as ε tends to 0. In the first integral on the other hand, we find that

$$\begin{aligned} S_R^{-1}(s,T)T\mathbf{v} &= (\overline{s}\mathcal{I} - T)\mathcal{Q}_s(T)^{-1}T\mathbf{v} \\ =&\overline{s}T\mathcal{Q}_s(T)^{-1}\mathbf{v} - T^2\mathcal{Q}_s(T)^{-1}\mathbf{v} \\ =&\overline{s}T\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T)\mathcal{Q}_s(T)^{-1}\mathbf{v} + \left(-2s_0T + |s|^{-1}\mathcal{I}\right)\mathcal{Q}_s(T)^{-1}\mathbf{v} \\ =&-\mathbf{v} + \overline{s}T\mathcal{Q}_s(T)^{-1}\mathbf{v} - \overline{s}T\mathcal{Q}_s(T)^{-1}\mathbf{v} + s(\overline{s}\mathcal{I} - T)\mathcal{Q}_s(T)^{-1}\mathbf{v} \\ =&-\mathbf{v} + sS_R^{-1}(s,T)\mathbf{v}. \end{aligned} \tag{7.32}$$

Hence, the function $s \mapsto S_R^{-1}(s,T)T\mathbf{v}$ for $s \in \partial(\Sigma_\varphi \cap \mathbb{C}_{\mathbf{i}})$ is bounded at 0 because of (6.1). Since it decays as $|s|^{-1}$ as $s \to \infty$ and since the function $s \mapsto s(s+\varepsilon)^{-1}$ is uniformly bounded in ε on $\partial(\Sigma_\varphi \cap \mathbb{C}_{\mathbf{i}})$, we can apply Lebesgue's theorem also in the first integral in order to take the limit as $\varepsilon \to 0$ and obtain

$$T^\alpha \mathbf{v} = \frac{1}{2\pi}\int_{\partial(\Sigma_\varphi \cap \mathbb{C}_{\mathbf{i}})} s^{\alpha-1}\, ds_{\mathbf{i}}\, S_R^{-1}(s,T)T\mathbf{v}.$$

Choosing the standard parametrisation of the path of integration, we thus find

$$\begin{aligned} T^\alpha\mathbf{v} =& \frac{1}{2\pi}\int_{-\infty}^{0} \left(-te^{\mathbf{i}\varphi}\right)^{\alpha-1} e^{\mathbf{i}\varphi}\mathbf{i} S_R^{-1}\left(te^{\mathbf{i}\varphi}, T\right) T\mathbf{v}\, dt \\ &+ \frac{1}{2\pi}\int_{0}^{+\infty} \left(te^{-\mathbf{i}\varphi}\right)^{\alpha-1} e^{-\mathbf{i}\varphi}(-\mathbf{i}) S_R^{-1}\left(te^{-\mathbf{i}\varphi}, T\right) T\mathbf{v}\, dt. \end{aligned}$$

Once more (6.1) and the fact that $S_R^{-1}(s,T)T\mathbf{v}$ is bounded at 0 allow us to apply Lebesgue's theorem in order to take the limit as φ tends to π. We finally find after a change of variables in the first integral that

$$\begin{aligned} T^\alpha\mathbf{v} =& \frac{1}{2\pi}\int_{0}^{+\infty} t^{\alpha-1}\left(-e^{\mathbf{i}\pi\alpha}\right)(-\mathbf{i}) S_R^{-1}\left(-te^{\mathbf{i}\pi}, T\right) T\mathbf{v}\, dt \\ &+ \frac{1}{2\pi}\int_{0}^{+\infty} t^{\alpha-1} e^{-\mathbf{i}\pi\alpha}(-\mathbf{i}) S_R^{-1}\left(te^{-\mathbf{i}\pi}, T\right) T\mathbf{v}\, dt \\ =& -\frac{\sin(\alpha\pi)}{\pi}\int_{0}^{+\infty} t^{\alpha-1} S_R^{-1}(-t,T)T\mathbf{v}\, dt, \end{aligned}$$

which equals (7.30) as $S_R^{-1}(-t,T) = (-t\mathcal{I} - T)^{-1} = -(t\mathcal{I}+T)^{-1}$ for $t \in \mathbb{R}$.

Let us now prove (7.31) and let us for now assume that $n-1 < \alpha < n$ and $n = m$. For $\mathbf{v} \in \operatorname{dom}(T^n)$, we then have

$$T^\alpha\mathbf{v} = T^{\alpha-(n-1)}T^{n-1}\mathbf{v} = \frac{\sin((\alpha-n+1)\pi)}{\pi}\int_0^{+\infty} t^{\alpha-n}(t\mathcal{I}+T)^{-1}T^n\mathbf{v}\, dt.$$

Integrating $n-1$ times by parts, we find

$$\begin{aligned} T^\alpha \mathbf{v} &= \frac{(n-1)!\sin((\alpha-n+1)\pi)}{\pi(\alpha-n+1)\cdots(\alpha-1)} \int_0^{+\infty} t^{\alpha-1}(t\mathcal{I}+T)^{-n}T^n\mathbf{v}\,dt \\ &= \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^{+\infty} t^{\alpha-1}(t\mathcal{I}+T)^{-n}T^n\mathbf{v}\,dt, \end{aligned} \tag{7.33}$$

where the second identity follows from the identities $\sin(z\pi)/\pi = 1/(\Gamma(z)\Gamma(1-z))$ and $z\Gamma(z) = \Gamma(z+1)$ for the gamma function. Hence, the identity (7.31) holds true if $n-1 < \alpha < n = m$.

Now observe that, because of (6.1) and because $(t\mathcal{I}+T)^{-n}T^n\mathbf{v} = ((t\mathcal{I}+T)^{-1}T)\,\mathbf{v}$ is bounded near 0 due to (7.32), the integral (7.33) defines a real analytic function in α on the entire interval $(0,n)$. From Proposition 7.22 and the identity principle for real analytic functions, we conclude that (7.31) holds also if $0 < \alpha < n = m$.

Finally, let us show by induction on m that (7.31) holds true for any $m \geq n$. For $m = n$ we have just shown this identity, so let us assume that it holds true for some $m \geq n$. We introduce the notation

$$c_m := \frac{\Gamma(m)}{\Gamma(m-\alpha)\Gamma(\alpha)} \quad \text{and} \quad I_m := \int_0^{+\infty} t^{\alpha-1}\left[T(t\mathcal{I}+T)^{-1}\right]^m \mathbf{v}\,dt$$

so that $T^\alpha\mathbf{v} = c_m I_m$. We want to show that $T^\alpha\mathbf{v} = c_{m+1}I_{m+1}$. By integration by parts, we deduce

$$\begin{aligned} I_m &= \left(\frac{t^\alpha}{\alpha}\left[T(t\mathcal{I}+T)^{-1}\right]^m\mathbf{v}\right)\Bigg|_0^{+\infty} + \frac{m}{\alpha}\int_0^{+\infty} t^\alpha\left[T(t\mathcal{I}+T)^{-1}\right]^m(t\mathcal{I}+T)^{-1}\mathbf{v}\,dt \\ &= \frac{m}{\alpha}\int_0^{+\infty} t^\alpha\left[T(t\mathcal{I}+T)^{-1}\right]^m(t\mathcal{I}+T)^{-1}\mathbf{v}\,dt \\ &= \frac{m}{\alpha}\int_0^{+\infty} t^{\alpha-1}\left(\left[T(t\mathcal{I}+T)^{-1}\right]^m\mathbf{v} - \left[T(t\mathcal{I}+T)^{-1}\right]^{m+1}\mathbf{v}\right)\,dt \\ &= \frac{m}{\alpha}(I_m - I_{m+1}). \end{aligned}$$

Hence $I_m = \frac{m}{m-\alpha}I_{m+1}$ and so

$$T^\alpha\mathbf{v} = c_m I_m = c_m\frac{m}{m-\alpha}I_{m+1} = c_{m+1}I_{m+1}.$$

The induction is complete.

□

7.2.1 Fractional Powers with Negative Real Part

If $\alpha < 0$ the fractional power p^α has polynomial limit infinity at 0 in any sector Σ_φ with $\varphi > \pi$. Because of Lemma 6.30, it does therefore not belong to $\mathcal{M}_L[\Sigma_\omega]_T$ if T is not injective. If on the other hand T is injective, then it is regularisable by some power of $p\,(1+p)^{-2}$ such that $p^\alpha \in \mathcal{M}_L[\Sigma_\omega]_T$. We can thus define T^α for injective sectorial operators via the H^∞-functional calculus.

Definition 7.29. Let $T \in \mathrm{Sect}(\omega)$ be injective. For any $\alpha \in \mathbb{R}$, we call the operator $T^\alpha := (p^\alpha)(T)$ the fractional power with exponent α of T.

The properties of the fractional powers of T in this case are again analogue to the complex case, cf. [59]. We state the most important properties for the sake of completeness, but we omit the proofs since they are either immediate consequences of the preceding results or can be shown with exactly the same arguments as in the complex case without making use of any quaternionic techniques.

Proposition 7.30. *Let $T \in \mathrm{Sect}(\omega)$ be injective and let $\alpha, \beta \in \mathbb{R}$.*

(i) The operator T^α is injective and $(T^\alpha)^{-1} = T^{-\alpha} = (T^{-1})^\alpha$.

(ii) We have $T^\alpha T^\beta \subset T^{\alpha+\beta}$ with $\mathrm{dom}\left(T^\alpha T^\beta\right) = \mathrm{dom}\left(T^\beta\right) \cap \mathrm{dom}\left(T^{\alpha+\beta}\right)$.

(iii) If $\overline{\mathrm{dom}(T)} = V = \overline{\mathrm{ran}(T)}$, then $T^{\alpha+\beta} = \overline{T^\alpha T^\beta}$.

(iv) If $0 < \alpha < 1$, then

$$T^{-\alpha}\mathbf{v} = \frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} t^{-\alpha}(t\mathcal{I} + T)^{-1}\mathbf{v}\, dt \quad \forall \mathbf{v} \in \mathrm{ran}(T).$$

(v) If $\alpha \in \mathbb{R}$ with $|\alpha| < \pi/\omega$, then $T^\alpha \in \mathrm{Sect}(|\alpha|\omega)$ and for all $\beta \in \mathbb{R}$

$$(T^\alpha)^\beta = \left(T^{\alpha\beta}\right).$$

(vi) If $0 < \alpha_1, \alpha_2$, then $\mathrm{dom}(T^{\alpha_2}) \cap \mathrm{ran}(T^{\alpha_1}) \subset \mathrm{dom}(T^\alpha)$ for each $\alpha \in (-\alpha_1, \alpha_2)$, the mapping $\alpha \mapsto T^\alpha \mathbf{v}$ is analytic on $(-\alpha_1, \alpha_2)$ for any $\mathbf{v} \in \mathrm{dom}(T^{-\alpha_2}) \cap \mathrm{ran}(T^{\alpha_1})$.

Remark 7.31. Observe that (iv) of Proposition 7.30 and Corollary 7.11 together with the semigroup property imply that the direct approach in Section 7.1 and the approach via the H^∞-functional calculus are consistent.

Proposition 7.32 (Komatsu Representation)**.** *Let $T \in \mathrm{Sect}(\omega)$ be injective. For any $\mathbf{v} \in \mathrm{dom}(A) \cap \mathrm{ran}(A)$ and any $\alpha \in (-1, 1)$, one has*

$$\begin{aligned} T^\alpha \mathbf{v} = \frac{\sin(\alpha\pi)}{\pi} \Bigg[& \frac{1}{\alpha}\mathbf{v} - \frac{1}{1+\alpha} T^{-1}\mathbf{v} \\ & + \int_0^1 t^{\alpha+1}(t\mathcal{I} + T)^{-1} T^{-1}\mathbf{v}\, dt + \int_1^{+\infty} t^{\alpha-1}(t\mathcal{I} + T)^{-1} T\mathbf{v}\, dt \Bigg] \\ = \frac{\sin(\alpha\pi)}{\pi} \Bigg[& \frac{1}{\alpha}\mathbf{v} + \int_0^1 t^{-\alpha}(\mathcal{I} + tT)^{-1} T\mathbf{v}\, dt - \int_0^1 t^\alpha \left(\mathcal{I} + tT^{-1}\right)^{-1} T^{-1}\mathbf{v}\, dt \Bigg]. \end{aligned}$$

7.3 Kato's Formula for the S-Resolvents

Kato showed in [63] that certain fractional powers of generators of analytic semigroups are again generators of analytic semigroups. Analogue results can be shown for quaternionic linear operators, but therefore we need a modified definition of fractional powers of an operator.

Definition 7.33. A densely defined closed operator T is of type (M, ω) with $M > 0$ and $\omega \in (0, \pi)$ if $T \in \mathrm{Sect}(\omega)$ and M is the uniform bound of $\|tS_R^{-1}(-t, T)\|$ on the negative real axis, that is

$$\|tS_R^{-1}(t, T)\| \leq M, \qquad \text{for } t \in (-\infty, 0). \tag{7.34}$$

Proposition 7.34. *Let T be of type (M,ω) with $M > 0$ and $\omega \in (0,\pi)$. Let $0 < \alpha < 1$ and let $\pi > \phi_0 > \max(\alpha\pi, \omega)$. The parameter integral*

$$F_\alpha(p,T) = \frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} t^\alpha (p^2 - 2pt^\alpha \cos(\alpha\pi) + t^{2\alpha})^{-1} S_R^{-1}(-t,T)\, dt. \tag{7.35}$$

defines a $\mathcal{B}(V)$-valued function on $\mathbb{H} \setminus \Sigma_{\phi_0}$ in p that is left slice hyperholomorphic.

Proof. For any compact subset K of $\mathbb{H} \setminus \Sigma_{\phi_0}$, we have $\min_{p\in K} \arg(p) > \alpha\pi$ and thus there exists some $\delta_K > 0$ such that

$$\left|p^2 - 2pt^\alpha \cos(\alpha\pi) + t^{2\alpha}\right| = \left|p - t^\alpha e^{\mathsf{i}_p \alpha\pi}\right| \left|p - t^\alpha e^{-\mathsf{i}_p \alpha\pi}\right| \geq \delta_K \tag{7.36}$$

for $p \in K$ and $t \geq 0$. For the same reason, we can find a constant $C_K > 0$ such that

$$\sup_{\substack{t\in[0,+\infty)\\ p\in K}} \left|p^2 - 2pt^\alpha \cos(\alpha\pi) + t^{2\alpha}\right|^{-1} t^{2\alpha} = \sup_{\substack{t\in[0,+\infty)\\ p\in K}} \frac{1}{\left|\frac{p}{t^\alpha} - e^{\mathsf{i}_p\alpha\pi}\right|} \frac{1}{\left|\frac{p}{t^\alpha} - e^{-\mathsf{i}_p\alpha\pi}\right|} < C_K$$

and hence

$$\left|p^2 - 2pt^\alpha \cos(\alpha\pi) + t^{2\alpha}\right|^{-1} \leq C_K t^{-2\alpha} \quad t \in [1,\infty),\, p \in K. \tag{7.37}$$

Now consider $p \in \mathbb{H} \setminus \Sigma_{\phi_0}$ and let K be a compact neighborhood of p. The integral in (7.35) converges absolutely and hence defines a bounded operator: because of (7.34) and the above estimates, we have for $s \in K$, and thus in particular for p itself, that

$$\begin{aligned}\|F_\alpha(s,T)\| \leq& \frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} t^\alpha \left|s^2 - 2st^\alpha \cos(\alpha\pi) + t^{2\alpha}\right|^{-1} \frac{M}{t}\, dt \\ \leq& \frac{M\sin(\alpha\pi)}{\delta_K \pi} \int_0^1 t^{\alpha-1}\, dt + \frac{M\sin(\alpha\pi) C_K}{\pi} \int_1^{+\infty} t^{-\alpha-1}\, dt < +\infty.\end{aligned}$$

Using (7.36) and (7.37), one can derive analogous estimates for the partial derivatives of the integrand $p \mapsto t^\alpha(p^2 - 2st^\alpha\cos(\alpha\pi) + t^{2\alpha})^{-1} S_R^{-1}(-t,T)$ with respect to p_0 and p_1.

Since these estimates are uniform on the neighborhood K of p, we can exchange differentiation and integration in order to compute the partial derivatives $\frac{\partial}{\partial p_0} F_\alpha(p,T)$ and $\frac{\partial}{\partial p_1} F_\alpha(p,T)$ of $F_\alpha(\cdot,T)$ at p. The integrand is however left slice hyperholomorphic and therefore also $F_\alpha(p,T)$ is left slice hyperholomorphic.

□

Lemma 7.35. *Let T be of type (M,ω) with $M > 0$ and $\omega \in (0,\pi)$, let $0 < \alpha < 1$ and assume that $0 \in \rho_S(T)$. Moreover, let ϕ_0 and $F_\alpha(p,T)$ be defined as in Proposition 7.34. If Γ is a piecewise smooth path that goes from $\infty e^{\mathsf{i}\theta}$ to $\infty e^{-\mathsf{i}\theta}$ in $(\mathbb{H}\setminus\Sigma_{\phi_0})\cap\mathbb{C}_{\mathsf{i}}$ and avoids the negative real axis $(-\infty,0]$ for some $\mathsf{i} \in \mathbb{S}$ and some $\theta \in (\phi_0,\pi]$, then*

$$F_\alpha(p,T) = \frac{1}{2\pi} \int_\Gamma S_R^{-1}(p,s^\alpha)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T). \tag{7.38}$$

Proof. First of all observe that the function $s \mapsto S_R^{-1}(p,s^\alpha)$ is the composition of the intrinsic function $s \mapsto s^\alpha$ defined on $\mathbb{H} \setminus (-\infty,0]$ and the right slice hyperholomorphic

function $s \mapsto S_R^{-1}(p,s)$ defined on $\mathbb{H}\setminus[p]$. This composition is in particular well defined on all of $\mathbb{H}\setminus(-\infty,0]$, because s^α maps $\mathbb{H}\setminus(-\infty,0]$ to the set $\{s\in\mathbb{H} : \arg(s) < \alpha\pi\}$, which is contained in the domain of $S_R^{-1}(p,s^\alpha)$ because $\arg(p) > \phi_0 > \alpha\pi$ by assumption. By Corollary 2.7, the function $s \mapsto S_R^{-1}(p,s^\alpha)$ is therefore right slice hyperholomorphic on $\mathbb{H}\setminus(-\infty,0]$.

An estimate similar to the one in the proof of Proposition 7.34 moreover assures that the integral in (7.38) converges absolutely. It thus follows from Theorem 2.27 that the value of the integral in (7.38) is the same for any choice of Γ and any choice of θ. Let us denote the value of this integral by $\mathfrak{I}_\alpha(p,T)$.

Since $0\in\rho_S(T)$, the open ball $B_\varepsilon(0)$ is contained in $\rho_S(T)$ if $\varepsilon>0$ is small enough. For $\theta\in(\phi_0,\pi)$, we set $U(\varepsilon,\theta) := \Sigma_\theta \setminus B_\varepsilon(0)$. Then

$$\mathfrak{I}_\alpha(p,T) = \frac{1}{2\pi}\int_{\partial(U(\varepsilon,\theta)\cap\mathbb{C}_{\mathsf{i}})} S_R^{-1}(p,s^\alpha)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T).$$

We assumed that $0\in\rho_S(T)$, and hence the right S-resolvent is bounded near 0, which allows us to take the limit $\varepsilon\to 0$. We obtain

$$\begin{aligned}
&\mathfrak{I}_\alpha(p,T) = \frac{1}{2\pi}\int_{\partial(\Sigma_\theta\cap\mathbb{C}_{\mathsf{i}})} S_R^{-1}(p,s^\alpha)\, ds_{\mathsf{i}}\, S_R^{-1}(s,T)\\
=& -\frac{1}{2\pi}\int_0^{+\infty} S_R^{-1}\left(p, t^\alpha e^{\mathsf{i}\alpha\theta}\right) e^{\mathsf{i}\theta}(-\mathsf{i}) S_R^{-1}\left(te^{\mathsf{i}\theta},T\right)\, dt\\
&+\frac{1}{2\pi}\int_0^{+\infty} S_R^{-1}\left(p, t^\alpha e^{-\mathsf{i}\alpha\theta}\right) e^{-\mathsf{i}\theta}(-\mathsf{i}) S_R^{-1}\left(te^{\mathsf{i}\theta},T\right)\, dt\\
=& -\frac{1}{2\pi}\int_0^{+\infty} \left(p^2 - 2t^\alpha\cos(\alpha\theta) + t^{2\alpha}\right)^{-1}\left(p - t^\alpha e^{-\mathsf{i}\alpha\theta}\right) e^{\mathsf{i}\theta}(-\mathsf{i}) S_R^{-1}\left(te^{\mathsf{i}\theta},T\right)\, dt\\
&+\frac{1}{2\pi}\int_0^{+\infty} \left(p^2 - 2t^\alpha\cos(\alpha\theta) + t^{2\alpha}\right)^{-1}\left(p - t^\alpha e^{\mathsf{i}\alpha\theta}\right) e^{-\mathsf{i}\theta}(-\mathsf{i}) S_R^{-1}\left(te^{-\mathsf{i}\theta},T\right)\, dt.
\end{aligned}$$

Again an estimate analogue to the one in the proof of Proposition 7.34 allows us to take the limit as θ tends to π and we obtain

$$\begin{aligned}
&\mathfrak{I}_\alpha(p,T) =\\
=& -\frac{1}{2\pi}\int_0^{+\infty} \left(p^2 - 2t^\alpha\cos(\alpha\pi) + t^{2\alpha}\right)^{-1}\left(p - t^\alpha e^{-\mathsf{i}\alpha\pi}\right) e^{\mathsf{i}\pi}(-\mathsf{i}) S_R^{-1}\left(te^{\mathsf{i}\pi},T\right)\, dt\\
&+\frac{1}{2\pi}\int_0^{+\infty} \left(p^2 - 2t^\alpha\cos(\alpha\pi) + t^{2\alpha}\right)^{-1}\left(p - t^\alpha e^{\mathsf{i}\alpha\pi}\right) e^{-\mathsf{i}\pi}(-\mathsf{i}) S_R^{-1}\left(te^{-\mathsf{i}\pi},T\right)\, dt\\
=&\frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty} t^\alpha(p^2 - 2pt^\alpha\cos(\alpha\pi) + t^{2\alpha})^{-1} S_R^{-1}(-t,T)\, dt = F_\alpha(p,T).
\end{aligned}$$

□

Lemma 7.36. *Let T be of type (M,ω) with $M>0$ and $\omega\in(0,\pi)$. Let $0<\alpha<1$ and let ϕ_0 and $F_\alpha(p,T)$ be defined as in Proposition 7.34. We have*

$$F_\alpha(\mu,T) - F_\alpha(\lambda,T) = (\lambda-\mu)F_\alpha(\mu,T)F_\alpha(\lambda,T) \quad \text{for } \lambda,\mu\in(-\infty,0]. \tag{7.39}$$

Proof. Assume first that $0 \in \rho_S(T)$. Any real λ commutes with $S_R^{-1}(-t,T)$ and thus we have

$$F_\alpha(\lambda,T) = \frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty} S_L^{-1}(-t,T)(\lambda^2 - 2\lambda t^\alpha\cos(\alpha\pi) + t^{2\alpha})^{-1}t^\alpha\,dt$$

because $S_R^{-1}(-t,T) = (-t\mathcal{I}-T)^{-1} = S_L^{-1}(-t,T)$ as t is real. Computations analogue to those in the proof of Lemma 7.35 show that $F_\alpha(\lambda,T)$ can thus be represented as

$$F_\alpha(\lambda,T) = \frac{1}{2\pi}\int_\Gamma S_L^{-1}(s,T)\,ds_{\mathsf{i}}\,S_L^{-1}(\lambda,s^\alpha), \tag{7.40}$$

where Γ is any path as in Lemma 7.35.

Now let $\varepsilon > 0$ such that $cl(B_\varepsilon(0)) \subset \rho_S(T)$, choose $\mathsf{i} \in \mathbb{S}$ and set

$$U_s := \Sigma_{\theta_s} \setminus B_{\varepsilon_s}(0) \quad \text{and} \quad U_p := \Sigma_{\theta_p} \setminus B_{\varepsilon_p}(0)$$

with $0 < \varepsilon_s < \varepsilon_p < \varepsilon$ and $\phi_0 < \theta_p < \theta_s < \pi$. Then $cl(U_p) \subset U_s$ and $\Gamma_s = \partial(U_s \cap \mathbb{C}_{\mathsf{i}})$ and $\Gamma_p = \partial(U_p \cap \mathbb{C}_{\mathsf{i}})$ are paths as in Lemma 7.35. Moreover, since T is of type (M,ω) with $0 \in \rho_S(T)$, we can find a constant C such that $\|S_R^{-1}(s,T)\| < C/(1+|s|)$ for $s \in (-\infty,0]$. By Lemma 7.5 we may choose ε_p, ε_s, θ_p and θ_s such that

$$\|S_R^{-1}(s,T)\| \le \frac{M_1}{1+|s|}, s \in \Gamma_s \quad \text{and} \quad \|S_L^{-1}(p,T)\| \le \frac{M_1}{1+|p|}, p \in \Gamma_p \tag{7.41}$$

for some constant $M_1 > 0$. Lemma 7.35 and (7.40) then imply

$$\begin{aligned} &F_\alpha(\mu,T)F_\alpha(\lambda,T) = \\ =&\frac{1}{(2\pi)^2}\int_{\Gamma_s}\int_{\Gamma_p} S_R^{-1}(\mu,s^\alpha)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)S_L^{-1}(p,T)\,dp_{\mathsf{i}}\,S_L^{-1}(\lambda,p^\alpha). \end{aligned}$$

Applying the S-resolvent equation (2.30) yields

$$\begin{aligned} &F_\alpha(\mu,T)F_\alpha(\lambda,T) \\ =&\frac{1}{(2\pi)^2}\int_{\Gamma_s}\int_{\Gamma_p} S_R^{-1}(\mu,s^\alpha)\,ds_{\mathsf{i}}\,S_R^{-1}(s,T)p(p^2-2s_0p+|s|^2)^{-1}\,dp_{\mathsf{i}}\,S_L^{-1}(\lambda,p^\alpha) \\ &-\frac{1}{(2\pi)^2}\int_{\Gamma_s}\int_{\Gamma_p} S_R^{-1}(\mu,s^\alpha)\,ds_{\mathsf{i}}\,S_L^{-1}(p,T)p(p^2-2s_0p+|s|^2)^{-1}\,dp_{\mathsf{i}}\,S_L^{-1}(\lambda,p^\alpha) \\ &-\frac{1}{(2\pi)^2}\int_{\Gamma_s}\int_{\Gamma_p} S_R^{-1}(\mu,s^\alpha)\,ds_{\mathsf{i}}\,\overline{s}S_R^{-1}(s,T)(p^2-2s_0p+|s|^2)^{-1}\,dp_{\mathsf{i}}\,S_L^{-1}(\lambda,p^\alpha) \\ &+\frac{1}{(2\pi)^2}\int_{\Gamma_s}\int_{\Gamma_p} S_R^{-1}(\mu,s^\alpha)\,ds_{\mathsf{i}}\,\overline{s}S_L^{-1}(p,T)(p^2-2s_0p+|s|^2)^{-1}\,dp_{\mathsf{i}}\,S_L^{-1}(\lambda,p^\alpha). \end{aligned}$$

Since $p \mapsto (p^2-2s_0p+|s|^2)^{-1}S_L^{-1}(\lambda,p^\alpha)$ and $p \mapsto p(p^2-2s_0p+|s|^2)^{-1}S_L^{-1}(\lambda,p^\alpha)$ are holomorphic on $cl(U_p \cap \mathbb{C}_{\mathsf{i}})$ and tend uniformly to zero as p tends to infinity in U_p, we deduce from Cauchy's integral theorem that the first and the third of the above integrals equal zero. The estimate (7.41) allows us to apply Fubini's theorem in order

to exchange the order of integration so that we are left with

$$F_\alpha(\mu,T)F_\alpha(\lambda,T) = \frac{1}{2\pi}\int_{\Gamma_p}\left[\frac{1}{2\pi}\int_{\Gamma_s} S_R^{-1}(\mu,s^\alpha)\,ds_{\mathsf{i}}\right.$$
$$\left.\cdot\left(\overline{s}S_L^{-1}(p,T) - S_L^{-1}(p,T)p\right)(p^2-2s_0p+|s|^2)^{-1}\right]dp_{\mathsf{i}}\,S_L^{-1}(\lambda,p^\alpha). \quad (7.42)$$

We want to apply Lemma 4.18 and thus define the set $U_{s,r} := U_s \cap B_r(0)$ for $r > 0$, which is a bounded slice Cauchy domain. Its boundary $\partial(U_{s,r}\cap\mathbb{C}_{\mathsf{i}})$ in $\mathbb{C}_{\mathsf{i}}$ consists of $\Gamma_{s,r} := \Gamma_s \cap B_r(0)$ and the set $C_r := \{re^{\mathsf{i}\varphi} : -\theta_s \leq \varphi \leq \theta_s\}$. If $p \in \Gamma_p$, then $p \in U_{s,r}$ for sufficiently large r because $\overline{U_p} \subset U_s$. Since the function $s \mapsto S_R^{-1}(\mu,s^\alpha) = (\mu - s^\alpha)^{-1}$ is intrinsic because μ is real, we can therefore apply Lemma 4.18 and obtain for any such r

$$\begin{aligned}
&S_L^{-1}(p,T)S_R^{-1}(\mu,p^\alpha)\\
=&\frac{1}{2\pi}\int_{\partial(U_{s,r}\cap\mathbb{C}_{\mathsf{i}})} S_R^{-1}(\mu,s^\alpha)\,ds_{\mathsf{i}}\left(\overline{s}S_L^{-1}(p,T) - S_L^{-1}(p,T)p\right)(p^2-2s_0p+|s|^2)^{-1}\\
=&\frac{1}{2\pi}\int_{\Gamma_{s,r}} S_R^{-1}(\mu,s^\alpha)\,ds_{\mathsf{i}}\left(\overline{s}S_L^{-1}(p,T) - S_L^{-1}(p,T)p\right)(p^2-2s_0p+|s|^2)^{-1}\\
&+\frac{1}{2\pi}\int_{C_r} S_R^{-1}(\mu,s^\alpha)\,ds_{\mathsf{i}}\left(\overline{s}S_L^{-1}(p,T) - S_L^{-1}(p,T)p\right)(p^2-2s_0p+|s|^2)^{-1}.
\end{aligned}$$

As r tends to infinity the integral over C_r vanishes and hence

$$\begin{aligned}
&S_L^{-1}(p,T)S_R^{-1}(\mu,p^\alpha)\\
=&\lim_{r\to+\infty}\frac{1}{2\pi}\int_{\Gamma_{s,r}} S_R^{-1}(\mu,s^\alpha)\,ds_{\mathsf{i}}\left(\overline{s}S_L^{-1}(p,T) - S_L^{-1}(p,T)p\right)(p^2-2s_0p+|s|^2)^{-1}\\
=&\frac{1}{2\pi}\int_{\Gamma_s} S_R^{-1}(\mu,s^\alpha)\,ds_{\mathsf{i}}\left(\overline{s}S_L^{-1}(p,T) - S_L^{-1}(p,T)p\right)(p^2-2s_0p+|s|^2)^{-1}.
\end{aligned}$$

Applying this identity in (7.42), we obtain

$$F_\alpha(\mu,T)F_\alpha(\lambda,T) = \frac{1}{2\pi}\int_{\Gamma_p} S_L^{-1}(p,T)\,dp_{\mathsf{i}}\,S_R^{-1}(\mu,p^\alpha)S_L^{-1}(\lambda,p^\alpha)$$

because $S_R^{-1}(\mu,p^\alpha)$ and dp_{i} commute as μ is real. Since also λ is real, we have

$$S_R^{-1}(\mu,p^\alpha)S_L^{-1}(\lambda,p^\alpha) = \frac{1}{\mu-p^\alpha}\frac{1}{\lambda-p^\alpha}$$
$$=\frac{1}{\lambda-\mu}\left(\frac{1}{\mu-p^\alpha} - \frac{1}{\lambda-p^\alpha}\right) = (\lambda-\mu)^{-1}\left(S_L^{-1}(\mu,p^\alpha) - S_L^{-1}(\lambda,p^\alpha)\right)$$

and thus, recalling (7.40), we obtain

$$\begin{aligned}
&F_\alpha(\mu,T)F_\alpha(\lambda,T)\\
=&(\lambda-\mu)^{-1}\left(\frac{1}{2\pi}\int_{\Gamma_p} S_L^{-1}(p,T)\,dp_{\mathsf{i}}\,S_L^{-1}(\mu,p^\alpha) - \frac{1}{2\pi}\int_{\Gamma_p} S_L^{-1}(p,T)\,dp_{\mathsf{i}}\,S_L^{-1}(\lambda,p^\alpha)\right)\\
=&(\lambda-\mu)^{-1}\left(F_\alpha(\mu,T) - F_\alpha(\lambda,T)\right).
\end{aligned}$$

If $0 \notin \rho_S(T)$, then we consider the operator $T + \varepsilon\mathcal{I}$ for small $\varepsilon > 0$. This operator satisfies $0 \in \rho_S(T + \varepsilon\mathcal{I}) = \rho_S(T) + \varepsilon$ and hence (7.39) applies. Moreover, for real t, we have

$$S_R^{-1}(-t, T + \varepsilon\mathcal{I}) = S_R^{-1}(-(t + \varepsilon), T).$$

The estimate

$$\|S_R^{-1}(-t, T + \varepsilon\mathcal{I})\| \leq \frac{M}{t + \varepsilon} \leq \frac{M}{t}$$

therefore allows us to apply Lebesgue's dominated convergence theorem to see that

$$\begin{aligned}
F_\alpha(p, T + \varepsilon\mathcal{I}) &= \frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} t^\alpha (p^2 - 2pt^\alpha \cos(\alpha\pi) + t^{2\alpha})^{-1} S_R^{-1}(-t, T + \varepsilon\mathcal{I})\, dt \\
&\overset{\varepsilon\to 0}{\longrightarrow} \frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} t^\alpha (p^2 - 2pt^\alpha \cos(\alpha\pi) + t^{2\alpha})^{-1} S_R^{-1}(-t, T)\, dt = F_\alpha(p, T).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
F_\alpha(\mu, T) - F_\alpha(\lambda, T) &= \lim_{\varepsilon\to 0} F_\alpha(\mu, T + \varepsilon\mathcal{I}) - F_\alpha(\lambda, T + \varepsilon\mathcal{I}) \\
&= \lim_{\varepsilon\to 0} (\lambda - \mu) F_\alpha(\mu, T + \varepsilon\mathcal{I}) F_\alpha(\lambda, T + \varepsilon\mathcal{I}) = (\lambda - \mu) F_\alpha(\mu, T) F_\alpha(\lambda, T)
\end{aligned}$$

for $\lambda, \mu \in (-\infty, 0]$ also in this case.

□

Theorem 7.37. *Let T be of type (M, ω), let $\alpha \in (0, 1)$ and let $\phi_0 > \max(\alpha\pi, \omega)$. There exists a densely defined closed operator B_α such that*

$$S_R^{-1}(p, B_\alpha) = F_\alpha(p, T) \qquad \textit{for} \quad p \in \mathbb{H} \setminus \Sigma_{\phi_0},$$

where $F_\alpha(p, T)$ is the operator-valued function defined by the integral (7.38). *Moreover, B_α is of type $(M, \alpha\omega)$.*

Proof. From identity (7.39) it follows immediately that $F_\alpha(\mu, T)$ and $F_\alpha(\lambda, T)$ commute and have the same kernel. Rewriting this equation in the form

$$F_\alpha(\mu, T) = F_\alpha(\lambda, T) \left(\mathcal{I} + (\lambda - \mu) F_\alpha(\mu, T)\right) \tag{7.43}$$

shows that $\operatorname{ran}(F_\alpha(\mu, T)) \subset \operatorname{ran}(F_\alpha(\lambda, T))$ and exchanging the roles of μ and λ yields $\operatorname{ran}(F_\alpha(\mu, T)) = \operatorname{ran}(F_\alpha(\lambda, T))$. Hence, $\operatorname{ran}(F_\alpha(\mu, T))$ does not depend on μ and so we denote it by $\operatorname{ran}(F_\alpha(\cdot, T))$.

We show now that

$$\lim_{\mathbb{R}\ni\mu\to-\infty} \mu F_\alpha(\mu, T)\mathbf{v} = \mathbf{v} \quad \text{for all } \mathbf{v} \in V, \tag{7.44}$$

where $\lim_{\mathbb{R}\ni\mu\to-\infty} \mu F_\alpha(\mu, T)\mathbf{v}$ denotes the limit as μ tends to $-\infty$ in $\mathbb{R}$. From (7.44), we easily deduce that $\operatorname{ran}(F_\alpha(\cdot, T))$ is dense in V because

$$V = \overline{\bigcup_{\mu\in(-\infty,0]} \operatorname{ran}(F_\alpha(\mu, T))} = \overline{\operatorname{ran}(F_\alpha(\cdot, T))}.$$

We consider first $\mathbf{v} \in \operatorname{dom}(T)$. Since

$$\int_0^{+\infty} \frac{t^{\alpha-1}}{\mu^2 - 2\mu t^\alpha \cos(\alpha\pi) + t^{2\alpha}}\, dt = -\frac{\pi}{\mu \sin(\alpha\pi)} \quad \text{for } \mu \leq 0, \tag{7.45}$$

it is

$$\begin{aligned}&\mu F_\alpha(\mu, T)\mathbf{v} - \mathbf{v}\\ &= = -\frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} \frac{-\mu t^{\alpha-1}}{\mu^2 - 2\mu t^\alpha \cos(\alpha\pi) + t^{2\alpha}} \left(t S_R^{-1}(-t, T)\mathbf{v} + \mathbf{v} \right)\, dt.\end{aligned}$$

For $-\mu \geq 1$ and $t \in (0, +\infty)$, we can estimate

$$\begin{aligned}&\frac{-\mu t^{\alpha-1}}{\mu^2 - 2\mu t^\alpha \cos(\alpha\pi) + t^{2\alpha}} =\\ &= \frac{-\mu t^{\alpha-1}}{\mu^2 \sin(\alpha\pi)^2 + (\mu\cos(\alpha\pi) - t^\alpha)^2} \leq \frac{-\mu t^{\alpha-1}}{\mu^2 \sin(\alpha\pi)^2} \leq \frac{t^{\alpha-1}}{\sin(\alpha\pi)^2}\end{aligned}$$

and due to (7.34) we have $\|t S_R^{-1}(-t, T)\mathbf{v} + \mathbf{v}\| \leq (M+1)\|\mathbf{v}\|$. On the other hand, since $\mathbf{v} \in \operatorname{dom}(T)$, it is

$$t S_R^{-1}(-t, T)\mathbf{v} + \mathbf{v} = -S_R^{-1}(-t, T)T\mathbf{v} \tag{7.46}$$

and in turn, again due to (7.34), we can also estimate

$$\|t S_R^{-1}(-t, T)\mathbf{v} + \mathbf{v}\| \leq \frac{\|T\mathbf{v}\|}{t}$$

so that we can apply Lebesgue's dominated convergence theorem with dominating function

$$f(t) = \begin{cases} \dfrac{K}{\sin(\alpha\pi)^2} t^{\alpha-1}, & \text{for } t \in (0,1) \\ \dfrac{K}{\sin(\alpha\pi)^2} t^{-\alpha-1}, & \text{for } t \in [1, +\infty) \end{cases},$$

with $K > 0$ large enough, in order to exchange the integral with the limit for $\mu \to -\infty$ in $\mathbb{R}$. In view of (7.46), we obtain

$$\begin{aligned}&\lim_{\mathbb{R} \ni \mu \to -\infty} \mu F_\alpha(\mu, T)\mathbf{v} - \mathbf{v}\\ &= -\frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} \lim_{\mathbb{R} \ni \mu \to -\infty} \frac{\mu t^{\alpha-1}}{\mu^2 - 2\mu t^\alpha \cos(\alpha\pi) + t^{2\alpha}} S_R^{-1}(-t, T)T\mathbf{v}\, dt = \mathbf{0}.\end{aligned}$$

For arbitrary $\mathbf{v} \in V$ and $\varepsilon > 0$ consider a vector $\mathbf{v}_\varepsilon \in \operatorname{dom}(T)$ with $\|\mathbf{v} - \mathbf{v}_\varepsilon\| < \varepsilon$. Because of (7.34) and (7.45), we have the uniform estimate

$$\|\mu F_\alpha(\mu, T)\| \leq \frac{-\mu \sin(\alpha\pi)}{\pi} \int_0^{+\infty} \frac{t^\alpha}{\mu^2 - 2\mu t^\alpha \cos(\alpha\pi) + t^{2\alpha}} \frac{M}{t} dt = M. \tag{7.47}$$

Therefore

$$\begin{aligned}&\lim_{\mathbb{R} \ni \mu \to -\infty} \|\mu F_\alpha(\mu, T)\mathbf{v} - \mathbf{v}\|\\ &\leq \lim_{\mathbb{R} \ni \mu \to -\infty} \|\mu F_\alpha(\mu, T)\| \|\mathbf{v} - \mathbf{v}_\varepsilon\| + \|F_\alpha(\mu, T)\mathbf{v}_\varepsilon - \mathbf{v}_\varepsilon\| + \|\mathbf{v}_\varepsilon - \mathbf{v}\|\\ &\leq (M+1)\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we deduce that (7.44) also holds true for arbitrary $\mathbf{v} \in V$.

Overall, we obtain that $\operatorname{ran}(F_\alpha(\cdot, T))$ is dense in V. The identity (7.44) moreover also implies that $\ker(F_\alpha(\cdot, T)) = \{\mathbf{0}\}$ because $\mathbf{v} = \lim_{\mathbb{R}\ni\mu\to-\infty} F_\alpha(\mu, T)\mathbf{v} = \mathbf{0}$ for any $\mathbf{v} \in \ker(F_\alpha(\cdot, T))$.

We consider now an arbitrary point $\mu_0 \in (-\infty, 0)$. By the above arguments, the mapping $F_\alpha(\mu_0, T) : V \to \operatorname{ran}(F_\alpha(\cdot, T))$ is invertible. Hence, we can define the operator $B_\alpha := \mu_0\mathcal{I} - F_\alpha(\mu_0, T)^{-1}$ that maps $\operatorname{dom}(B_\alpha) = \operatorname{ran}(F_\alpha(\mu_0, T))$ to V. Apparently, B_α has dense domain and $S_R^{-1}(\mu_0, B_\alpha) = F_\alpha(\mu_0, B_\alpha)$. For $\mu \in (-\infty, 0]$, we can apply (7.43) and (7.39) in order to obtain

$$\begin{aligned}
&(\mu\mathcal{I} - B_\alpha)F_\alpha(\mu, T) = \\
=&((\mu - \mu_0)\mathcal{I} + (\mu_0\mathcal{I} - B_\alpha))F_\alpha(\mu_0, T)(\mathcal{I} + (\mu_0 - \mu)F_\alpha(\mu, T)) \\
=&\mathcal{I} + (\mu - \mu_0)\big(F_\alpha(\mu_0, T) + (\mu_0 - \mu)F_\alpha(\mu_0, T)F_\alpha(\mu, T) - F_\alpha(\mu, T)\big) = \mathcal{I}.
\end{aligned}$$

A similar calculation shows that $F_\alpha(\mu, T)(\mu\mathcal{I} - B_\alpha)\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \operatorname{dom}(B_\alpha)$. We conclude that $S_R^{-1}(\mu, B_\alpha) = F_\alpha(\mu, T)$ for any $\mu \in (-\infty, 0)$. Since $p \mapsto F_\alpha(p, T)$ and $p \mapsto S_R^{-1}(p, B_\alpha)$ are left slice hyperholomorphic and agree on $(-\infty, 0)$, Theorem 2.8 implies $S_R^{-1}(p, B_\alpha) = F_\alpha(p, T)$ for any $p \in \mathbb{H} \setminus cl(\Sigma_{\phi_0})$.

Finally, in order to show that B_α is of type $(M, \alpha\omega)$, we choose an arbitrary imaginary unit $\mathsf{i} \in \mathbb{S}$ and consider the restriction of $S_R^{-1}(\cdot, B_\alpha)$ to the plane $\mathbb{C}_{\mathsf{i}}$. This restriction is a holomorphic function with values in the left-vector space $\mathcal{B}(V)$ over $\mathbb{C}_{\mathsf{i}}$. We show now that this restriction has a holomorphic continuation to the sector $(\mathbb{H} \setminus cl(\Sigma_{\alpha\omega})) \cap \mathbb{C}_{\mathsf{i}}$. Since this sector is symmetric with respect to the real axis, we can apply Corollary 2.10 and obtain a left slice hyperholomorphic continuation of $S_R^{-1}(p, B_\alpha)$ to the sector $\mathbb{H} \setminus cl(\Sigma_{\alpha\omega})$. By Theorem 3.12, this implies in particular that $\mathbb{H} \setminus cl(\Sigma_{\alpha\omega}) \subset \rho_S(T)$.

The above considerations showed that we can represent $S_R^{-1}(p, B_\alpha)$ for any point $p \in (\mathbb{H} \setminus cl(\Sigma_\omega)) \cap \mathbb{C}_{\mathsf{i}}$ using Kato's formula (7.35). Rewriting this formula as a path integral over the path $\gamma_0(t) = te^{\mathsf{i}\pi}, t \in [0, +\infty)$, we obtain

$$S_R^{-1}(p, B_\alpha) = -\frac{\sin(\alpha\pi)}{\pi} \int_{\gamma_0} \frac{z^\alpha e^{-\mathsf{i}\pi\alpha}}{(z^\alpha e^{-\mathsf{i}2\alpha\pi} - p)(z^\alpha - p)} S_R^{-1}(z, T)\, dz,$$

where z denotes a complex variable in $\mathbb{C}_{\mathsf{i}}$ and $z \mapsto z^\alpha$ is a branch of a complex α-th power of z that is holomorphic on $\mathbb{C}_{\mathsf{i}} \setminus [0, \infty)$. To be more precise, let us choose $\left(re^{\mathsf{i}\theta}\right)^\alpha = r^\alpha e^{\mathsf{i}\theta\alpha}$ with $\theta \in (0, 2\pi)$. (This is however not the restriction of the quaternionic function $s \mapsto s^\alpha$ defined in Definition 7.3 to the plane $\mathbb{C}_{\mathsf{i}}$, cf. Remark 7.2!)

Observe that for fixed p the integrand is holomorphic on $D_0 := (\mathbb{H} \setminus\ cl(\Sigma_\omega)) \cap \mathbb{C}_{\mathsf{i}}$. Hence, by applying Cauchy's Integral Theorem, we can exchange the path of integration γ_0 by a suitable path $\gamma_\kappa(t) = te^{\mathsf{i}(\pi-\kappa)}, t \in [0, +\infty)$ and obtain

$$S_R^{-1}(p, B_\alpha) = -\frac{\sin(\alpha\pi)}{\pi} \int_{\gamma_\kappa} \frac{z^\alpha e^{-\mathsf{i}\pi\alpha}}{(z^\alpha e^{-\mathsf{i}2\alpha\pi} - p)(z^\alpha - p)} S_R^{-1}(z, T)\, dz. \tag{7.48}$$

On the other hand, for any $\kappa \in (-\omega, \omega)$, such integral defines a holomorphic function on the sector $D_\kappa := \{p \in \mathbb{C}_{\mathsf{i}} : \alpha(\pi - \kappa) < \arg p < 2\pi - \alpha(\pi + \kappa)\}$, where the convergence of the integral is guaranteed because the operator T is of type (M, ω). The

above argument showed that this function coincides with $S_R^{-1}(p, B_\alpha)$ on the common domain $D_0 \cap D_\kappa$, and hence $p \mapsto S_R^{-1}(p, B_\alpha)$ has a holomorphic continuation F_{i} to

$$D = \bigcup_{\kappa \in (-\omega,\omega)} D_\kappa = \{p \in \mathbb{C}_{\mathsf{i}} : \alpha(\pi - \kappa) < \arg_{\mathbb{C}_{\mathsf{i}}}(p) < 2\pi - \alpha(\pi - \kappa)\}.$$

This set is symmetric with respect to the real axis and, as mentioned above, we deduce from Corollary 2.10 that there exists a left slice hyperholomorphic extension F of F_{i} to the axially symmetric hull $[D] = \mathbb{H} \setminus cl(\Sigma_{\alpha\omega})$ of D. Consequently, $\mathbb{H} \setminus cl(\Sigma_{\alpha\omega}) \subset \rho_S(B_\alpha)$ and F coincides with $S_R^{-1}(\cdot, B_\alpha)$ on $\mathbb{H} \setminus cl(\Sigma_{\alpha\omega})$.

In order to show that $\|pS_R(p, B_\alpha)\|$ is bounded on every sector $\mathbb{H} \setminus cl(\Sigma_\theta)$ with $\theta \in (\omega\alpha, 0)$, we consider first a set

$$D_{\kappa,\delta} := \{p \in \mathbb{C}_{\mathsf{i}} : \delta + \alpha(\pi - \kappa) < \arg p < 2\pi - \alpha(\pi + \kappa) - \delta\}$$

with $\kappa \in (-\omega, \omega)$ and small $\delta > 0$. For $p \in D_{\kappa,\delta}$ with $\phi = \arg_{\mathbb{C}_{\mathsf{i}}}(p) \in (0, 2\pi)$, we may represent $pS_R^{-1}(p, B_\alpha)$ by means of (7.48) and estimate

$$\begin{aligned}
&\frac{\pi}{\sin(\alpha\pi)}\|pS_R^{-1}(p, B_\alpha)\| \leq \\
\leq &|p| \int_0^{+\infty} \frac{r^\alpha}{|(r^\alpha e^{-\mathsf{i}(\pi+\kappa)\alpha} - p)(r^\alpha e^{\mathsf{i}(\pi-\kappa)\alpha} - p)|}\|S_R^{-1}(re^{\mathsf{i}(\pi-\kappa)}, T)\|\, dr \\
= &|p| \int_0^{+\infty} \frac{r^\alpha}{|(r^\alpha - |p|e^{\mathsf{i}(\phi+(\pi+\kappa)\alpha)})(r^\alpha - |p|e^{\mathsf{i}(\phi-(\pi-\kappa)\alpha)})|}\|S_R^{-1}(re^{\mathsf{i}(\pi-\kappa)}, T)\|\, dr.
\end{aligned}$$

The operator T is of type (ω, M) and hence exists a constant $M_\kappa > 0$ such that $\|S_R^{-1}(re^{\mathsf{i}(\pi-\kappa)}, T)\| \leq M/r$. Substituting $\tau = r^\alpha/|p|$ yields

$$\|pS_R^{-1}(p, B_\alpha)\| \leq \frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} \frac{M_\kappa}{|(\tau - e^{\mathsf{i}(\phi+(\pi+\kappa)\alpha)})(\tau - e^{\mathsf{i}(\phi-(\pi-\kappa)\alpha)})|}\, d\tau.$$

This integral is uniformly bounded for

$$\phi \in (\delta + \alpha(\pi - \kappa), 2\pi - \alpha(\pi + \kappa) - \delta)$$

such that there exists a constant that depends only on κ and δ such that

$$\|pS(p, B_\alpha)\| \leq C(\kappa, \delta) \quad \text{for } p \in D_{\kappa,\delta}.$$

Now consider a sector $\mathbb{H} \setminus cl(\Sigma_\theta)$ with $\theta \in (\omega\alpha, \pi)$. Then there exist $(\kappa_\ell, \delta_\ell)$ with $\ell = 1, \ldots, n$ such that $\mathbb{H} \setminus cl(\Sigma_\theta) \cap \mathbb{C}_{\mathsf{i}} \subset \bigcup_{\ell=1}^n D_{\kappa_\ell,\delta_\ell}$ and hence

$$\|pS_R^{-1}(p, B_\alpha)\| \leq C := \max_{1 \leq i \leq n} C(\kappa_\ell, \delta_\ell) \quad \text{for } p \in \mathbb{H} \setminus cl(\Sigma_\theta) \cap \mathbb{C}_{\mathsf{i}}.$$

For arbitrary $p = p_0 + \mathsf{i}_p p_1 \in \mathbb{H} \setminus cl(\Sigma_\theta) \cap \mathbb{C}_{\mathsf{i}}$, set $p_{\mathsf{i}} = p_0 + \mathsf{i} p_1$. Then the Representation Formula, Theorem 2.9, implies

$$\|pS_R^{-1}(p, T)\| \leq \frac{1}{2}\|(1 - \mathsf{i}_p\mathsf{i})p_{\mathsf{i}}S_R^{-1}(p_{\mathsf{i}}, B_\alpha)\| + \frac{1}{2}\|(1 + \mathsf{i}_p\mathsf{i})\overline{p_{\mathsf{i}}}S_R^{-1}(\overline{p_{\mathsf{i}}}, B_\alpha)\| \leq 2C.$$

Finally, the estimate $\|tS_R^{-1}(-t, B_\alpha)\| \leq M/t$ follows immediately from (7.47).

$\square$

Definition 7.38. Let $T \in \mathcal{K}(V)$ be of type (ω, M). For $\alpha \in (0,1)$ we define $T^\alpha := B_\alpha$.

Corollary 7.39. *Definition 7.38 is consistent with Definition 7.18 and Definition 7.20.*

Proof. Let $T \in \mathcal{K}(V)$, let $\alpha \in (0,1)$ and let T^α be the operator obtained from Definition 7.38. If $\|S_R^{-1}(s,T)\| \leq K/(1+|s|)$ for $s \in (-\infty, 0]$, then we can apply Lebesgue's dominated convergence theorem in order to pass to the limit as p tends to 0 in Kato's formula (7.35) for the right S-resolvent of T^α. We obtain

$$(T^\alpha)^{-1} = -S_R^{-1}(0, T^\alpha) = -\frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} t^{-\alpha} S_R^{-1}(-t, T), \, dt = T^{-\alpha}$$

where the last equality follows from Corollary 7.11 resp. from (7.30).

□

CHAPTER 8

On the Minimal Necessary Structure: Operator Theory on One-Sided Spaces

Quaternionic operator theory is usually formulated on two-sided Banach spaces, that is under the assumption that there exists a left multiplication on the considered Banach space. Indeed, this seemed a technical requirement since the formulas (2.25) and (2.26) for the S-resolvents contain the multiplication of operators with non-real scalars. As we pointed out in Remark 2.45, the multiplication of operators with nonreal scalars is only a meaningful operation if they act on a two-sided Banach space—whereas the space of bounded operators on a one-sided quaternionic Banach space is of a real and not a quaternionic Banach space.

A right linear operator is via the linearity-condition however only related to the right multiplication on the Banach space and hence its spectral properties should be independent of any left multiplication. This could be considered a philosophical problem, but for instance on quaternionic Hilbert spaces only a right sided structure is defined a priori. In order to define the S-functional calculus of a closed operator on quaternionic Hilbert space, one has to choose a random left multiplication in order to turn it into a two-sided space. The spectral properties of the operator should then be independent of this left-multiplication. Similarly, the proofs of the spectral theorem for normal quaternionic quaternionic linear operators introduce left multiplications on the considered quaternionic Hilbert space (in [5] it is only partially defined, in [51] it is defined for all quaternions). This left multiplication is chosen to suite the considered operator in a certain sense, but it is only partially determined by the operator and then extended randomly. The authors point however out that the spectral measure associated with a normal operator is independent of the chosen extension.

In this chapter, which contains results published in [47], we show that the essential parts of quaternionic operator theory can be formulated without the assumption of a

left multiplication on the respective Banach space. We derive—at least for intrinsic function—an integral representation of the operator $f(T)$ obtained via the S-functional calculus that is independent of the left multiplication because it only contains the multiplication of vectors with nonreal scalars from the right. We then use this integral representation to define the S-functional calculus for intrinsic functions on a quaternionic right Banach space.

We also show that these parts of quaternionic theory are consistent with the complex theory, which allows us see quaternionic operator theory from a new perspective. We can then embed the complex numbers into the quaternions by identifying them with one of the complex planes $\mathbb{C}_{\mathsf{i}}$. As pointed out in Remark 2.37, we can consider the quaternionic Banach space, on which we work, as a complex Banach space and any quaternionic linear operator as complex linear. We can then obtain results from quaternionic operator theory simply by applying results from the theory of complex linear operators if we make sure that a certain symmetry condition is satisfied. This symmetry condition guarantees compatibility with the S-spectrum and assures that we stay within a quaternionic framework so that all results are independent of the chosen imbedding of the complex numbers into the quaternion.

8.1 The Relation Between Complex and Quaternionic Theory

We consider operators on a quaternionic right Banach space V_R. As pointed out in Remark 2.45, the space $\mathcal{B}(V_R)$ of all bounded right linear operators on V_R is only a real Banach space and does not admit a multiplication with non-real quaternionic scalars. However, operator

$$\mathcal{Q}_s(T) := T^2 - 2s_0T + |s|^2\mathcal{I}$$

does not contain any non-real scalar. We can hence define the S-spectrum of T just as in the case of an operator on a two-sided Banach space.

Definition 8.1. Let $T \in \mathcal{K}(V_R)$. We define the S-resolvent set of T as

$$\rho_S(T) := \left\{s \in \mathbb{H} : \mathcal{Q}_s(T)^{-1} \in \mathcal{B}(V_R)\right\}$$

and the S-spectrum of T as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

For $s \in \rho_S(T)$, we call $\mathcal{Q}_s(T)^{-1}$ the pseudo-resolvent (or spherical resolvent) of T at s. Furthermore, we define the extended S-spectrum as

$$\sigma_{SX}(T) := \begin{cases} \sigma_S(T) & \text{if } T \text{ is bounded} \\ \sigma_S(T) \cup \{\infty\} & \text{if } T \text{ is unbounded.} \end{cases}$$

In order to show the boundedness of $\sigma_S(T)$ by $\|T\|$ for a bounded operator T on a two-sided quaternionic Banach space, one usually proceeds as follows, cf. [36]. One starts from the series expansion of the left S-resolvent

$$S_L^{-1}(s,T) = \sum_{n=0}^{+\infty} T^n s^{-1-n} \quad |T\| \ < |s|$$

and shows the identity

$$\mathcal{Q}_s(T)S_L^{-1}(s,T) = (T^2 - 2s_0T + |s|^2\mathcal{I})\sum_{n=0}^{+\infty} T^n s^{-1-n} = \overline{s}\mathcal{I} - T.$$

For $\|T\| < |s|$, the operator $\overline{s}\mathcal{I} - T$ is invertible with $(\overline{s}\mathcal{I} - T)^{-1} = s^{-1}\sum_{n=0}^{+\infty}(Ts^{-1})^n$, and so one finds

$$\mathcal{Q}_s(T)S_L^{-1}(s,T)(\overline{s}\mathcal{I} - T)^{-1} = \mathcal{I}.$$

Similarly, one can show that $(\overline{s}\mathcal{I} - T)^{-1}S_R^{-1}(s,T)\mathcal{Q}_s(T) = \mathcal{I}$ and hence $\mathcal{Q}_s(T)$ is invertible if $\|T\| < |s|$.

These computations obviously require the multiplication of T and $\mathcal{I}$ with s resp. $\overline{s}$, which are in general not real. In Section 4 of [22], we however introduced a series expansion for the pseudo-resolvent. This series expansion has real coefficients and hence also holds if the operator acts only on a right Banach space, so that we can show that the properties of the S-spectrum are the same in this setting.

Theorem 8.2. *Let $T \in \mathcal{K}(V_R)$.*

(i) The S-spectrum $\sigma_S(T)$ is a closed subset of $\mathbb{H}$ and the extended S-spectrum $\sigma_{SX}(T)$ is a closed and hence compact subset of $\mathbb{H}_\infty$.

(ii) If T is bounded, then $\sigma_S(T)$ is bounded by the norm of T, that is

$$\sigma_S(T) \subset cl(B_{\|T\|}(0)).$$

For s with $\|T\| < |s|$, the pseudo-resolvent of T at s is given by

$$\mathcal{Q}_s(T)^{-1} = \sum_{n=0}^{+\infty} T^n a_n \quad \text{with} \quad a_n := \sum_{k=0}^{n} \overline{s}^{-k-1}s^{-n+k-1}, \tag{8.1}$$

where this series converges in the operator norm.

Proof. The space $\mathcal{B}(X)$ as a real Banach algebra. The set $\mathrm{Inv}(\mathcal{B}(X))$ of invertible elements of this real Banach algebra is open (see for example Theorem 10.12 in [74]). Since $\tau : s \mapsto \mathcal{Q}_s(T)$ is a continuous function with values in $\mathcal{B}(X)$, we find that $\rho_S(T) = \tau^{-1}(\mathrm{Inv}(\mathcal{B}(X)))$ is open in $\mathbb{H}$ and that $\sigma_S(T)$ in turn is closed. If $\sigma_S(T)$ is unbounded, then (ii) implies that T is unbounded and hence $\sigma_{SX}(T) = \sigma_S(T) \cup \{\infty\}$ is closed in $\mathbb{H}_\infty$. Otherwise, $\sigma_S(T)$ is already closed in $\mathbb{H}$ so that $\sigma_{SX}(T)$ is closed in $\mathbb{H}_\infty$. These arguments are analogue to the case of operators on two-sided Banach spaces in [36], but we repeated them for the sake of completeness.

Let us now consider the series (8.1). First of all, observe that the coefficients a_n are actually real so that this series is meaningful since

$$\overline{a_n} = \overline{\sum_{k=0}^{n} \overline{s}^{-k-1}s^{-n+k-1}} = \sum_{k=0}^{n} \overline{s}^{-n+k-1}s^{-k-1} = a_n.$$

We furthermore have

$$|a_n| \leq \sum_{k=0}^{n} |\overline{s}|^{-k-1}|s|^{-n+k-1} = (n+1)|s|^{-n-2}$$

so that

$$\sum_{n=0}^{+\infty} \|T^n a_n\| \leq \sum_{n=0}^{+\infty} \|T\|^n |s|^{-n-2}(n+1).$$

Since $\|T\| < |s|$, the ratio test implies the convergence of this series as

$$\lim_{n\to\infty} \frac{\|T\|^{n+1}|s|^{-n-3}(n+2)}{\|T\|^n|s|^{-n-2}(n+1)} = \lim_{n\to\infty} \frac{(n+2)\|T\|}{(n+1)|s|} = \frac{\|T\|}{|s|} < 1.$$

We furthermore have

$$\begin{aligned}
\mathcal{Q}_s(T)\sum_{n=1}^{+\infty} T^n a_n &= (T^2 - 2s_0T + |s|^2\mathcal{I})\sum_{n=0}^{+\infty} T^n a_n\\
=&\sum_{n=2}^{+\infty} T^n a_{n-2} - \sum_{n=1}^{+\infty} T^n a_{n-1} 2s_0 + \sum_{n=0}^{+\infty} T^n a_n |s|^2\\
=&\sum_{n=2}^{+\infty} T^n (a_{n-2} - a_{n-1}2s_0 + a_n|s|^2)\\
&+ T(a_0 2s_0 + a_1|s|^2) + \mathcal{I}a_0|s|^2.
\end{aligned}$$

Since $2s_0 = s + \overline{s}$ and $|s|^2 = s\overline{s} = \overline{s}s$, we have for $n \geq 2$ that

$$\begin{aligned}
&a_{n-2} - a_{n-1}2s_0 + a_n|s|^2 =\\
=&\sum_{k=0}^{n-2} \overline{s}^{-k-1}s^{-n+1+k} - \sum_{k=0}^{n-1} \overline{s}^{-k-1}2s_0s^{-n+k} + \sum_{k=0}^{n} \overline{s}^{-k-1}|s|^2 s^{-n+k-1}\\
=&\sum_{k=1}^{n-1} \overline{s}^{-k}s^{-n+k} - \sum_{k=0}^{n-1} \overline{s}^{-k}s^{-n+k} - \sum_{k=0}^{n-1} \overline{s}^{-k-1}s^{-n+k+1} + \sum_{k=0}^{n} \overline{s}^{-k}s^{-n+k}\\
=& - s^{-n} + s^{-n} \ = 0.
\end{aligned}$$

Similarly, we also find because of $s^{-1} = |s|^{-2}\overline{s}$ and $\overline{s}^{-1} = |s|^{-2}s$ that

$$\begin{aligned}
a_0 2s_0 - a_1|s|^2 &= |s|^{-2}(s+\overline{s}) - \left(\overline{s}^{-1}s^{-2} + \overline{s}^{-2}s^{-1}\right)|s|^2\\
&= \overline{s}^{-1} + s^{-1} - |s|^{-2}\left(s^{-1} + \overline{s}^{-1}\right)|s|^2 = 0
\end{aligned}$$

and so altogether

$$\mathcal{Q}_s(T)\sum_{n=0}^{+\infty} T^n a_n = \mathcal{I}a_0|s|^2 = \mathcal{I}.$$

Since the coefficients are real, they commute with T an $\mathcal{Q}_s(T)$ and therefore also

$$\sum_{n=0}^{+\infty} T^n a_n \mathcal{Q}_s(T) = \mathcal{Q}_s(T)\sum_{n=0}^{+\infty} T^n a_n = \mathcal{I}.$$

Therefore $\mathcal{Q}_s(T)$ is invertible for any s with $\|T\| \ < |s|$ and so $\sigma_S(T) \subset cl(B_{\|T\|}(0))$.

□

We continue by investigating the relations between some of the fundamental notions of operator theory defined by the quaternionic structure and those defined by the various complex structures on the quaternionic right Banach space V_R. As pointed out in Remark 2.37, any quaternionic right Banach space V_R can in a natural way be considered a complex Banach space over any of the complex planes $\mathbb{C}_{\mathsf{i}}$ by restricting the multiplication with quaternionic scalars from the right to $\mathbb{C}_{\mathsf{i}}$. In order to deal with the different structures on V_R, we introduce the following notation.

Notation 8.3. Let V_R be a quaternionic right Banach space. For $\mathsf{i} \in \mathbb{S}$, we denote the space V_R considered as a complex Banach space over the complex field $\mathbb{C}_{\mathsf{i}}$ by $V_{R,\mathsf{i}}$. If T is a quaternionic right linear operator on V_R, then $\rho_{\mathbb{C}_{\mathsf{i}}}(T)$ and $\sigma_{\mathbb{C}_{\mathsf{i}}}(T)$ shall denote its resolvent set and spectrum as a $\mathbb{C}_{\mathsf{i}}$-complex linear operator on $V_{R,\mathsf{i}}$. If A is a $\mathbb{C}_{\mathsf{i}}$-complex linear, but not quaternionic linear operator on $V_{R,\mathsf{i}}$, then we denote its spectrum as usual by $\sigma(A)$.

If we want to distinguish between the identity operator on V_R and the identity operator on $V_{R,\mathsf{i}}$ we denote them by $\mathcal{I}_{V_R}$ and $\mathcal{I}_{V_{R,\mathsf{i}}}$. We point out that the operator $\lambda\mathcal{I}_{V_{R,\mathsf{i}}}$ for $\lambda \in \mathbb{C}_{\mathsf{i}}$ acts as $\lambda\mathcal{I}_{V_{R,\mathsf{i}}}\mathbf{v} = \mathbf{v}\lambda$ because the multiplication with scalars on $V_{R,\mathsf{i}}$ is defined as the quaternionic right scalar multiplication on V_R restricted to $\mathbb{C}_{\mathsf{i}}$.

We shall now specify an observation made in [54, Theorem 5.9], where the authors showed that $\rho_S(T)$ is the axially symmetric hull of $\rho_{\mathbb{C}_{\mathsf{i}}}(T) \cap \overline{\rho_{\mathbb{C}_{\mathsf{i}}}(T)}$.

Theorem 8.4. *Let $T \in \mathcal{K}(V_R)$ and choose $\mathsf{i} \in \mathbb{S}$. The spectrum $\sigma_{\mathbb{C}_{\mathsf{i}}}(T)$ of T considered as a closed complex linear operator on $V_{R,\mathsf{i}}$ equals $\sigma_S(T) \cap \mathbb{C}_{\mathsf{i}}$, i.e.*

$$\sigma_{\mathbb{C}_{\mathsf{i}}}(T) = \sigma_S(T) \cap \mathbb{C}_{\mathsf{i}}. \tag{8.2}$$

For any λ in the resolvent set $\rho_{\mathbb{C}_{\mathsf{i}}}(T)$ of T as a complex linear operator on $V_{R,\mathsf{i}}$, the $\mathbb{C}_{\mathsf{i}}$-linear resolvent of T is given by $R_\lambda(T) = \left(\overline{\lambda}\mathcal{I}_{V_{R,\mathsf{i}}} - T\right)\mathcal{Q}_\lambda(T)^{-1}$, i.e.

$$R_\lambda(T)\mathbf{v} := \mathcal{Q}_\lambda(T)^{-1}\mathbf{v}\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}\mathbf{v}. \tag{8.3}$$

For any $\mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$, we moreover have

$$R_{\overline{\lambda}}(T)\mathbf{v} = -[R_\lambda(T)(\mathbf{v}\mathsf{j})]\mathsf{j}. \tag{8.4}$$

Proof. Let $\lambda \in \rho_S(T) \cap \mathbb{C}_{\mathsf{i}}$. The resolvent $(\lambda\mathcal{I}_{V_{R,\mathsf{i}}} - T)^{-1}$ of T as a $\mathbb{C}_{\mathsf{i}}$-linear operator on $V_{R,\mathsf{i}}$ is then given by (8.3). Indeed, since T and $Q_\lambda(T)^{-1}$ commute, we have for $\mathbf{v} \in \operatorname{dom}(T)$ that

$$\begin{aligned}
&=(\overline{\lambda}\mathcal{I}_{V_{R,\mathsf{i}}} - T)\mathcal{Q}_\lambda(T)^{-1}(\mathbf{v}\lambda - T\mathbf{v})\\
&=(\overline{\lambda}\mathcal{I}_{V_{R,\mathsf{i}}} - T)\left(\mathcal{Q}_\lambda(T)^{-1}\mathbf{v}\lambda - T\mathcal{Q}_\lambda(T)^{-1}\mathbf{v}\right)\\
&=\mathcal{Q}_\lambda(T)^{-1}\mathbf{v}\lambda\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}\mathbf{v}\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}\mathbf{v}\lambda + T^2\mathcal{Q}_\lambda(T)^{-1}\mathbf{v}\\
&=(|\lambda|^2\mathcal{I}_{V_{R,\mathsf{i}}} - 2\lambda_0 T + T^2)\mathcal{Q}_\lambda(T)^{-1}\mathbf{v} = \mathbf{v}.
\end{aligned}$$

Similarly, for $\mathbf{v} \in V_{R,\mathsf{i}} = V_R$, we have

$$\begin{aligned}
&(\lambda\mathcal{I}_{V_{R,\mathsf{i}}} - T)R_\lambda(T)\mathbf{v}\\
&=(\lambda\mathcal{I}_{V_{R,\mathsf{i}}} - T)\left(\mathcal{Q}_\lambda(T)^{-1}\mathbf{v}\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}\mathbf{v}\right)\\
&=\mathcal{Q}_\lambda(T)^{-1}\mathbf{v}\overline{\lambda}\lambda - T\mathcal{Q}_\lambda(T)^{-1}\mathbf{v}\lambda - T\mathcal{Q}_\lambda(T)^{-1}\mathbf{v}\overline{\lambda} + T^2\mathcal{Q}_\lambda(T)^{-1}\mathbf{v}\\
&=(|\lambda|^2\mathcal{I}_{V_{R,\mathsf{i}}} - 2\lambda_0 T + T^2)\mathcal{Q}_\lambda(T)^{-1}\mathbf{v} = \mathbf{v}.
\end{aligned}$$

Since $\mathcal{Q}_\lambda(T)^{-1}$ maps $V_{R,\mathsf{i}}$ to $\operatorname{dom}(T^2) \subset \operatorname{dom}(T)$, we find that the operator $R_\lambda(T) = (\lambda \mathcal{I}_{V_{R,\mathsf{i}}} - T)\mathcal{Q}_\lambda(T)^{-1}$ is bounded and so λ belongs to the resolvent set $\rho_{\mathbb{C}_\mathsf{i}}(T)$ of T considered as a $\mathbb{C}_\mathsf{i}$-linear operator on $V_{R,\mathsf{i}}$. Hence, $\rho_S(T) \cap \mathbb{C}_\mathsf{i} \subset \rho_{\mathbb{C}_\mathsf{i}}(T)$ and in turn $\sigma_{\mathbb{C}_\mathsf{i}}(T) \subset \sigma_S(T) \cap \mathbb{C}_\mathsf{i}$. Together with the axial symmetry of the S-spectrum, this further implies

$$\sigma_{\mathbb{C}_\mathsf{i}}(T) \cup \overline{\sigma_{\mathbb{C}_\mathsf{i}}(T)} \subset (\sigma_S(T) \cap \mathbb{C}_\mathsf{i}) \cup (\overline{\sigma_S(T) \cap \mathbb{C}_\mathsf{i}}) = \sigma_S(T) \cap \mathbb{C}_\mathsf{i}, \tag{8.5}$$

where $\overline{A} = \{\overline{z} : z \in A\}$.

If λ and $\overline{\lambda}$ both belong to $\rho_{\mathbb{C}_\mathsf{i}}(T)$, then $[\lambda] \subset \rho_S(T)$ because

$$\begin{aligned}
&(\lambda \mathcal{I}_{V_{R,\mathsf{i}}} - T)(\overline{\lambda} \mathcal{I}_{V_{R,\mathsf{i}}} - T)\mathbf{v} \\
=&(\mathbf{v}\overline{\lambda})\lambda - (T\mathbf{v})\lambda - T(\mathbf{v}\overline{\lambda}) + T^2\mathbf{v} \\
=&(T^2 - 2\lambda_0 T + |\lambda|^2)\mathbf{v}
\end{aligned}$$

and hence $\mathcal{Q}_\lambda(T)^{-1} = R_\lambda(T)R_{\overline{\lambda}}(T) \in \mathcal{B}(V_R)$. Thus $\rho_S(T) \cap \mathbb{C}_\mathsf{i} \supset \rho_{\mathbb{C}_\mathsf{i}}(T) \cap \overline{\rho_{\mathbb{C}_\mathsf{i}}(T)}$ and in turn

$$\sigma_S(T) \cap \mathbb{C}_\mathsf{i} \subset \sigma_{\mathbb{C}_\mathsf{i}}(T) \cup \overline{\sigma_{\mathbb{C}_\mathsf{i}}(T)}. \tag{8.6}$$

The two relations (8.5) and (8.6) yield together

$$\sigma_S(T) \cap \mathbb{C}_\mathsf{i} = \sigma_{\mathbb{C}_\mathsf{i}}(T) \cup \overline{\sigma_{\mathbb{C}_\mathsf{i}}(T)}. \tag{8.7}$$

What remains to show is that $\rho_{\mathbb{C}_\mathsf{i}}(T)$ and $\sigma_{\mathbb{C}_\mathsf{i}}(T)$ are symmetric with respect to the real axis, which then implies

$$\sigma_S(T) \cap \mathbb{C}_\mathsf{i} = \sigma_{\mathbb{C}_\mathsf{i}}(T) \cup \overline{\sigma_{\mathbb{C}_\mathsf{i}}(T)} = \sigma_{\mathbb{C}_\mathsf{i}}(T). \tag{8.8}$$

Let $\lambda \in \rho_{\mathbb{C}_\mathsf{i}}(T)$ and choose $\mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$. We show that $R_{\overline{\lambda}}(T)$ equals the mapping $A\mathbf{v} := -\left[R_\lambda(T)(\mathbf{v}\mathsf{j})\right]\mathsf{j}$. As $\lambda\mathsf{j} = \mathsf{j}\overline{\lambda}$ and $\mathsf{j}\lambda = \overline{\lambda}\mathsf{j}$, we have for $\mathbf{v} \in \operatorname{dom}(T)$ that

$$\begin{aligned}
&A\left(\overline{\lambda}\mathcal{I}_{V_{R,\mathsf{i}}} - T\right)\mathbf{v} = A\left(\mathbf{v}\overline{\lambda} - T\mathbf{v}\right) \\
=&-\left[R_\lambda(T)\left((\mathbf{v}\overline{\lambda})\mathsf{j} - (T\mathbf{v})\mathsf{j}\right)\right]\mathsf{j} \\
=&-\left[R_\lambda(T)((\mathbf{v}\mathsf{j})\lambda - T(\mathbf{v}\mathsf{j}))\right]\mathsf{j} \\
=&-\left[R_\lambda(T)(\lambda\mathcal{I}_{V_{R,\mathsf{i}}} - T)(\mathbf{v}\mathsf{j})\right]\mathsf{j} = -\mathbf{v}\mathsf{j}\mathsf{j} = \mathbf{v}.
\end{aligned}$$

Similarly, for arbitrary $\mathbf{v} \in V_{R,\mathsf{i}} = V_R$, we have

$$\begin{aligned}
&\left(\overline{\lambda}\mathcal{I}_{V_{R,\mathsf{i}}} - T\right)A\mathbf{v} = (A\mathbf{v})\,\overline{\lambda} - T\,(A\mathbf{v}) \\
=&-\left[R_\lambda(T)(\mathbf{v}\mathsf{j})\right]\mathsf{j}\overline{\lambda} + T\left(\left[R_\lambda(T)(\mathbf{v}\mathsf{j})\right]\mathsf{j}\right) \\
=&-\left[R_\lambda(T)(\mathbf{v}\mathsf{j})\lambda - T(R_\lambda(T)(\mathbf{v}\mathsf{j}))\right]\mathsf{j} \\
=&-\left[(\lambda\mathcal{I}_{V_{R,\mathsf{i}}} - T)R_\lambda(T)(\mathbf{v}\mathsf{j})\right]\mathsf{j} = -\mathbf{v}\mathsf{j}\mathsf{j} = \mathbf{v}.
\end{aligned}$$

Hence, if $\lambda \in \rho_{\mathbb{C}_\mathsf{i}}(T)$, then $R_{\overline{\lambda}}(T) = -\left[R_\lambda(T)(\mathbf{v}\mathsf{j})\right]\mathsf{j}$ such that in particular $\overline{\lambda} \in \rho_{\mathbb{C}_\mathsf{i}}(T)$. Consequently $\rho_{\mathbb{C}_\mathsf{i}}(T)$ and in turn also $\sigma_{\mathbb{C}_\mathsf{i}}(T)$ are symmetric with respect to the real axis such that (8.8) holds true.

□

Remark 8.5. The relation (8.7) has already been observed in [54, Theorem 5.9]. For the sake of completeness, we repeated the respective arguments here. Also the relation $R_\lambda(T)R_{\overline{\lambda}}(T) = \mathcal{Q}_\lambda(T)^{-1}$, which is a consequence of (8.3), was understood by the authors. The novelty in the above theorem is hence the fact that for a quaternionic linear operator T automatically $\sigma_{\mathbb{C}_\mathsf{i}}(T) = \overline{\sigma_{\mathbb{C}_\mathsf{i}}(T)}$ due to (8.4). For unitary operators, this symmetry was already understood by Sharma and Coulson, as the I learnt only recently. In [78] they showed the following: if T is a unitary quaternionic linear operator on a quaternionic Hilbert space $\mathcal{H}$, then the spectrum $\sigma(T_s)$ of the operator T_s induced by T on the symplectic image $\mathcal{H}_s$ satisfies $\sigma(T_s) = \overline{\sigma(T_s)}$. Their strategy for showing this was however different. Since the S-spectrum and the associated pseudo-resolvent had not yet been developed, they showed this symmetry for the set of approximate eigenvalues of T_s using (1.1). Since with T also T_s is unitary, $\sigma(T_s)$ only consists of approximate eigenvalues and the statement follows.

Definition 8.6. Let $T \in \mathcal{K}(V_R)$. We define the V_R-valued function

$$\mathcal{R}_s(T;\mathbf{v}) = \mathcal{Q}_s(T)^{-1}\mathbf{v}\overline{s} - T\mathcal{Q}_s(T)^{-1}\mathbf{v} \qquad \forall \mathbf{v} \in V_R,\ s \in \rho_S(T).$$

Remark 8.7. By Theorem 8.4, the mapping $\mathbf{v} \mapsto \mathcal{R}_s(T;\mathbf{v})$ coincides with the resolvent of T at s applied to $\mathbf{v}$ if T is considered a $\mathbb{C}_{\mathsf{i}_s}$-linear operator on V_{R,i_s}.

Corollary 8.8. *Let $T \in \mathcal{K}(V_R)$. For any $\mathbf{v} \in V_R$, the function $\mathbf{f}(s) := \mathcal{R}_s(T;\mathbf{v})$ is right slice hyperholomorphic on $\rho_S(T)$.*

Proof. Obviously, we have $\mathbf{f}(s) = \boldsymbol{\alpha}(s_0,s_1) + \boldsymbol{\beta}(s_0,s_1)\mathsf{i}_s$ with

$$\boldsymbol{\alpha}(s_0,s_1) = \mathcal{Q}_s(T)^{-1}\mathbf{v}s_0 - T\mathcal{Q}_s(T)^{-1}\mathbf{v} \quad \text{and} \quad \boldsymbol{\beta}(s_0,s_1) = -\mathcal{Q}_s(T)^{-1}\mathbf{v}s_1,$$

which satisfy the compatibility conditions (2.4). The property that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy the Cauchy-Riemann-equations (2.6) is equivalent to $\mathbf{f}|_{\rho_S(T)\cap\mathbb{C}_\mathsf{i}}$ being (right) holomorphic for any $\mathsf{i} \in \mathbb{S}$. But this is true because $\mathbf{f}|_{\rho_S(T)\cap\mathbb{C}_\mathsf{i}}$ coincides with the resolvent of T as an operator on $V_{R,\mathsf{i}}$ applied to $\mathbf{v}$ by Theorem 8.4, which is a holomorphic function on $\rho_{\mathbb{C}_\mathsf{i}}(T) = \rho_S(T) \cap \mathbb{C}_\mathsf{i}$.

□

The above result allows us to write the operator $f(T)$ obtained via the S-functional calculus for an intrinsic slice hyperholomorphic function without making use of a quaternionic multiplication on $\mathcal{B}(V_R)$.

Theorem 8.9. *Let $T \in \mathcal{K}(V)$ be a closed operator on a two-sided quaternionic Banach space V with $\rho_S(T) \neq \emptyset$ and let $f \in \mathcal{SH}(\sigma_{SX}(T)))$. For any $\mathsf{i} \in \mathbb{S}$ and any unbounded slice Cauchy domain U with $\sigma_S(T) \subset U$ and $cl(U) \subset \mathcal{D}(F)$, the operator $f(T)$ obtained via the S-functional calculus satisfies*

$$f(T)\mathbf{v} = \mathbf{v}f(\infty) + \int_{\partial(U\cap\mathbb{C}_\mathsf{i})} \mathcal{R}_z(T;\mathbf{v})f(z)\,dz\frac{-\mathsf{i}}{2\pi} \quad \forall \mathbf{v} \in V. \tag{8.9}$$

Proof. Let U be a slice Cauchy domain such that $\sigma_S(T) \subset U$ and $cl(U) \subset \operatorname{dom}(f)$. We then have for any $\mathsf{i} \in \mathbb{S}$ and any $\mathbf{v} \in V$ that

$$f(T)\mathbf{v} = f(\infty)\mathbf{v} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_\mathsf{i})} f(s)\,ds_\mathsf{i}\,S_R^{-1}(s,T)\mathbf{v}. \tag{8.10}$$

If $\gamma_\ell : [0,1] \to \mathbb{C}_{\mathbf{i}}^+$, $\ell = 1,\dots N$ is the part of $\partial(U\cap\mathbb{C}_{\mathbf{i}})$ that lies in $\mathbb{C}_{\mathbf{i}}^{\geq}$ as in Definition 4.4, then we have by Lemma 4.11 that

$$\begin{aligned}
&\int_{\partial(U\cap\mathbb{C}_{\mathbf{i}})} f(s)\,ds_{\mathbf{i}}\,S_R^{-1}(s,T)\mathbf{v}\\
=&\sum_{\ell=1}^{N}\int_0^1 2\mathrm{Re}\left(f(\gamma_\ell(t))(-\mathbf{i})\gamma_\ell'(t)\overline{\gamma_\ell(t)}\mathbf{v}\right)\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\mathbf{v}\,dt\\
&-\sum_{\ell=1}^{N}\int_0^1 2\mathrm{Re}\Big(f(\gamma_\ell(t))(-\mathbf{i})\gamma_\ell'(t)\Big)T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\mathbf{v}\,dt.
\end{aligned}\tag{8.11}$$

Since $\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\mathbf{v}$ and $TQ_{\gamma_\ell(t)}(T)^{-1}\mathbf{v}$ commute with real numbers, we furthermore have

$$\begin{aligned}
&\int_{\partial(U\cap\mathbb{C}_{\mathbf{i}})} f(s)\,ds_{\mathbf{i}}\,S_R^{-1}(s,T)\mathbf{v}\\
=&\sum_{\ell=1}^{N}\int_0^1 \mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\mathbf{v}\,2\mathrm{Re}\left(f(\gamma_\ell(t))(-\mathbf{i})\gamma_\ell'(t)\overline{\gamma_\ell(t)}\right)\,dt\\
&-\sum_{\ell=1}^{N}\int_0^1 T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\mathbf{v}\,2\mathrm{Re}\Big(f(\gamma_\ell(t))(-\mathbf{i})\gamma_\ell'(t)\Big)\,dt\\
=&\sum_{\ell=1}^{N}\int_0^1 \left(\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\mathbf{v}\overline{\gamma_\ell(t)}-T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\mathbf{v}\right)f(\gamma_\ell(t))\gamma_\ell'(t)\,dt(-\mathbf{i})\\
&-\sum_{\ell=1}^{N}\int_0^1 \left(\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\mathbf{v}\gamma_\ell(t)-T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\mathbf{v}\right)\overline{f(\gamma_\ell(t))\gamma_\ell'(t)}\,dt(-\mathbf{i}).
\end{aligned}$$

Recalling that $f(\overline{x}) = \overline{f(x)}$ because f is intrinsic, that $\mathcal{Q}_{\overline{s}}(T)^{-1} = \mathcal{Q}_s(T)^{-1}$ for any $s\in\rho_S(T)$ and that $(-\overline{\gamma_\ell})(t) = -\overline{\gamma_\ell'(1-t)}$, we thus find

$$\begin{aligned}
&\int_{\partial(U\cap\mathbb{C}_{\mathbf{i}})} f(s)\,ds_{\mathbf{i}}\,S_R^{-1}(s,T)\mathbf{v}\\
=&\sum_{\ell=1}^{N}\int_{\gamma_\ell}\left(\mathcal{Q}_z(T)^{-1}\mathbf{v}\overline{z}-T\mathcal{Q}_z(T)^{-1}\mathbf{v}\right)f(z)\,dz(-\mathbf{i})\\
&+\sum_{\ell=1}^{N}\int_{-\overline{\gamma_\ell}}\left(\mathcal{Q}_z(T)^{-1}\mathbf{v}\overline{z}-T\mathcal{Q}_z(T)^{-1}\mathbf{v}\right)f(z)\,dt(-\mathbf{i})\\
=&\int_{\partial(U\cap\mathbb{C}_{\mathbf{i}})}\left(\mathcal{Q}_z(T)^{-1}\mathbf{v}\overline{z}-T\mathcal{Q}_z(T)^{-1}\mathbf{v}\right)f(z)\,dz(-\mathbf{i})\\
=&\int_{\partial(U\cap\mathbb{C}_{\mathbf{i}})}\mathcal{R}_z(T;\mathbf{v})f(z)\,dz(-\mathbf{i}).
\end{aligned}$$

Finally, observe that $f(\infty) = \lim_{s\to\infty} f(s)\in\mathbb{R}$ because, as an intrinsic function, f takes only real values on the real line. Since any vector commutes with real numbers,

we can hence rewrite (8.10) as

$$f(T)\mathbf{v} = \mathbf{v}f(\infty) + \int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} \mathcal{R}_z(T;\mathbf{v})f(z)\,dz\frac{(-\mathsf{i})}{2\pi}.$$

□

The above theorem shows once again that the S-functional calculus is the proper generalisation of the holomorphic Riesz-Dunford functional calculus. Indeed, combining it with Theorem 8.4 reveals another deep relation between these two techniques.

Corollary 8.10. *Let $T \in \mathcal{K}(V)$, let $f \in \mathcal{SH}(\sigma_{SX}(T))$ and let $\mathsf{i} \in \mathbb{S}$. Applying the S-functional calculus for T to f or considering T as a $\mathbb{C}_{\mathsf{i}}$-linear operator on V_{i} and applying the holomorphic Riesz-Dunford functional calculus to $f_{\mathsf{i}} := f|_{\mathcal{D}(f)\cap\mathbb{C}_{\mathsf{i}}}$ yield the same operator. Both techniques are compatible.*

Another important observation is the independence of the S-functional calculus of the left multiplication if f is intrinsic.

Corollary 8.11. *Let V_R be a quaternionic right Banach space and let $T \in \mathcal{K}(V_R)$. Let $f \in \mathcal{SH}(\sigma_{SX}(T))$ and assume that it is possible to endow V_R with two different left scalar multiplications that turn it into a two-sided quaternionic Banach space. We denote these two-sided Banach spaces by V_1 and V_2 and we denote the operators obtained by applying the S-functional calculus of T on V_1 resp. V_2 to f by $[f(T)]_1$ and $[f(T)]_2$. We then have*

$$[f(T)]_1\mathbf{v} = [f(T)]_2\mathbf{v} \quad \forall \mathbf{v} \in V_R\ = V_1 = V_2.$$

Proof. The operators $[f(T)]_1$ and $[f(T)]_2$ can both be represented by (8.9). This formula does however not involve any multiplication with scalars from the left such that we obtain the statement.

□

Remark 8.12. We point out that the representation (8.9) and also Corollary 8.11 only hold for intrinsic functions. Indeed, the symmetry $f(\overline{x}) = \overline{f(x)}$, which is satisfied only by intrinsic functions, is crucial in the proof of Theorem 8.9. For left or right slice-hyperholomorphic functions that are not intrinsic, the operator $f(T)$ will in general depend on the left multiplication. Any left slice hyperholomorphic function f can for instance be written as

$$f(s) = f_0(s) + f_1(s)e_1 + f_2(s)e_2 + f_3(s)e_3$$

with intrinsic slice hyperholomorphic components $f_\ell, \ell = 0, \dots, 3$, where $e_\ell, \ell = 1, 2, 3$ is the generating basis of the quaternions. The operators $f_\ell(T), \ell = 0, \dots, 3$ are then determined by the right multiplication on the space, but

$$f(T) = f_0(T) + f_1(T)e_1 + f_2(T)e_2 + f_3(T)e_3$$

obviously depends on how the imaginary units $e_\ell, \ell = 1, 2, 3$ are multiplied onto vectors from the left.

8.2 Intrinsic S-Functional Calculus on Right-Sided Spaces

We observe once again, that (8.9) does only contain multiplications of operators with real numbers and multiplications of vectors with quaternionic scalars from the right—operations that are also available on a right Banach space. It does not contain any operation that requires a two-sided Banach space, i.e. neither multiplications of vectors with scalars from the left nor multiplications of operators with non-real scalars. We can hence use this formula to define the S-functional calculus for operators on right Banach spaces.

Definition 8.13. Let V_R be a quaternionic right Banach space and let $T \in \mathcal{K}(V_R)$. For $f \in \mathcal{SH}(\sigma_{SX}(T))$, we define

$$f(T)\mathbf{v} := \mathbf{v}f(\infty) + \int_{\partial(U\cap\mathbb{C}_{\mathsf{i}})} \mathcal{R}_s(T;\mathbf{v})f(z)\,dz\frac{-\mathsf{i}}{2\pi},$$

where $\mathsf{i} \in \mathbb{S}$ is an arbitrary imaginary unit and U is an unbounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $cl(U) \subset \mathcal{D}(f)$.

The operator is well-defined. Cauchy's integral theorem guarantees the independence of the choice of U and if we perform the computations in the proof of Theorem 8.9 in the inverse order, we arrive at the representation (8.11) for $f(T)$. This formula does not contain any imaginary units but only real scalars so that $f(T)$ is also independent of the choice of $\mathsf{i} \in \mathbb{S}$. In particular this representation also guarantees that the obtained operator is again quaternionic right linear. Moreover, because of Theorem 8.4, the operator $f(T)$ coincides with the operator that we obtain if we consider T as a closed $\mathbb{C}_{\mathsf{i}}$-linear operator on $V_{R,\mathsf{i}}$ and apply the Riesz-Dunford functional calculus to $f_{\mathsf{i}} = f|_{\mathcal{D}(f)\cap\mathbb{C}_{\mathsf{i}}}$. The following lemma shows that the intrinsic S-functional calculus of an operator on a right Banach space has the same properties as the S-functional calculus defined with the usual approach.

Lemma 8.14. *Let $T \in \mathcal{K}(V_R)$.*

(i) We have $(af+g)(T) = af(T)+g(T)$ and $(fg)(T) = f(T)g(T)$ for all functions $f, g \in \mathcal{SH}(\sigma_{SX}(T))$ and all $a \in \mathbb{R}$.

(ii) For any $f(T) \in \mathcal{SH}(\sigma_S(T))$, the operator $f(T)$ commutes with T and moreover also with any bounded operator $A \in \mathcal{B}(V_R)$ that commutes with T.

(iii) The spectral mapping theorem holds

$$f(\sigma_{SX}(T)) = \sigma_S(f(T)) = \sigma_{SX}(f(T)) \tag{8.12}$$

for all $f \in \mathcal{SH}(\sigma_{SX}(T))$. Moreover, if $g \in \mathcal{SH}(f(\sigma_{SX}(T))$, then $(g \circ f)(T) = g(f(T))$.

(iv) If $\Delta \subset \sigma_{SX}(T)$ is a spectral set (i.e. open and closed in $\sigma_{SX}(T)$), then let $\chi_\Delta \in \mathcal{SH}(\sigma_{SX}(T))$ be equal to 1 on a neighborhood of Δ and equal to 0 on a neighborhood of $\sigma_{SX}(T) \setminus \Delta$. The operator $\chi_\Delta(T)$ is a continuous projection that commutes with T and the right linear subspace $V_\Delta := \operatorname{ran}\chi_\Delta(T)$ of V_R is invariant under T. Finally, if we denote $T_\Delta := T|_{V_\Delta}$, then $\sigma_S(T) = \Delta$ and

$$f(T_\Delta) = f(T)|_\Delta. \tag{8.13}$$

(v) Assume that T is bounded and assume that $N \in \mathcal{B}(V_R)$ is another bounded operator that commutes with T. If $\sigma_S(N) \subset cl(B_\varepsilon(0))$ for some $\varepsilon > 0$, then

$$\sigma_S(T+N) \subset cl(B_\varepsilon(\sigma_S(T))),$$

where

$$B_\varepsilon(\sigma_S(T)) = \{s \in \mathbb{H} : \operatorname{dist}(s, \sigma_S(T)) < \varepsilon\}.$$

For any $f \in \mathcal{SH}\Big(cl(B_\varepsilon(\sigma_S(T)))\Big)$, we furthermore have

$$f(T) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!}(\partial_S{}^n f)(T) = \sum_{n=0}^{+\infty} \left(\frac{1}{n!}(\partial_S{}^n f)(T)\right) N^n,$$

where this series converges in the operator norm.

Proof. For neatness let us denote by $f_{\mathsf{i}}[T]$ the operator obtained from the Riesz-Dunford functional calculus considering T as a $\mathbb{C}_{\mathsf{i}}$-linear operator on $V_{R,\mathsf{i}}$ with $\mathsf{i} \in \mathbb{S}$. The class of intrinsic functions is closed under addition, pointwise multiplication and multiplication with real numbers. Since moreover $f(T) = f_{\mathsf{i}}[T]$, the properties in (i) and (ii) follow immediately from the properties of the Riesz-Dunford-functional calculus as

$$(af+g)(T) = (af+g)_{\mathsf{i}}[T] = af_{\mathsf{i}}[T] + g_{\mathsf{i}}[T] = af(T) + g(T)$$

as well as

$$(fg)(T) = (fg)_{\mathsf{i}}[T] = f_{\mathsf{i}}[T]g_{\mathsf{i}}[T] = f(T)g(T)$$

and

$$f(T)T = f_{\mathsf{i}}(T)T \subset Tf_{\mathsf{i}}(T) = Tf(T).$$

Any operator $A \in \mathcal{B}(V_R)$ that commutes with T is also a bounded $\mathbb{C}_{\mathsf{i}}$-linear operator on $V_{R,\mathsf{i}}$ that commutes with T and hence the consistency with the Riesz-Dunford functional calculus yields again

$$f(T)A = f_{\mathsf{i}}[T]A = Af_{\mathsf{i}}[T] = Af(T).$$

Since f is intrinsic, we have $f(\operatorname{dom}(f) \cap \mathbb{C}_{\mathsf{i}}) \subset \mathbb{C}_{\mathsf{i}}$ for all $\mathsf{i} \in \mathbb{S}$ and $f([s]) = [f(s)]$. The spectral mapping theorem in (iii) holds because Theorem 8.4 implies

$$\sigma_S(f(T)) \cap \mathbb{C}_{\mathsf{i}} = \sigma_{\mathbb{C}_{\mathsf{i}}}(f(T)) = \sigma(f_{\mathsf{i}}[T]) = f_{\mathsf{i}}(\sigma_{\mathbb{C}_{\mathsf{i}}X}(T)) = f(\sigma_{SX}(T)) \cap \mathbb{C}_{\mathsf{i}}.$$

Taking the axially symmetric hull yields (8.12). If $g \in \mathcal{SH}(f(\sigma_S(T)))$, we thus also have $g_{\mathsf{i}} \in \sigma(f_{\mathsf{i}}[T])$ and

$$(g \circ f)(T) = (g \circ f)_{\mathsf{i}}[T] = (g_{\mathsf{i}} \circ f_{\mathsf{i}})[T] = g_{\mathsf{i}}[f_{\mathsf{i}}[T]] = g(f(T)).$$

If Δ is a spectral set, then $\chi_\Delta(T)$ is a projection that commutes with T as $\chi_\Delta(T)^2 = \chi_\Delta^2(T) = \chi_\Delta(T)$. Moreover, $\Delta_{\mathsf{i}} := \Delta \cap \mathbb{C}_{\mathsf{i}}$ is a spectral set of T as a $\mathbb{C}_i$-linear operator on $V_{R,\mathsf{i}}$ and $(\chi_\Delta)_{\mathsf{i}} = \chi_{\Delta_{\mathsf{i}}}$. Hence, $\chi_\Delta(T) = \chi_{\Delta_{\mathsf{i}}}[T]$ and so $V_\Delta = \operatorname{ran} \chi_\Delta(T) = \operatorname{ran} \chi_{\Delta_{\mathsf{i}}}[T]$. The properties of the Riesz-Dunford functional calculus imply

$$\sigma_S(T_\Delta) \cap \mathbb{C}_{\mathsf{i}} = \sigma_{\mathbb{C}_{\mathsf{i}}}(T_\Delta) = \Delta_{\mathsf{i}} = \Delta \cap \mathbb{C}_{\mathsf{i}}.$$

Taking the axially symmetric hull yields $\sigma_S(T_\Delta) = \Delta$. Furthermore

$$f(T_\Delta) = f_{\mathsf{i}}[T_\Delta] = f_{\mathsf{i}}[T]|_{\operatorname{ran}\chi_{\Delta_{\mathsf{i}}}[T]} = f_{\mathsf{i}}[T]|_{\operatorname{ran}\chi_{\Delta}(T)} = f(T)|_{V_\Delta}.$$

Let finally T be bounded and let $N \in \mathcal{B}(V_R)$ with $\sigma_S(N) \subset cl(B_\varepsilon(0))$ commute with T. By Theorem 8.4 we have $\sigma_{\mathbb{C}_{\mathsf{i}}}(N) \subset cl(B_\varepsilon(0) \cap \mathbb{C}_{\mathsf{i}})$ and hence the properties of the Riesz-Dunford-functional calculus (precisely Theorem 10 in [38, Chapter VII.6]) imply

$$\begin{aligned}\sigma_S(T+N) \cap \mathbb{C}_{\mathsf{i}} &= \sigma_{\mathbb{C}_{\mathsf{i}}}(T+N)\\ &\subset \{z \in \mathbb{C}_{\mathsf{i}} : \operatorname{dist}(z, \sigma_{\mathbb{C}_{\mathsf{i}}}(T)) \leq \varepsilon\} = cl(B_\varepsilon(\sigma_S(T))) \cap \mathbb{C}_{\mathsf{i}}\end{aligned}$$

and

$$f_{\mathsf{i}}[T+N] = \sum_{n=0}^{+\infty} N^n \frac{1}{n!} f_{\mathsf{i}}^{(n)}[T],$$

where this series converges in the operator norm. We therefore find

$$\sigma_S(T+N) = [\sigma_S(T+N) \cap \mathbb{C}_{\mathsf{i}}] = [cl(B_\varepsilon(\sigma_S(T))) \cap \mathbb{C}_{\mathsf{i}}] = cl(B_\varepsilon(\sigma_S(T)))$$

and, using the fact that $(\partial_S f)_{\mathsf{i}} = (f_{\mathsf{i}})'$, we also find that

$$f(T+N) = f_{\mathsf{i}}[T+N] = \sum_{n=0}^{+\infty} N^n \frac{1}{n!} f_{\mathsf{i}}^{(n)}[T] = \sum_{n=0}^{+\infty} N^n \frac{1}{n!} (\partial_S{}^n f)(T).$$

□

Remark 8.15. We point out that (8.13) can not be shown as a property of the S-functional calculus with the usual approach: since invariant subspaces of right linear operators are only right linear subspace, it is in general not possible to define $f(T_\Delta)$. This would require V_Δ to be a two-sided subspace of the Banach space. This fact might cause technical difficulties as it happened for instance in the proof of Theorem 6.44. The approach in Definition 8.13 provides a work-around for this problem.

8.3 Concluding Remarks

The results presented in this chapter, in particular Theorem 8.4 and Theorem 8.9, showed a deep relation between complex and quaternionic operator theory and this relation will become even more apparent in part II of this thesis, cf. Lemma 9.26 and Theorem 10.19. If we choose any imaginary unit $\mathsf{i} \in \mathbb{S}$ and consider our quaternionic (right) vector space V_R as a complex vector space over $\mathbb{C}_{\mathsf{i}}$, then the quaternionic results coincide with the complex linear counterparts after suitable identifications: the spectrum $\sigma_{\mathbb{C}_{\mathsf{i}}}(T)$ of T as a $\mathbb{C}_{\mathsf{i}}$-linear operator on V_R equals $\sigma_S(T) \cap \mathbb{C}_{\mathsf{i}}$, the $\mathbb{C}_{\mathsf{i}}$-linear resolvent can be computed from the quaternionic pseudo-resolvent and vice versa and the quaternionic S-functional calculus for intrinsic slice hyperholomorphic functions coincides with the $\mathbb{C}_{\mathsf{i}}$-linear Riesz-Dunford functional calculus. Similarly, we will see that spectral integration with respect to a quaternionic spectral system on V_R is equivalent to spectral integration with respect to a suitably constructed $\mathbb{C}_{\mathsf{i}}$-linear spectral measure and a quaternionic linear operator is a quaternionic spectral operator if and

only if it is a spectral operator when considered as a $\mathbb{C}_{\mathsf{i}}$-linear operator. Once again these relations show that the theory based on the S-spectrum and slice hyperholomorphic functions developed by Colombo, Sabadini and different co-authors is actually the right approach towards a mathematically rigorous extension of classical operator theory to the noncommutative setting. The relation with the complex theory presented above offers moreover the possibility of using the powerful tools of complex operator theory to study quaternionic linear operators. In Chapter 11, we will for instance use the complex Fourier transform in order to study spectral properties of the nabla operator, which is the quaternionification of the gradient and the divergence operator.

We showed the correspondence of complex and quaternionic functional calculi only for intrinsic slice functions. This is however the best one can get: let T be a quaternionic linear operator, let $\mathsf{i} \in \mathbb{S}$ be an arbitrary imaginary unit and let $f : U \subset \mathbb{C}_{\mathsf{i}} \to \mathbb{C}_{\mathsf{i}}$ be a function suitable for the $\mathbb{C}_{\mathsf{i}}$-linear version of whichever functional calculus we want to apply. If $\mathbf{v}$ is an eigenvector of T associated with $\lambda \in \mathbb{C}_{\mathsf{i}}$ and $\mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$, then $\mathbf{v}\mathsf{j}$ is an eigenvector of T associated with $\overline{\lambda}$ as

$$T(\mathbf{v}\mathsf{j}) = (T\mathbf{v})\mathsf{j} = (\mathbf{v}\lambda)\mathsf{j} = (\mathbf{v}\mathsf{j})\overline{\lambda}.$$

The basic intuition of a functional calculus is that $f(T)$ should be obtained by applying f to the eigenvalues resp. spectral values of T. Hence $\mathbf{v}$ is an eigenvector of $f(T)$ associated with $f(\lambda)$ and $\mathbf{v}\mathsf{j}$ is an eigenvector of $f(T)$ associated with $f\left(\overline{\lambda}\right)$. If however $f(T)$ is quaternionic linear, then again

$$f(T)(\mathbf{v}\mathsf{j}) = (f(T)\mathbf{v})\mathsf{j} = (\mathbf{v}f(\lambda))\mathsf{j} = (\mathbf{v}\mathsf{j})\overline{f(\lambda)}.$$

Hence, we must have $f\left(\overline{\lambda}\right) = \overline{f(\lambda)}$ in order to obtain a quaternionic linear operator. The slice extension of f is then an intrinsic slice function. For any function f that does not satisfy this symmetry, the operator $f(T)$ will in general not be quaternionic linear. Hence we cannot expect any accordance with the quaternionic theory for such functions.

The class of intrinsic slice functions is however sufficient to recover all the spectral information about an operator: projections onto invariant subspaces obtained via the S-functional calculus are generated by characteristic functions of spectral sets, which are intrinsic slice functions. The continuous functional calculus is defined using intrinsic slice functions and even the spectral measure in the spectral theorem for normal operators can be constructed using intrinsic slice functions (cf. Section 2.4 and Section 9.4 as well as [4, 5, 49] for more details). Even more, any class of functions that can be used to recover spectral information via a functional calculus, necessarily consists of intrinsic slice functions. This is a consequence of the symmetry (1.1) resp. (1.2) of the eigenvalues resp. the S-spectra of quaternionic linear operators. Indeed, as already mentioned above, the very fundamental intuition of a functional calculus is that $f(T)$ should be defined by action of f on the spectral values of T. If in particular $\mathbf{v}$ is an eigenvalue of T associated with the eigenvalue s, then $\mathbf{v}$ should be an eigenvector of $f(T)$ associated with the eigenvalue $f(s)$, i.e.

$$T\mathbf{v} = \mathbf{v}s \qquad \text{implies} \qquad f(T)\mathbf{v} = \mathbf{v}f(s). \tag{8.14}$$

Let now $\mathsf{i} \in \mathbb{S}$ be an arbitrary imaginary unit, let $s_{\mathsf{i}} = s_0 + \mathsf{i}s_1$ and let $h \in \mathbb{H} \setminus \{0\}$ be such that $s_{\mathsf{i}} = h^{-1}sh$ as in Lemma 2.1. Then $\mathbf{v}h$ is an eigenvalue of T associated with

s_{i} as

$$T(\mathbf{v}h) = (T\mathbf{v})h = \mathbf{v}sh = (\mathbf{v}h)(h^{-1}sh) = (\mathbf{v}h)s_{\mathsf{i}}.$$

If (8.14) holds true, then

$$f(T)\mathbf{v}h = (\mathbf{v}h)f(s_{\mathsf{i}}).$$

However, we also have

$$f(T)(\mathbf{v}h) = (f(T)\mathbf{v})h = \mathbf{v}f(s)h = (\mathbf{v}h)(h^{-1}f(s)h)$$

and hence necessarily

$$f(h^{-1}sh) = f(s_{\mathsf{i}}) = h^{-1}f(s)h. \tag{8.15}$$

If we choose $h_{\mathsf{i}} = \mathsf{i}_s$, we find $f(s) = \mathsf{i}_s^{-1}f(s)\mathsf{i}_s$ and in turn $\mathsf{i}_s f(s) = f(s)\mathsf{i}_s$. A quaternion commutes with the imaginary unit i_s if and only if it belongs to $\mathbb{C}_{\mathsf{i}_s}$. Hence, $f(s) = \alpha + \mathsf{i}_s\beta$ with $\alpha, \beta \in \mathbb{R}$. Now let again $\mathsf{i} \in \mathbb{S}$ be arbitrary, set $s_{\mathsf{i}} = s_0 + \mathsf{i}s_1$ and choose $h \in \mathbb{H} \setminus \{0\}$ such that $\mathsf{i} = h^{-1}\mathsf{i}_s h$ and in turn $s_{\mathsf{i}} = s_0 + \mathsf{i}s_1 = s_0 + h^{-1}\mathsf{i}_s h s_1 = h^{-1}sh$. Equation (8.15) implies then

$$f(s_{\mathsf{i}}) = f(h^{-1}sh) = h^{-1}f(s)h = \alpha + h^{-1}\mathsf{i}_s h\beta = \alpha + \mathsf{i}\beta$$

so that f is an intrinsic slice function.

The symmetry of the set of right eigenvalues and the relation (1.1) between eigenvectors associated with eigenvalues in the same eigensphere therefore require that the class of functions to which a quaternionic functional applies consists of intrinsic slice functions, if this functional calculus respects the fundamental intuition of a functional calculus given in (8.14). This observation is reflected by resp. explains several known phenomena in the theory.

- The S-functional calculus for left and for right slice hyperholomorphic functions are not always consistent. If f is left and right slice hyperholomorphic, then both functional calculi will in general not yield the same operator unless f is intrinsic. Any such function is intrinsic up to addition with a locally constant function, but for locally constant functions the two functional calculi disagree. This is due to the fact that invariant subspaces of right linear operators are in general only right linear subspaces and not closed under multiplication with scalars from the left as we discussed in detail in Chapter 4.

- Even if the functional calculus is defined for a class of functions that does not only contain intrinsic slice functions, the product rule only holds for the intrinsic slice functions in this class [3, 4, 8, 36, 49]. An exception is [49, Theorem 7.8], where the continuous functional calculus for $\mathbb{C}_{\mathsf{i}}$-slice functions satisfies a product rule. This functional calculus is however not quaternionic in nature. Instead, it considers the operator as the quaternionic linear extension of a $\mathbb{C}_{\mathsf{i}}$-linear operator on a suitably chosen $\mathbb{C}_{\mathsf{i}}$-complex component space. The functional calculus can then be interpreted as applying the continuous functional calculus to the $\mathbb{C}_{\mathsf{i}}$-linear operator on the component space and extending the obtained operator to the entire quaternionic linear space.

- Similarly, the spectral mapping theorem only holds for those functions in the class of admissible functions that are intrinsic slice functions, cf. [8, 36, 49]. Again [49,

Theorem 7.8] obtains a generalized spectral mapping property, that holds because this functional calculus actually considers T as a complex linear operator on a suitably chosen complex component space.

Part II

Spectral Integration and Spectral Operators

CHAPTER 9

Spectral Integration in the Quaternionic Setting

Before we start the study of quaternionic spectral operators, we shall discuss spectral integration in the quaternionic setting. As we saw in Section 2.4, there exist different approaches to this topic in the literature, but we find them unsatisfying reasons that we explain in Section 9.4. In particular, these approaches do not generalise to the Banach space setting, in which we want to develop the theory of quaternionic spectral operators in Chapter 10.

In this chapter we therefore develop an approach to spectral integration of intrinsic slice functions on a quaternionic right Banach space. This integration is done with respect to a spectral system instead of a spectral measure, a concept that specifies ideas of Viswanath in [86, 87]. It has a clear and intuitive interpretation in terms of the right linear structure of the space and it is compatible with the complex theory in the sense of the results in Chapter 8. Moreover, it is also compatible with the existing approaches presented in Section 2.4. In particular, the proof of the spectral theorem for bounded normal operators in [5] easily translates into the language of spectral systems and does then not require to introduce any random structure that is not determined by the operator itself. The results in this chapter can be found in [47].

9.1 Spectral Integrals of Real-Valued Slice Functions

The basic idea of spectral integration is well known: it generates a multiplication operator in a way that generalizes the multiplication with eigenvalues in the discrete case. If for instance $\lambda \in \sigma(A)$ for some normal operator $A : \mathbb{C}^n \to \mathbb{C}^n$, then we can define $E(\{\lambda\})$ to be the orthogonal projection of $\mathbb{C}^n$ onto the eigenspace associated with λ and we find $A = \sum_{\lambda \in \sigma(A)} \lambda\, E(\{\lambda\})$. Setting $E(\Delta) = \sum_{\lambda \in \Delta} E(\{\lambda\})$ one obtains a discrete measure on $\mathbb{C}$, the values of which are orthogonal projections on $\mathbb{C}^n$, and A is

the integral of the identity function with respect to this measure. Changing the notation accordingly we have

$$A = \sum_{\lambda \in \sigma(A)} \lambda\, E(\{\lambda\}) \qquad \Longrightarrow \qquad A = \int_{\sigma(A)} \lambda\, dE(\lambda). \tag{9.1}$$

Via functional calculus it is possible to define functions of an operator. The fundamental intuition of a functional calculus is that $f(A)$ should be defined by the action of f on the spectral values (resp. the eigenvalues) of A. For our normal operator A on $\mathbb{C}^n$ the operator $f(A)$ is the operator with the following property: if $\mathbf{v} \in \mathbb{C}^n$ is an eigenvector of A with respect to λ, then $\mathbf{v}$ is an eigenvector of $f(A)$ with respect to $f(\lambda)$, just as it happens for instance naturally for powers and polynomials of A. Using the above notation we thus have

$$f(A) = \sum_{\lambda \in \sigma(A)} f(\lambda)\, E(\{\lambda\}) \qquad \Longrightarrow \qquad f(A) = \int_{\sigma(A)} f(\lambda)\, dE(\lambda). \tag{9.2}$$

In infinite dimensional Hilbert spaces, the spectrum of a normal operator might be not discrete such that the expressions on the left-hand side of (9.1) and (9.2) do not make sense. If E however is a suitable projection-valued measure, then it is possible to give the expression (9.2) a meaning by following the usual way of defining integrals via the approximation of f by simple functions. The spectral theorem then shows that for every normal operator T there exists a spectral measure such that (9.1) holds true.

If we want to introduce similar concepts in the quaternionic setting, we are—even in the discrete case—faced with several unexpected phenomena.

(P1) The space of bounded linear operators on a quaternionic Banach space V_R is only a real Banach space and not a quaternionic one as pointed out in Remark 2.45. Hence, the expressions in (9.1) and (9.2) are a priori only defined if λ resp. $f(\lambda)$ are real. Otherwise one needs to give meaning to the multiplication of the operator $E(\{\lambda\})$ with nonreal scalars.

(P2) The multiplication with a (non-real) scalar on the right is not linear such that $aE(\{\lambda\})$ can for $a \in \mathbb{H}$ not be defined as $(aE(\{\lambda\}))\mathbf{v} = (E(\{\lambda\})\mathbf{v})a$. Moreover, the set of eigenvectors associated with a specific eigenvalue does not constitute a linear subspace of V_R: if for instance $T\mathbf{v} = \mathbf{v}s$ with $s = s_0 + \mathbf{i}_s s_1$ and $\mathbf{j} \in \mathbb{S}$ with $\mathbf{i}_s \perp \mathbf{j}$, then $T(\mathbf{v}\mathbf{j}) = (T\mathbf{v})\mathbf{j} = (\mathbf{v}s)\mathbf{j} = (\mathbf{v}\mathbf{j})\overline{s} \neq (\mathbf{v}\mathbf{j})s$.

(P3) Finally, the set of eigenvalues is in general not discrete: if $s \in \mathbb{H}$ is an eigenvalue of T with $T\mathbf{v} = \mathbf{v}s$ for some $\mathbf{v} \neq \mathbf{0}$ and $s_{\mathbf{i}} = s_0 + \mathbf{i}s_1 \in [s]$, then there exists $h \in \mathbb{H} \setminus \{0\}$ such that $s_{\mathbf{i}} = h^{-1}sh$ and so

$$T(\mathbf{v}h) = T(\mathbf{v})h = \mathbf{v}sh = (\mathbf{v}h)h^{-1}sh = (\mathbf{v}h)s_{\mathbf{i}}. \tag{9.3}$$

Thus, any $s_{\mathbf{i}} \in [s]$ is also an eigenvalue of T.

As a first consequence of (P2) and (P3) the notion of eigenvalue and eigenspace have to be adapted: linear subspaces are in the quaternionic setting not associated with individual eigenvalues s but with spheres $[s]$ of equivalent eigenvalues.

Definition 9.1. Let $T \in \mathcal{K}(V_R)$ and let $s \in \mathbb{H} \setminus \mathbb{R}$. We say that $[s]$ is an eigensphere of T if there exists a vector $\mathbf{v} \in V_R \setminus \{\mathbf{0}\}$ such that

$$(T^2 - 2s_0 T + |s|^2 \mathcal{I})\mathbf{v} = \mathcal{Q}_s(T)\mathbf{v} = \mathbf{0}. \tag{9.4}$$

The eigenspace of T associated with $[s]$ consists of all those vectors that satisfy (9.4).

Remark 9.2. For real values, things remain as we know them from the classical case: a quaternion $s \in \mathbb{R}$ is an eigenvalues of T if $T\mathbf{v} - \mathbf{v}s = 0$ for some $\mathbf{v} \neq \mathbf{0}$. The quaternionic right linear subspace $\ker(T - s\mathcal{I})$ is then called the eigenspace of T associated with s.

Any eigenvector $\mathbf{v}$ that satisfies $T(\mathbf{v}) = \mathbf{v}s_{\mathsf{i}}$ with $s_{\mathsf{i}} = s_0 + \mathsf{i}s_1$ for some $\mathsf{i} \in \mathbb{S}$ belongs to the eigenspace associated with the eigensphere $[s]$. Note however that the eigenspace associated with an eigensphere $[s]$ does not only consist of eigenvectors. It contains also linear combinations of eigenvectors associated with different eigenvalues in $[s]$ as the next lemma shows.

Lemma 9.3. *Let $T \in \mathcal{K}(V_R)$, let $[s]$ be an eigensphere of T and let $\mathsf{i} \in \mathbb{S}$. A vector $\mathbf{v}$ belongs to the eigenspace associated with $[s]$ if and only if $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ such that $T\mathbf{v}_1 = \mathbf{v}_1 s_{\mathsf{i}}$ and $T\mathbf{v}_2 = \mathbf{v}_2\overline{s_{\mathsf{i}}}$ where $s_{\mathsf{i}} = s_0 + \mathsf{i}s_1$.*

Proof. Observe that

$$\mathcal{Q}_s(T)\mathbf{v} = T^2\mathbf{v} - T\mathbf{v}2s_0 + \mathbf{v}|s|^2 = T(T\mathbf{v} - \mathbf{v}\overline{s_{\mathsf{i}}}) - (T\mathbf{v} - \mathbf{v}\overline{s_{\mathsf{i}}})s_{\mathsf{i}} \tag{9.5}$$

and

$$\mathcal{Q}_s(T)\mathbf{v} = T^2\mathbf{v} - T\mathbf{v}2s_0 + \mathbf{v}|s|^2 = T(T\mathbf{v} - \mathbf{v}s_{\mathsf{i}}) - (T\mathbf{v} - \mathbf{v}s_{\mathsf{i}})\overline{s_{\mathsf{i}}}. \tag{9.6}$$

Hence, $\mathcal{Q}_s(T)\mathbf{v} = \mathbf{0}$ for any eigenvector associated with s_{i} or $\overline{s_{\mathsf{i}}}$ and in turn also for any $\mathbf{v}$ that is the sum of two such vectors.

If conversely $\mathbf{v}$ satisfies (9.4), then we deduce from (9.5), that $T\mathbf{v} - \mathbf{v}\overline{s_{\mathsf{i}}}$ is a right eigenvector associated with s_{i} and that $T\mathbf{v} - \mathbf{v}s_{\mathsf{i}}$ is a right eigenvector of T associated with $\overline{s_{\mathsf{i}}}$. Since s_{i} and i commute, the vectors $\mathbf{v}_1 = (T\mathbf{v} - \mathbf{v}\overline{s_{\mathsf{i}}})\frac{-\mathsf{i}}{2s_1}$ and $\mathbf{v}_2 = (T\mathbf{v} - \mathbf{v}s_{\mathsf{i}})\frac{\mathsf{i}}{2s_1}$ are right eigenvectors associated with s resp. $\overline{s_{\mathsf{i}}}$, too. Hence we found the desired decomposition as

$$\mathbf{v}_1 + \mathbf{v}_2 = (T\mathbf{v} - \mathbf{v}\overline{s_{\mathsf{i}}})\frac{-\mathsf{i}}{2s_1} + (T\mathbf{v} - \mathbf{v}s_{\mathsf{i}})\frac{\mathsf{i}}{2s_1} = \mathbf{v}(\overline{s_{\mathsf{i}}} - s_{\mathsf{i}})\frac{\mathsf{i}}{2s_1} = \mathbf{v}.$$

□

Remark 9.4. If $\mathsf{j} \in \mathbb{S}$ with $\mathsf{j} \perp \mathsf{i}$, then $\tilde{\mathbf{v}}_2 := \mathbf{v}_2(-\mathsf{j})$ is an eigenvector of T associated with s. Hence we can write $\mathbf{v}$ also as $\mathbf{v} = \mathbf{v}_1 + \tilde{\mathbf{v}}_2\mathsf{j}$, where $\mathbf{v}_1, \tilde{\mathbf{v}}_2$ are both eigenvectors associated with s_{i}.

Since the eigenspaces of quaternionic linear operators are not associated with individual eigenvalues but instead with eigenspheres, quaternionic spectral measures must not be defined on arbitrary subsets of the quaternions. Instead their natural domains of definition consist of axially symmetric subsets of $\mathbb{H}$ so that they associate subspaces of V_R not to sets of spectral values but to sets of spectral spheres. This is also consistent with the fact that the S-spectrum of an operator is axially symmetric.

Definition 9.5. We denote the sigma-algebra of axially symmetric Borel sets on $\mathbb{H}$ by $\mathsf{B}_\mathsf{S}(\mathbb{H})$. Furthermore, we denote the set of all real-valued $\mathsf{B}_\mathsf{S}(\mathbb{H})$-$\mathsf{B}(\mathbb{R})$-measurable functions defined on $\mathbb{H}$ by $\mathcal{M}_s(\mathbb{H},\mathbb{R})$ and the set of all such functions that are bounded by $\mathcal{M}_s^\infty(\mathbb{H},\mathbb{R})$.

Remark 9.6. The restrictions of functions in $\mathcal{M}_s^\infty(\mathbb{H},\mathbb{R})$ to a complex halfplane $\mathbb{C}_{\mathsf{i}}^{\geq}$ are exactly the functions that were used to construct the spectral measure of a quaternionic normal operator in [5], cf. Remark 2.79.

Definition 9.7. A quaternionic spectral measure on a quaternionic right Banach space V_R is a function $E : \mathsf{B}_\mathsf{S}(\mathbb{H}) \to \mathcal{B}(V_R)$ that satisfies

(i) $E(\Delta)$ is a continuous projection and $\|E(\Delta)\| \leq K$ for any $\Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H})$ with a constant $K > 0$ independent of Δ,

(ii) $E(\emptyset) = 0$ and $E(\mathbb{H}) = \mathcal{I}$,

(iii) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for any $\Delta_1, \Delta_2 \in \mathsf{B}_\mathsf{S}(\mathbb{H})$ and

(iv) for any sequence $(\Delta_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathsf{B}_\mathsf{S}(\mathbb{H})$ we have

$$E\left(\bigcup_{n\in\mathbb{N}} \Delta_n\right)\mathbf{v} = \sum_{n=1}^{+\infty} E(\Delta_n)\mathbf{v} \qquad \text{for all } \mathbf{v} \in V_R.$$

Corollary 9.8. *Let E be a spectral measure on V_R and let V_R^* be its dual space, the left Banach space consisting of all continuous right linear mappings from V_R to $\mathbb{H}$. For any $\mathbf{v} \in V_R$ and any $\mathbf{v}^* \in V_R^*$, the mapping $\Delta \mapsto \langle \mathbf{v}^*, E(\Delta)\mathbf{v}\rangle$ is a quaternion-valued measure on* $\mathsf{B}_\mathsf{S}(\mathbb{H})$.

Remark 9.9. Following [87], most authors considered quaternionic spectral measures that were defined on the Borel sets $\mathsf{B}\big(\mathbb{C}_{\mathsf{i}}^{\geq}\big)$ of one of the closed complex halfplanes $\mathbb{C}_{\mathsf{i}}^{\geq} = \{s_0 + \mathsf{i}s_1 : s_0 \in \mathbb{R}, s_1 \geq 0\}$. This is equivalent to E being defined on $\mathsf{B}_\mathsf{S}(\mathbb{H})$. Indeed, if $\tilde{E}$ is defined on $\mathsf{B}\big(\mathbb{C}_{\mathsf{i}}^{\geq}\big)$, then setting

$$E(\Delta) := \tilde{E}\big(\Delta \cap \mathbb{C}_{\mathsf{i}}^{\geq}\big) \qquad \forall \Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H})$$

yields a spectral measure in the sense of Definition 9.7 that is defined on $\mathsf{B}_\mathsf{S}(\mathbb{H})$. If on the other hand we start with a spectral measure E defined on $\mathsf{B}_\mathsf{S}(\mathbb{H})$, then setting

$$\tilde{E}(\Delta) := E([\Delta]) \qquad \forall \Delta \in \mathsf{B}\big(\mathbb{C}_{\mathsf{i}}^{\geq}\big)$$

yields the respective measure on $\mathsf{B}\big(\mathbb{C}_{\mathsf{i}}^{\geq}\big)$. Although both definitions are equivalent, we prefer $\mathsf{B}_\mathsf{S}(\mathbb{H})$ as the domain of E because it does not suggest a dependence on the imaginary unit i.

For a function $f \in \mathcal{M}_s^\infty(\mathbb{H},\mathbb{R})$, we can now define the spectral integral with respect to a spectral measure E as in the classical case [38]. If f is a simple function, i.e. $f(s) = \sum_{k=1}^n \alpha_k \chi_{\Delta_k}(s)$ with pairwise disjoint sets $\Delta_k \in \mathsf{B}_\mathsf{S}(\mathbb{H})$, where χ_{Δ_k} denotes the characteristic function of Δ_k, then we set

$$\int_{\mathbb{H}} f(s)\, dE(s) := \sum_{k=1}^n \alpha_k E(\Delta_k). \tag{9.7}$$

There exists a constant $C_E > 0$ that depends only on E such that

$$\left\| \int_{\mathbb{H}} f(s)\, dE(s) \right\| \leq C_E \|f\|_\infty, \tag{9.8}$$

where $\|.\|_\infty$ denotes the supremum-norm. If $f \in \mathcal{M}_s^\infty(\mathbb{H}, \mathbb{R})$ is arbitrary, then we can find a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $\|f - f_n\|_\infty \to 0$ as $n \to +\infty$. In this case we can set

$$\int_{\mathbb{H}} f(s)\, dE(s) := \lim_{n\to+\infty} \int_{\mathbb{H}} f_n(s)\, dE(s), \tag{9.9}$$

where this sequence converges in the operator norm because of (9.8).

Lemma 9.10. *Let E be a quaternionic spectral measure on V_R. The mapping $f \mapsto \int_{\mathbb{H}} f(s)\, dE(s)$ is a continuous homomorphism from $\mathcal{M}_s^\infty(\mathbb{H}, \mathbb{R})$ to $\mathcal{B}(V_R)$. Moreover, if T commutes with E, i.e. it satisfies $TE(\Delta) = E(\Delta)T$ for all $\Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H})$, then T commutes with $\int_{\mathbb{H}} f(s)\, dE(s)$ for any $f \in \mathcal{M}_s^\infty(\mathbb{H}, \mathbb{R})$.*

Corollary 9.11. *Let E be a quaternionic spectral measure on V_R and let f be a function in $\mathcal{M}_s^\infty(\mathbb{H}, \mathbb{R})$. For any $\mathbf{v} \in V_R$ and any $\mathbf{v}^* \in V_R^*$, we have*

$$\left\langle \mathbf{v}^*, \left[\int_{\mathbb{H}} f\, dE \right] \mathbf{v} \right\rangle = \int_{\mathbb{H}} f(s)\, d\langle \mathbf{v}^*, E(s)\mathbf{v}\rangle.$$

Proof. Let $f_n = \sum_{k=1}^{N_n} \alpha_{n,k}\chi_{\Delta_{n,k}} \in \mathcal{M}_s^\infty(\mathbb{H}, \mathbb{R})$ be such that $\|f - f_n\| \to 0$ as $n \to +\infty$. Since all coefficients $\alpha_{n,k}$ are real, we have

$$\begin{aligned} \left\langle \mathbf{v}^*, \left[\int_{\mathbb{H}} f\, dE \right] \mathbf{v} \right\rangle &= \lim_{n\to\infty} \left\langle \mathbf{v}^*, \left[\sum_{k=1}^{N_n} \alpha_{n,k} E(\Delta_{n,k}) \right] \mathbf{v} \right\rangle \\ &= \lim_{n\to\infty} \sum_{k=1}^{N_n} \alpha_{n,k} \langle \mathbf{v}^*, E(\Delta_{n,k})\mathbf{v}\rangle = \int_{\mathbb{H}} f(s)\, d\langle \mathbf{v}^*, E(s)\mathbf{v}\rangle. \end{aligned}$$

□

Remark 9.12. The fact that the above definitions are well-posed and the properties given in Lemma 9.10 can be shown as in the classical case, so we omit their proofs. One can also deduce them directly from the classical theory: if we consider V_R as a real Banach space and E as a spectral measure with values in the space $\mathcal{B}_{\mathbb{R}}(V_R)$ of bounded $\mathbb{R}$-linear operators on V_R, which obviously contains $\mathcal{B}(V_R)$, then $\int_{\mathbb{H}} f(s)\, dE(s)$ defined in (9.7) resp. (9.9) is nothing but the spectral integral of f with respect to E in the classical sense. Since any α_k in (9.7) is real and since each $E(\Delta)$ is a quaternionic right linear projection, the integral of any simple function f with respect to E is a quaternionic right linear operator and hence belongs to $\mathcal{B}(V_R)$. The space $\mathcal{B}(V_R)$ is closed in $\mathcal{B}_{\mathbb{R}}(V_R)$ and hence the property of being quaternionic linear survives the approximation by simple functions such that $\int_{\mathbb{H}} f(s)\, dE(s) \in \mathcal{B}(V_R)$ for any $f \in \mathcal{M}_s^\infty(\mathbb{H}, \mathbb{R})$ even if we consider it as the integral with respect to a (real) spectral measure on the real Banach space V_R.

9.2 Imaginary Operators

The techniques introduced so far allow us to integrate real-valued functions with respect to a spectral measure. This is obviously insufficient, even for formulating the statement corresponding to (9.1) in the quaternionic setting unless $\sigma_S(T)$ is real. In order to define spectral integrals for functions that are not real-valued, we need additional information.

This fits another observation: in contrast to the complex case, even for the simple case of a normal operator on a finite-dimensional quaternionic Hilbert space, a decomposition of the space V_R into the eigenspaces of T is not sufficient to recover the entire operator T. Let $\mathsf{i}, \mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \neq \mathsf{j}$ and consider for instance the operators T_1, T_2 and T_3 on $\mathbb{H}^2$, which are given by their matrix representations

$$T_1 = \begin{pmatrix} \mathsf{i} & 0 \\ 0 & \mathsf{i} \end{pmatrix} \qquad T_2 = \begin{pmatrix} \mathsf{i} & 0 \\ 0 & \mathsf{j} \end{pmatrix} \qquad T_3 = \begin{pmatrix} \mathsf{j} & 0 \\ 0 & \mathsf{j} \end{pmatrix}. \tag{9.10}$$

For each $\ell = 1, 2, 3$, we have $\sigma_S(T_\ell) = \mathbb{S}$ and that the only eigenspace of T_ℓ is the entire space $\mathbb{H}^2$. The spectral measure E that is associated with T_ℓ is hence given by $E(\Delta) = 0$ if $\mathbb{S} \not\subset \Delta$ and $E(\Delta) = \mathcal{I}$ if $\mathbb{S} \subset \Delta$. Obviously, the spectral measures associated with these operators agree, although these operators do not the coincide.

Since the eigenspace of an operator T that is associated with some eigensphere $[s]$ contains eigenvectors associated with different eigenvalues, we need some additional information to understand 'how to multiply the eigensphere onto the associated eigenspace', i.e. to understand which vector in the eigenspace must be multiplied with which eigenvalue in the corresponding eigensphere $[s]$. This information will be provided by a suitable imaginary operator. Such operators generalise the properties of the anti-selfadjoint partially unitary operator J_0 in the decomposition (2.40) of a normal operator on a Hilbert space to the Banach space setting.

Definition 9.13. An operator $J \in \mathcal{B}(V_R)$ is called imaginary if $-J^2$ is the projection of V_R onto $\operatorname{ran} J$ along $\ker J$. We call J fully imaginary if $-J^2 = \mathcal{I}$, i.e. if in addition $\ker J = \{0\}$.

Corollary 9.14. *An operator $J \in \mathcal{B}(V_R)$ is an imaginary operator if and only if*

(i) $-J^2$ *is a projection and*

(ii) $\ker J = \ker J^2$.

Proof. If J is an imaginary operator, then obviously (i) and (ii) hold true. Assume on the other hand that (i) and (ii) hold. Obviously $\operatorname{ran}(-J^2) \subset \operatorname{ran} J$. For any $\mathbf{u} \in V_R$, we have $(-J^2)\mathbf{u} - \mathbf{u} \in \ker(-J^2) = \ker J$ because

$$(-J^2)\left((-J^2)\mathbf{u} - \mathbf{u}\right) = (-J^2)^2\mathbf{u} - (-J)^2\mathbf{u} = (-J^2)\mathbf{u} - (-J)^2\mathbf{u} = \mathbf{0}$$

as $(-J^2)$ is a projection. Therefore

$$\mathbf{0} = J(-J^2\mathbf{u} - \mathbf{u}) = (-J^2)J\mathbf{u} - J\mathbf{u}$$

and hence $\mathbf{v} = (-J^2)\mathbf{v}$ for any $\mathbf{v} = J\mathbf{u} \in \operatorname{ran} J$. Consequently, $\operatorname{ran}(-J^2) \supset \operatorname{ran} J$ and in turn $\operatorname{ran} J = \operatorname{ran}(-J^2)$. Since $\ker J = \ker(-J^2)$, we find that $-J^2$ is the projection of V_R onto $\operatorname{ran} J$ along $\ker J$, i.e. that J is an imaginary operator.

□

Remark 9.15. The above implies that any anti-selfadjoint partially unitary operator J_0 on a quaternionic Hilbert space $\mathcal{H}$ is an imaginary operator. Indeed, for any $\mathbf{v} \in \ker \mathsf{J}_0$, we obviously have $-\mathsf{J}_0^2\mathbf{v} = \mathbf{0}$. Since the restriction of J_0 to $\mathcal{H}_0 := \operatorname{ran} \mathsf{J}_0 = \ker \mathsf{J}_0^{\perp}$ is unitary and J_0 is anti-selfadjoint, we furthermore have for $\mathbf{v} \in \mathcal{H}_0$ that $-\mathsf{J}_0^2\mathbf{v} = \mathsf{J}_0^*\mathsf{J}_0\mathbf{v} = (\mathsf{J}_0|_{\mathcal{H}_0})^*(\mathsf{J}_0|_{\mathcal{H}_0})\mathbf{v} = (\mathsf{J}_0|_{\mathcal{H}_0})^{-1}(\mathsf{J}_0|_{\mathcal{H}_0})\mathbf{v} = \mathbf{v}$. Hence $-\mathsf{J}_0^2$ is the orthogonal projection onto $\mathcal{H}_0 = \operatorname{ran} \mathsf{J}_0$ and so J_0 is an imaginary operator. In particular any unitary anti-selfadjoint operator is fully imaginary.

Lemma 9.16. *If $J \in \mathcal{B}(V_R)$ is an imaginary operator, then $\sigma_S(T) \subset \{0\} \cup \{\mathbb{S}\}$.*

Proof. Since the operator $-J^2$ is a projection, its S-spectrum $\sigma_S(-J^2)$ is a subset of $\{0,1\}$. Indeed, for any projection $P \in \mathcal{B}(V_R)$, a simple calculation shows that the pseudo-resolvent of P at any $s \in \mathbb{H} \setminus \{0,1\}$ is given by

$$\mathcal{Q}_s(P)^{-1} = -\frac{1}{|s|^2}\left(\frac{1-2\mathrm{Re}(s)}{1-2\mathrm{Re}(s)+|s|^2}P - \mathcal{I}\right)$$

such that $s \in \rho_S(P)$. As a consequence of the spectral mapping theorem, we find that

$$-\sigma_S(J)^2 = \{-s^2 : s \in \sigma_S(J)\} = \sigma_S(-J^2) \subset \{0,1\}.$$

But if $-s^2 \in \{0,1\}$, then $s \in \{0\} \cup \mathbb{S}$ and hence $\sigma_S(J) \subset \{0\} \cup \mathbb{S}$.

□

Remark 9.17. If $J = 0$, then J is an imaginary operator with $\sigma_S(T) = \{0\}$. If on the other hand $\ker J = \{\mathbf{0}\}$ (i.e. if J is fully imaginary), then $\sigma_S(T) = \mathbb{S}$. In any other case we obviously have $\sigma_S(T) = \{0\} \cup \mathbb{S}$.

The following theorem gives a complete characterization of imaginary operators on V_R.

Theorem 9.18. *Let $J \in \mathcal{B}(V_R)$ be an imaginary operator. For any $\mathsf{i} \in \mathbb{S}$, the Banach space V_R admits a direct sum decomposition as*

$$V_R = V_{J,0} \oplus V_{J,\mathsf{i}}^{+} \oplus V_{J,\mathsf{i}}^{-} \tag{9.11}$$

with

$$\begin{aligned} V_{J,0} &= \ker(J),\\ V_{J,\mathsf{i}}^{+} &= \{\mathbf{v} \in V : J\mathbf{v} = \mathbf{v}\mathsf{i}\},\\ V_{J,\mathsf{i}}^{-} &= \{\mathbf{v} \in V : J\mathbf{v} = \mathbf{v}(-\mathsf{i})\}. \end{aligned} \tag{9.12}$$

The spaces $V_{J,\mathsf{i}}^{+}$ and $V_{J,\mathsf{i}}^{-}$ are complex Banach spaces over $\mathbb{C}_{\mathsf{i}}$ with the natural structure inherited from V_R and for each $\mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$ the map $\mathbf{v} \mapsto \mathbf{v}\mathsf{j}$ is a $\mathbb{C}_{\mathsf{i}}$-antilinear and isometric bijection between $V_{J,\mathsf{i}}^{+}$ and $V_{J,\mathsf{i}}^{-}$.

Conversely, let $\mathsf{i},\mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$ and assume that V_R is the direct sum $V_R = V_0 \oplus V_+ \oplus V_-$ of a closed ($\mathbb{H}$-linear) subspace V_0 and two closed $\mathbb{C}_{\mathsf{i}}$-linear subspaces V_+ and V_- of V_R such that $\Psi : \mathbf{v} \mapsto \mathbf{v}\mathsf{j}$ is a bijection between V_+ and V_-. Let E_+ and E_- be the $\mathbb{C}_{\mathsf{i}}$-linear projections onto V_+ and V_- along $V_0 \oplus V_-$ resp. $V_0 \oplus V_+$. The operator $J\mathbf{v} := E_+\mathbf{v}\mathsf{i} + E_-\mathbf{v}(-\mathsf{i})$ for $\mathbf{v} \in V_R$ is an imaginary operator on V_R.

Proof. We first assume that J is an imaginary operator and show the existence of the corresponding decomposition of V_R. Let $\mathsf{i} \in \mathbb{S}$ and $V_{R,\mathsf{i}}$ denote the space V_R considered as a complex Banach over $\mathbb{C}_{\mathsf{i}}$. Furthermore, let us assume that $J \neq 0$ as the statement is obviously true in this case. Then J is a bounded $\mathbb{C}_{\mathsf{i}}$-linear operator on $V_{R,\mathsf{i}}$ and by Theorem 8.4 and Lemma 9.16 the spectrum of J as an element of $\mathcal{B}(V_{R,\mathsf{i}})$ is $\sigma_{\mathbb{C}_{\mathsf{i}}}(J) = \sigma_S(J) \cap \mathbb{C}_{\mathsf{i}} \subset \{0, \mathsf{i}, -\mathsf{i}\}$. We define now for $\tau \in \{0, \mathsf{i}, -\mathsf{i}\}$ the projection E_τ as the spectral projection associated with $\{\tau\}$ obtained from the Riesz-Dunford functional calculus. If we choose $0 < \varepsilon < \frac{1}{2}$, then the relation $R_z(J) = (\overline{z}\mathcal{I}_{V_{R,\mathsf{i}}} - J)\mathcal{Q}_z(J)^{-1}$ in Theorem 8.4 implies

$$E_\tau \mathbf{v} = \int_{\partial U_\varepsilon(\tau;\mathbb{C}_{\mathsf{i}})} R_z(J)\mathbf{v}\, dz \frac{1}{2\pi\mathsf{i}} = \int_{\partial U_\varepsilon(\tau;\mathbb{C}_{\mathsf{i}})} \mathcal{Q}_z(J)^{-1}(\mathbf{v}\overline{z} - J\mathbf{v})\, dz \frac{1}{2\pi\mathsf{i}},$$

where $U_\varepsilon(\tau;\mathbb{C}_{\mathsf{i}})$ denotes the ball of radius ε in $\mathbb{C}_{\mathsf{i}}$ that is centered at τ. (Since we assumed $\ker J \neq V$, the projections E_{i} and $E_{-\mathsf{i}}$ are not trivial. It might however happen that $E_0 = 0$, but this is not a problem in the following argumentation.)

We set

$$V_{J,0} = E_0 V_{R,\mathsf{i}}, \quad V_{J,\mathsf{i}}^+ = E_{\mathsf{i}} V_{R,\mathsf{i}} \quad \text{and} \quad V_{J,\mathsf{i}}^- = E_{-\mathsf{i}} V_{R,\mathsf{i}}.$$

Obviously these are closed $\mathbb{C}_{\mathsf{i}}$-linear subspaces of $V_{R,\mathsf{i}}$ resp. V_R and (9.11) holds true.

Let us now show that the relation (9.12) holds true. We first consider the subspace $V_{J,\mathsf{i}}^+$. Since it is the range of the Riesz-projector E_{i} associated with the spectral set $\{\mathsf{i}\}$, this is an $\mathbb{C}_{\mathsf{i}}$-linear subspace of $V_{R,\mathsf{i}}$ that is invariant under J and the restriction $J_+ := J|_{V_{J,\mathsf{i}}^+}$ has spectrum $\sigma(J_+) = \{\mathsf{i}\}$. Now observe that $-J_+^2 = -J^2|_{V_{J,\mathsf{i}}^+}$ is the restriction of a projection onto an invariant subspace and hence a projection itself. Since $0 \notin \sigma(-J_+^2) = -\sigma(J_+)^2 = \{1\}$, we find $\ker -J_+^2 = \{\mathbf{0}\}$ and in turn $\mathcal{I}_+ := \mathcal{I}_{V_{J,\mathsf{i}}^+} = -J_+^2$. For $\mathbf{v} \in V_{J,\mathsf{i}}^+$ we therefore have

$$\begin{aligned}-\mathbf{v} = J_+^2\mathbf{v} &= (J_+ - \mathsf{i}\mathcal{I}_+ + \mathsf{i}\mathcal{I}_+)^2\mathbf{v} = (J_+ - \mathsf{i}\mathcal{I}_+ + \mathsf{i}\mathcal{I}_+)((J_+ - \mathsf{i}\mathcal{I}_+)\mathbf{v} + \mathbf{v}\mathsf{i}) \\ &= (J_+ - \mathsf{i}\mathcal{I}_+)^2\mathbf{v} + (J_+ - \mathsf{i}\mathcal{I}_+)\mathbf{v}\mathsf{i} + (J_+ - \mathsf{i}\mathcal{I}_+)\mathbf{v}\mathsf{i} + \mathbf{v}\mathsf{i}^2.\end{aligned}$$

As $\mathsf{i}^2 = -1$ this is equivalent to

$$(J_+ - \mathsf{i}\mathcal{I}_+)^2\mathbf{v} = (J_+ - \mathsf{i}\mathcal{I}_+)\mathbf{v}(-2\mathsf{i}).$$

Hence $(J_+ - \mathsf{i}\mathcal{I}_+)\mathbf{v}$ is either $\mathbf{0}$ or an eigenvector of $J_+ - \mathsf{i}\mathcal{I}_+$ associated with the eigenvalue $-2\mathsf{i}$. By the spectral mapping theorem $\sigma(J_+ - \mathsf{i}\mathcal{I}_+) = \sigma(J_+) - \mathsf{i} = \{0\}$. Hence, $J_+ - \mathsf{i}\mathcal{I}_+$ cannot have an eigenvector with respect to the eigenvalue $-2\mathsf{i}$ and so $(J_+ - \mathsf{i}\mathcal{I}_+)\mathbf{v} = \mathbf{0}$. Therefore $J_+ = \mathcal{I}_+ i$ and $J\mathbf{v} = J_+\mathbf{v} = \mathbf{v}\mathsf{i}$ for all $\mathbf{v} \in V_{J,\mathsf{i}}^+$.

With similar arguments, one shows that $J\mathbf{v} = \mathbf{v}(-\mathsf{i})$ for any $\mathbf{v} \in V_{J,\mathsf{i}}^-$. Finally, $\sigma(-J_0^2) = -\sigma(J_0)^2 = \{0\}$ for $J_0 := J|_{V_{J,0}}$. Since $-J_0^2 = -J^2|_{V_{J,0}}$ is the restriction of a projection to an invariant subspace and thus a projection itself, we find that $-J_0^2$ is the zero operator and hence $V_{J,0} = \ker(-J_0)^2 \subset \ker(J^2) = \ker J$. On the other hand $\ker J \subset V_{J,0}$ as $V_{J,0}$ is the invariant subspace associated with the spectral value 0 of J. Thus $V_{J,0} = \ker J$ and so (9.12) is true.

Finally, if $\mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$ and $\mathbf{v} \in V_+$ then $(J\mathbf{v}\mathsf{j}) = J(\mathbf{v})\mathsf{j} = \mathbf{v}\mathsf{i}\mathsf{j} = (\mathbf{v}\mathsf{j})(-\mathsf{i})$. Hence $\Psi : \mathbf{v} \to \mathbf{v}\mathsf{j}$ maps $V_{J,\mathsf{i}}^+$ to $V_{J,\mathsf{i}}^-$. It is obviously $\mathbb{C}_{\mathsf{i}}$-antilinear, isometric and a bijection as $\mathbf{v} = -(\mathbf{v}\mathsf{j})\mathsf{j}$ so that the proof of the first statement is finished.

Now let $\mathbf{i}, \mathbf{j} \in \mathbb{S}$ with $\mathbf{i} \perp \mathbf{j}$ and assume that $V_R = V_0 \oplus V_+ \oplus V_-$ with subspaces V_0, V_+ and V_- as in the assumptions. We define $J\mathbf{v} := E_+\mathbf{v}\mathbf{i} + E_-\mathbf{v}(-\mathbf{i})$. Obviously, J is a continuous $\mathbb{C}_{\mathbf{i}}$-linear operator on $V_{R,\mathbf{i}}$. The mapping $\Psi : \mathbf{v} \mapsto \mathbf{v}\mathbf{j}$ maps V_+ bijectively to V_-, but since $\Psi^{-1} = -\Psi$ it also maps V_- bijectively to V_+. Moreover, as an $\mathbb{H}$-linear subspace, V_0 is invariant under Ψ. For $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_+ + \mathbf{v}_- \in V_0 \oplus V_+ \oplus V_- = V_R$, we therefore find

$$\begin{aligned} J(\mathbf{v}\mathbf{j}) =& E_+(\mathbf{v}\mathbf{j})\mathbf{i} + E_-(\mathbf{v}\mathbf{j})(-\mathbf{i}) = \mathbf{v}_-\mathbf{j}\mathbf{i} + \mathbf{v}_+\mathbf{j}(-\mathbf{i}) \\ =& \mathbf{v}_-(-\mathbf{i})\mathbf{j} + \mathbf{v}_+\mathbf{i}\mathbf{j} = \left(E_-\mathbf{v}(-\mathbf{i})\right)\mathbf{j} + \left(E_+\mathbf{v}\mathbf{i}\right)\mathbf{j} = (J\mathbf{v})\mathbf{j}. \end{aligned}$$

If now $a \in \mathbb{H}$, then we can write $a = a_1 + a_2\mathbf{j}$ with $a_1, a_2 \in \mathbb{C}_{\mathbf{i}}$ and find due to the $\mathbb{C}_{\mathbf{i}}$-linearity of J that

$$J(\mathbf{v}a) = J(\mathbf{v}a_1) + J(\mathbf{v}a_2\mathbf{j}) = J(\mathbf{v})a_1 + J(\mathbf{v})a_2\mathbf{j} = J(\mathbf{v})(a_1 + a_2\mathbf{j}) = J(\mathbf{v})a.$$

Hence, J is quaternionic linear and therefore belongs to $\mathcal{B}(V_R)$.

As $E_+E_- = E_-E_+ = 0$, we furthermore observe that

$$\begin{aligned} -J^2\mathbf{v} &= -J(E_+\mathbf{v}\mathbf{i} + E_-\mathbf{v}(-\mathbf{i})) \\ = -\left(E_+^2\mathbf{v}\mathbf{i}^2 + E_+E_-\mathbf{v}(-\mathbf{i}^2) + E_-E_+\mathbf{v}(-\mathbf{i}^2) + E_-^2\mathbf{v}(-\mathbf{i})^2\right) &= (E_+ + E_-)\mathbf{v}. \end{aligned}$$

Hence, $-J^2$ is the projection onto $V_+ \oplus V_- = \operatorname{ran}(J)$ along $\ker J = V_0$ such that J is actually an imaginary operator.

□

9.3 Spectral Systems and Spectral Integrals of Intrinsic Slice Functions

As pointed out already several times, invariant subspaces of an operator are in the quaternionic setting not associated with spectral values but with entire spectral spheres. Hence quaternionic spectral measures associate subspaces of V_R with sets of entire spectral spheres and not with arbitrary sets of spectral values. If we want to integrate a function f that takes non-real values with respect to a spectral measure E, then we need some additional information. We need to know how to multiply the different values that f takes on a spectral sphere onto the vectors associated with the different spectral values in this sphere. This information is given by a suitable imaginary operator. Similar to [87], we hence introduce now the notion of a spectral system.

Definition 9.19. A spectral system on V_R is a couple (E, J) consisting of a spectral measure E and an imaginary operator J such that

(i) E and J commute, i.e. $E(\Delta)J = JE(\Delta)$ for all $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$ and

(ii) $E(\mathbb{H} \setminus \mathbb{R}) = -J^2$, that is $E(\mathbb{R})$ is the projection onto $\ker J$ along $\operatorname{ran} J$ and $E(\mathbb{H} \setminus \mathbb{R})$ is the projection onto $\operatorname{ran} J$ along $\ker J$.

Definition 9.20. We denote by $\mathcal{SM}^{\infty}(\mathbb{H})$ the set of all bounded intrinsic slice functions on $\mathbb{H}$ that are measurable with respect to the usual Borel sets $\mathsf{B}(\mathbb{H})$ on $\mathbb{H}$.

Lemma 9.21. *A function $f : \mathbb{H} \to \mathbb{H}$ belongs to $\mathcal{SM}^{\infty}(\mathbb{H})$ if and only if it is of the form $f(s) = \alpha(s) + \mathbf{i}_s\beta(s)$ with $\alpha, \beta \in \mathcal{M}_s^{\infty}(\mathbb{H}, \mathbb{R})$ and $\beta(s) = 0$ for $s \in \mathbb{R}$.*

Proof. If $f(s) = \alpha(s) + \mathsf{i}_s\beta(s)$ with $\alpha, \beta \in \mathcal{M}_s^\infty(\mathbb{H}, \mathbb{R})$ and $\beta(s) = 0$ for $s \in \mathbb{R}$, then we can set $\alpha(s_0, s_1) := \alpha(s_0 + \mathsf{i}s_1)$ and $\beta(s_0, s_1) := \beta(s_0 + \mathsf{i}s_1)$ and $\beta(s_0, -s_1) := -\beta(s_0 + \mathsf{i}s_1)$ with $\mathsf{i} \in \mathbb{S}$ arbitrary. Since $\alpha(s)$ and $\beta(s)$ are $\mathsf{B}_\mathsf{S}(\mathbb{H})$-measurable, they are constant on each sphere $[s]$ and so this definition is independent of the chosen imaginary unit i. Since $\beta(s) = 0$ for real s, $\beta(s_0, s_1)$ is moreover well defined for $s_1 = 0$. We find that $f(s) = \alpha(s) + \mathsf{i}_s\beta(s) = \alpha(s_0, s_1) + \mathsf{i}_s\beta(s_0, s_1)$ with $\alpha(s_0, s_1)$ and $\beta(s_0, s_1)$ taking real values and satisfying (2.4) such that f is actually an intrinsic slice function. Moreover, the functions $\alpha(s)$ and $\beta(s)$ and the function $\varphi(s) := \mathsf{i}_s$ if $s \notin \mathbb{R}$ and $\varphi(s) := 0$ if $s \in \mathbb{R}$ are $\mathsf{B}(\mathbb{H})$-$\mathsf{B}(\mathbb{H})$-measurable. As $\beta(s) = 0$ if $s \in \mathbb{R}$, we have $f(s) = \alpha(s) + \mathsf{i}_s\beta(s) = \alpha(s) + \varphi(s)\beta(s)$ and hence the function f is $\mathsf{B}(\mathbb{H})$-$\mathsf{B}(\mathbb{H})$-measurable too.

If on the other hand $f \in \mathcal{SM}^\infty(\mathbb{H})$ with $f(s) = \alpha(s_0, s_1) + \mathsf{i}_s\beta(s_0, s_1)$, then also $\alpha(s) := \frac{1}{2}\left(f(s) + f(\overline{s})\right) = \alpha(s_0, s_1)$ and $\beta(s) := \frac{1}{2}\varphi(s)\left(f(\overline{s}) - f(s)\right) = \beta(s_0, s_1)$ with $\varphi(s)$ as above are $\mathsf{B}(\mathbb{H})$-$\mathsf{B}(\mathbb{H})$-measurable. Moreover $\beta(s) = 0$ if $s_1 = 0$. Since f is intrinsic, these functions take values in $\mathbb{R}$ and hence they are $\mathsf{B}(\mathbb{H})$-$\mathsf{B}(\mathbb{R})$-measurable. They are moreover constant on each sphere $[s]$ such that the preimages $\alpha^{-1}(A)$ and $\beta^{-1}(A)$ of each set $A \in \mathsf{B}(\mathbb{R})$ are axially symmetric Borel sets in $\mathbb{H}$. Consequently, α and β are $\mathsf{B}_\mathsf{S}(\mathbb{H})$-$\mathsf{B}(\mathbb{R})$-measurable. Finally, $|f|^2 = |\alpha|^2 + |\beta|^2$ such that f is bounded if and only if α and β are bounded.

□

Corollary 9.22. *Any function $f \in \mathcal{SM}^\infty(\mathbb{H})$ is $\mathsf{B}_\mathsf{S}(\mathbb{H})$-$\mathsf{B}_\mathsf{S}(\mathbb{H})$-measurable.*

Proof. Let $\Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H})$. Its inverse image $f^{-1}(\Delta)$ is a Borel set in $\mathbb{H}$ because f is $\mathsf{B}(\mathbb{H})$-$\mathsf{B}(\mathbb{H})$-measurable. If $s \in f^{-1}(\Delta)$, then $f(s) = \alpha(s_0, s_1) + \mathsf{i}_s\beta(s_0, s_1) \in \Delta$. The axially symmetry of Δ implies then that for any $s_\mathsf{i} = s_0 + \mathsf{i}s_1 \in [s]$ with $\mathsf{i} \in \mathbb{S}$ also $f(s_\mathsf{i}) = \alpha(s_0, s_1) + \mathsf{i}_s\beta(s_0, s_1) \in \Delta$ and hence $s_\mathsf{i} \in f^{-1}(\Delta)$. Thus $s \in f^{-1}(\Delta)$ implies $[s] \subset f^{-1}(\Delta)$ and so $f^{-1}(\Delta) \in \mathsf{B}_\mathsf{S}(\mathbb{H})$.

□

We observe that the Lemma 9.21 implies that the spectral integrals of the component functions α and β of any $f = \alpha + \mathsf{i}_s\beta \in \mathcal{SM}^\infty(\mathbb{H})$ are defined by Definition 9.7.

Definition 9.23. Let (E, J) be a spectral system on V_R. For $f \in \mathcal{SM}^\infty(\mathbb{H})$ with $f(s) = \alpha(s) + \mathsf{i}_s\beta(s)$ we define the spectral integral of f with respect to (E, J) as

$$\int_{\mathbb{H}} f(s)\, dE_J(s) := \int_{\mathbb{H}} \alpha(s)\, dE(s) + J \int_{\mathbb{H}} \beta(s)\, dE(s). \tag{9.13}$$

The estimate (9.8) generalizes to

$$\left\| \int_{\mathbb{H}} f(s)\, dE(s) \right\| \leq C_E\|\alpha\|_\infty + C_E\|J\|\|\beta\|_\infty \leq C_{E,J}\|f\|_\infty \tag{9.14}$$

with

$$C_{E,J} := C_E(1 + \|J\|).$$

As a consequence of Lemma 9.10 and the fact that J and E commute, we immediately obtain the following result.

Lemma 9.24. *Let (E, J) be a spectral system on V_R. The mapping*

$$f \mapsto \int_{\mathbb{H}} f(s)\, dE_J(s)$$

is a continuous homomorphism from $(\mathcal{SM}^\infty(\mathbb{H}), \|.\|_\infty)$ to $\mathcal{B}(V_R)$. Moreover, if an operator $T \in \mathcal{B}(V_R)$ commutes with E and J, then it commutes with $\int_{\mathbb{H}} f(s)\, dE_J(s)$ for any $f \in \mathcal{SM}^\infty(\mathbb{H})$.

From Corollary 9.11, we furthermore immediately obtain the following lemma, which is analogue to Lemma 2.77.

Corollary 9.25. *Let (E, J) be a spectral system on V_R and let $f = \alpha + \mathbf{i}\beta \in \mathcal{SM}^\infty(\mathbb{H})$. For any $\mathbf{v} \in V_R$ and any $\mathbf{v}^* \in V_R^*$, we have*

$$\left\langle \mathbf{v}^*, \left[\int_{\mathbb{H}} f(s)\, dE_J(s)\right] \mathbf{v} \right\rangle = \int_{\mathbb{H}} \alpha(s)\, d\left\langle \mathbf{v}^*, E(s)\mathbf{v}\right\rangle + \int_{\mathbb{H}} \beta(s)\, d\left\langle \mathbf{v}^*, E(s)J\mathbf{v}\right\rangle .$$

Similar to the what happens for the S-functional calculus, there exists a deep relation between quaternionic and complex spectral integrals on V_R.

Lemma 9.26. *Let (E, J) be a spectral system on V_R, let $\mathbf{i} \in \mathbb{S}$, let E_+ be the projection of V_R onto $V_{J,\mathbf{i}}^+$ along $V_{J,0} \oplus V_{J,\mathbf{i}}^-$ and let E_- be the projection of V_R onto $V_{J,\mathbf{i}}^-$ along $V_{J,0} \oplus V_{J,\mathbf{i}}^+$, cf. Theorem 9.18. For $\Delta \in \mathsf{B}(\mathbb{C}_\mathbf{i})$, we set*

$$E_\mathbf{i}(\Delta) := \begin{cases} E_+E([\Delta]) & \text{if } \Delta \subset \mathbb{C}_\mathbf{i}^+ \\ E(\Delta) & \text{if } \Delta \subset \mathbb{R} \\ E_-E(\Delta) & \text{if } \Delta \subset \mathbb{C}_\mathbf{i}^- \\ E_\mathbf{i}(\Delta \cap \mathbb{C}_\mathbf{i}^+) + E_\mathbf{i}(\Delta \cap \mathbb{R}) + E_\mathbf{i}(\Delta \cap \mathbb{C}_\mathbf{i}^-) & \text{otherwise,} \end{cases} \tag{9.15}$$

where $\mathbb{C}_\mathbf{i}^+$ and $\mathbb{C}_\mathbf{i}^-$ are the open upper and lower halfplane in $\mathbb{C}_\mathbf{i}$. Then $E_\mathbf{i}$ is a spectral measure on $V_{R,\mathbf{i}}$. For any $f \in \mathcal{SM}^\infty(\mathbb{H})$, we have with $f_\mathbf{i} := f|_{\mathbb{C}_\mathbf{i}}$ that

$$\int_{\mathbb{H}} f(s)\, dE_J(s) = \int_{\mathbb{C}_\mathbf{i}} f_\mathbf{i}(z)\, dE_\mathbf{i}(s). \tag{9.16}$$

Proof. Recall that E and J commute. For $\mathbf{v}_+ \in V_{J,\mathbf{i}}^+$, we thus have $JE(\Delta)\mathbf{v}_+ = E(\Delta)J\mathbf{v}_+ = E(\Delta)\mathbf{v}_+\mathbf{i}$ such that $E(\Delta)\mathbf{v}_+ \in V_{J,\mathbf{i}}^+$ and in turn $E_+E(\Delta)\mathbf{v}_+ = E(\Delta)\mathbf{v}_+$. Similarly, we see that $E(\Delta)\mathbf{v}_\sim \in V_{J,0} \oplus V_{J,\mathbf{i}}^-$ for any $\mathbf{v}_\sim \in V_{J,0} \oplus V_{J,\mathbf{i}}^-$ such that $E_+E(\Delta)\mathbf{v}_\sim = \mathbf{0}$. Hence, if we decompose $\mathbf{v} \in V_R$ as $\mathbf{v} = \mathbf{v}_+ + \mathbf{v}_\sim$ with $\mathbf{v}_+ \in V_{J,\mathbf{i}}^+$ and $\mathbf{v}_\sim \in V_{J,0} \oplus V_{J,\mathbf{i}}^-$ according to Theorem 9.18, then

$$E_+E(\Delta)\mathbf{v} = E_+E(\Delta)\mathbf{v}_+ + E_+E(\Delta)\mathbf{v}_\sim = E(\Delta)\mathbf{v}_+$$

and $E(\Delta)E_+\mathbf{v} = E(\Delta)\mathbf{v}_+$ such that altogether $E(\Delta)E_+\mathbf{v} = E_+E(\Delta)\mathbf{v}$. Analogous arguments show that $E_-E(\Delta) = E(\Delta)E_-$ and hence E_+, E_-, and $E(\Delta)$, $\Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H})$, commute mutually.

Let us now show that $E_\mathbf{i}$ is actually a $\mathbb{C}_\mathbf{i}$-complex linear spectral measure on $V_{R,\mathbf{i}}$. For each $\Delta \in \mathsf{B}(\mathbb{C}_\mathbf{i})$ set $\Delta_+ := \Delta \cap \mathbb{C}_\mathbf{i}^+$, $\Delta_- := \Delta \cap \mathbb{C}_\mathbf{i}^-$ and $\Delta_\mathbb{R} := \Delta \cap \mathbb{R}$ for neatness

and recall that $[\,\cdot\,]$ denotes the axially symmetric hull of a set. For any $\Delta, \sigma \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$, we have then

$$\begin{aligned} E([\Delta_+])E(\sigma_{\mathbb{R}}) = E(\Delta_{\mathbb{R}})E([\sigma_+]) = 0 \\ E([\Delta_-])E(\sigma_{\mathbb{R}}) = E(\Delta_{\mathbb{R}})E([\sigma_-]) = 0 \end{aligned} \tag{9.17}$$

because of (iii) in Definition 9.7. Moreover, E_+ and E_- as well as $E([\Delta+])$, $E([\Delta_-])$ and $E(\Delta_{\mathbb{R}})$ are projections that commute mutually, as we just showed. Since in addition $E_+E_- = E_-E_+ = 0$, we have

$$\begin{aligned} E_{\mathsf{i}}(\Delta)^2 =& \left(E_+E([\Delta_+]) + E(\Delta_{\mathbb{R}}) + E_-E([\Delta_-])\right)^2 \\ =& E_+^2E([\Delta_+])^2 + E_+E([\Delta_+])E(\Delta_{\mathbb{R}}) + E_+E_-E([\Delta_+])E([\Delta_-]) \\ &+ E_+E(\Delta_{\mathbb{R}})E([\Delta_+]) + E(\Delta_{\mathbb{R}})^2 + E_-E(\Delta_{\mathbb{R}})E([\Delta_-]) \\ &+ E_-E_+E([\Delta_-])E([\Delta_+]) + E_-E([\Delta_-])E(\Delta_{\mathbb{R}}) + E_-^2E([\Delta_-])^2 \\ =& E_+E([\Delta_+]) + E(\Delta_{\mathbb{R}}) + E_-E([\Delta_-]) = E_{\mathsf{i}}(\Delta). \end{aligned} \tag{9.18}$$

Hence, $E_{\mathsf{i}}(\Delta)$ is a projection that is moreover continuous as $\|E_{\mathsf{i}}(\Delta)\| \leq K(1 + \|E_+\| + \|E_-\|)$, where $K > 0$ is the constant in Definition 9.7. Althogether, we find that E has takes values that are uniformly bounded projections in $\mathcal{B}(V_{R,\mathsf{i}})$.

We obviously have $E_{\mathsf{i}}(\emptyset) = 0$. Since $E_+ + E_- = E(\mathbb{H} \setminus \mathbb{R})$ because of (ii) in Definition 9.19 also

$$\begin{aligned} E_{\mathsf{i}}(\mathbb{C}_{\mathsf{i}}) &= E_+E([\mathbb{C}_{\mathsf{i}}^+]) + E(\mathbb{R}) + E_-E([\mathbb{C}_{\mathsf{i}}^-]) \\ &= (E_+ + E_-)E(\mathbb{H} \setminus \mathbb{R}) + E(\mathbb{R}) = E(\mathbb{H}) = \mathcal{I}. \end{aligned}$$

Using the same properties of E_+, E_- and $E(\Delta)$ as in (9.18), we find that for $\Delta, \sigma \in \mathsf{B}(\mathbb{C}_{\mathsf{i}})$

$$\begin{aligned} &E_{\mathsf{i}}(\Delta)E(\sigma) = \\ =& \big(E_+E([\Delta_+]) + E(\Delta_{\mathbb{R}}) + E_-E([\Delta_-])\big)\big(E_+E([\sigma_+]) + E(\sigma_{\mathbb{R}}) + E_-E([\sigma_-])\big) \\ =& E_+^2E([\Delta_+])E([\sigma_+]) + E_+E([\Delta_+])E(\sigma_{\mathbb{R}}) + E_+E_-E([\Delta_+])E([\sigma_-]) \\ &+ E_+E(\Delta_{\mathbb{R}})E([\sigma_+]) + E(\Delta_{\mathbb{R}})E(\sigma_{\mathbb{R}}) + E_-E(\Delta_{\mathbb{R}})E([\sigma_-]) \\ &+ E_-E_+E([\Delta_-])E([\sigma_+]) + E_-E([\Delta_-])E(\sigma_{\mathbb{R}}) + E_-^2E([\Delta_-])E([\sigma_-]) \\ =& E_+E([\Delta_+] \cap [\sigma_+]) + E(\Delta_{\mathbb{R}} \cap \sigma_{\mathbb{R}}) + E_-E([\Delta_-] \cap [\sigma_-]). \end{aligned}$$

In general it is not true that $[A] \cap [B] = [A \cap B]$ for $A, B \subset \mathbb{C}_{\mathsf{i}}$. (Just think for instance about $A = \{\mathsf{i}\}$ and $B = \{-\mathsf{i}\}$ with $[A] \cap [B] = \mathbb{S} \cap \mathbb{S} = \mathbb{S}$ and $[A \cap B] = [\emptyset] = \emptyset$.) For any axially symmetric set C we have however

$$C = \left[C \cap \mathbb{C}_{\mathsf{j}}^{\geq}\right] \qquad \forall \mathsf{j} \in \mathbb{S}$$

If A and B belong to the same complex halfplane $\mathbb{C}_{\mathsf{j}}^{\geq}$, then

$$[A] \cap [B] = \left[([A] \cap [B]) \cap \mathbb{C}_{\mathsf{j}}^{\geq}\right] = \left[\left([A] \cap \mathbb{C}_{\mathsf{j}}^{\geq}\right) \cap \left([B] \cap \mathbb{C}_{\mathsf{j}}^{\geq}\right)\right] = [A \cap B]. \tag{9.19}$$

Hence $[\Delta_+] \cap [\sigma_+] = [(\Delta \cap \sigma)_+]$ and $[\Delta_-] \cap [\sigma_-] = [(\Delta \cap \sigma)_-]$ such that altogether

$$E_{\mathsf{i}}(\Delta)E_{\mathsf{i}}(\sigma) = E_+E([(\Delta \cap \sigma)_+]) + E(\Delta_{\mathbb{R}} \cap \sigma_{\mathbb{R}}) + E_-E([(\Delta \cap \sigma)_-]) = E_{\mathsf{i}}(\Delta \cap \sigma).$$

Finally, we find for $\mathbf{v} \in V_{R,\mathsf{i}} = V_R$ and any countable family $(\Delta_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets that

$$\begin{aligned}
&E_{\mathsf{i}}\left(\bigcup_{n\in\mathbb{N}} \Delta_n\right)\mathbf{v} = \\
=&E_+ E\left(\left[\bigcup_{n\in\mathbb{N}} \Delta_{n,+}\right]\right)\mathbf{v} + E\left(\bigcup_{n\in\mathbb{N}} \Delta_{n,\mathbb{R}}\right)\mathbf{v} + E_- E\left(\left[\bigcup_{n\in\mathbb{N}} \Delta_{n,-}\right]\right)\mathbf{v} \\
=&E_+ E\left(\bigcup_{n\in\mathbb{N}} [\Delta_{n,+}]\right)\mathbf{v} + E\left(\bigcup_{n\in\mathbb{N}} \Delta_{n,\mathbb{R}}\right)\mathbf{v} + E_- E\left(\bigcup_{n\in\mathbb{N}} [\Delta_{n,-}]\right)\mathbf{v}.
\end{aligned}$$

Since the sets $\Delta_{n,+}$, $n \in \mathbb{N}$, are pairwise disjoint sets in the upper halfplane $\mathbb{C}_{\mathsf{i}}^+$, also their axially symmetric hulls are pairwise disjoint because of (9.19). Similarly, also the axially symmetric hulls of the sets $\Delta_{n,-}$, $n \in \mathbb{N}$, are pairwise disjoint such that

$$\begin{aligned}
&E_{\mathsf{i}}\left(\bigcup_{n\in\mathbb{N}} \Delta_n\right)\mathbf{v} = \\
=&\sum_{n\in\mathbb{N}} E_+ E_{\mathsf{i}}\left([\Delta_{n,+}]\right)\mathbf{v} + \sum_{n\in\mathbb{N}} E\left(\Delta_{n,\mathbb{R}}\right)\mathbf{v} + \sum_{n\in\mathbb{N}} E_- E\left([\Delta_{n,-}]\right)\mathbf{v} \\
=&\sum_{n\in\mathbb{N}} E_{\mathsf{i}}(\Delta_n)\mathbf{v}.
\end{aligned}$$

Altogether, we see that E_{i} is actually a $\mathbb{C}_{\mathsf{i}}$-linear spectral measure on $V_{\mathbb{R},\mathsf{i}}$.

Now let us consider spectral integrals. We start with the simplest real-valued function possible: $f = \alpha\chi_{\Delta}$ with $a \in \mathbb{R}$ and $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$. As $f_{\mathsf{i}} = \alpha\chi_{\Delta\cap\mathbb{C}_{\mathsf{i}}}$ and $E(\Delta) = E_{\mathsf{i}}(\Delta_{\mathsf{i}} \cap \mathbb{C}_{\mathsf{i}})$, we have for such function

$$\int_{\mathbb{H}} f(s)\, dE(s) = \alpha E(\Delta) = \alpha E_{\mathsf{i}}(\Delta \cap \mathbb{C}_{\mathsf{i}}) = \int_{\mathbb{C}_{\mathsf{i}}} f_{\mathsf{i}}(z)\, dE(z).$$

By linearity we find that (9.16) holds true for any simple function $f(s) = \sum_{\ell=1}^{n} \alpha_k \chi_{\Delta(s)}$ in $\mathcal{M}_s^{\infty}(\mathbb{H},\mathbb{R})$. Since these functions are dense in $\mathcal{M}_s^{\infty}(\mathbb{H},\mathbb{R})$, it even holds true for any function in $\mathcal{M}_s^{\infty}(\mathbb{H},\mathbb{R})$. Now consider the function $\varphi(s) = \mathsf{i}_s$ if $s \in \mathbb{H}\backslash\mathbb{R}$ and $\varphi(s) = 0$ if $s \in \mathbb{R}$. Since $\varphi_{\mathsf{i}}(z) = \mathsf{i}\chi_{\mathbb{C}_{\mathsf{i}}^+} + (-\mathsf{i})\chi_{\mathbb{C}_{\mathsf{i}}^-}$ and $E_{\mathsf{i}}(\mathbb{C}_{\mathsf{i}}^+) = E_+$ and $E_{\mathsf{i}}^- = E_-$, the integral of φ_{i} with respect to E_{i} is

$$\begin{aligned}
\int_{\mathbb{C}_{\mathsf{i}}} \varphi(z)\, dE_{\mathsf{i}}(z)\mathbf{v} &= \left(\mathsf{i}E_{\mathsf{i}}(\mathbb{C}_{\mathsf{i}}^+)\right)\mathbf{v} + \left((-\mathsf{i})E_{\mathsf{i}}(\mathbb{C}_{\mathsf{i}}^-)\right)\mathbf{v} \\
&= E_+\mathbf{v}\mathsf{i} + E_-\mathbf{v}(-\mathsf{i}) = J\mathbf{v}
\end{aligned}$$

for all $\mathbf{v} \in V_{R,\mathsf{i}} = V_R$. If f is now an arbitrary function in $\mathcal{SM}^{\infty}(\mathbb{H})$, then $f(s) = \alpha(s) + \varphi(s)\beta(s)$ with $\alpha, \beta \in \mathcal{M}_s^{\infty}(\mathbb{H},\mathbb{R})$ and $\beta(s) = 0$ if $s \in \mathbb{R}$ by Lemma 9.21. By what we have shown so far and the homomorphism properties of both quaternionic and

complex spectral integrals, we thus find

$$\begin{aligned}
&\int_{\mathbb{H}} f(s)\, dE_J(s) = \\
&= \int_{\mathbb{H}} \alpha(s)\, dE(s) + J \int_{\mathbb{H}} \beta(s)\, dE(s) \\
&= \int_{\mathbb{C}_{\mathsf{i}}} \alpha_{\mathsf{i}}(z)\, dE_{\mathsf{i}}(z) + \left(\int_{\mathbb{C}_{\mathsf{i}}} \varphi_{\mathsf{i}}(z)\, dE_{\mathsf{i}}(z) \right) \left(\int_{\mathbb{C}_{\mathsf{i}}} \beta_{\mathsf{i}}(z)\, dE_{\mathsf{i}}(z) \right) \\
&= \int_{\mathbb{C}_{\mathsf{i}}} \alpha_{\mathsf{i}}(z) + \varphi_{\mathsf{i}}(z)\beta_{\mathsf{i}}(z)\, dE_{\mathsf{i}}(z) = \int_{\mathbb{C}_{\mathsf{i}}} f_{\mathsf{i}}(z)\, dE_{\mathsf{i}}(z).
\end{aligned}$$

□

Working on a quaternionic Hilbert space, one might consider only spectral measures whose values are orthogonal projections. If J is an anti-selfadjoint partially unitary operator as it happens for instance in the spectral theorem for normal operators in [5], then E_{i} has values that are orthogonal projections.

Corollary 9.27. *Let $\mathcal{H}$ be a quaternionic Hilbert space, let (E, J) be a spectral system on $\mathcal{H}$, let $\mathsf{i} \in \mathbb{S}$ and let E_{i} be the spectral measure defined in* (9.15)*. If $E(\Delta)$ is for any $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$ an orthogonal projection on $\mathcal{H}$ and J is an anti-selfadjoint partially unitary operator, then $E_{\mathsf{i}}(\Delta_{\mathsf{i}})$ is for any $\Delta_{\mathsf{i}} \in \mathsf{B}(\mathbb{C}_{\mathsf{i}})$ an orthogonal projection on $(\mathcal{H}, \langle \cdot, \cdot, \rangle_{\mathsf{i}})$, where $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathsf{i}} = \{\langle \mathbf{u}, \mathbf{v} \rangle\}_{\mathsf{i}}$ as in Remark 2.40.*

Proof. If $\mathbf{u}, \mathbf{v} \in \mathcal{H}_{J,\mathsf{i}}^{+}$, then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, -J^2 \mathbf{v} \rangle = \langle J\mathbf{u}, J\mathbf{v} \rangle = \langle \mathbf{u}\mathsf{i}, \mathbf{v}\mathsf{i} \rangle = (-\mathsf{i}) \langle \mathbf{u}, \mathbf{v} \rangle \mathsf{i}$$

such that $\mathsf{i}\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle \mathsf{i}$. Since a quaternion commutes with $\mathsf{i} \in \mathbb{S}$ if and only if it belongs to $\mathbb{C}_{\mathsf{i}}$, we have $\langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{C}_{\mathsf{i}}$. Hence, if we choose $\mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$, then $\langle \mathbf{u}, \mathbf{v}\mathsf{j} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle \mathsf{j} \in \mathbb{C}_{\mathsf{i}}\mathsf{j}$ such that in turn $\langle \mathbf{u}, \mathbf{v}\mathsf{j} \rangle_{\mathsf{i}} = \{\langle \mathbf{u}, \mathbf{v} \rangle\}_{\mathsf{i}} = 0$ for $\mathbf{u}, \mathbf{v} \in \mathcal{H}_{J,\mathsf{i}}^{+}$. Since $\mathcal{H}_{J,\mathsf{i}}^{-} = \{\mathbf{v}\mathsf{j} : \mathbf{v} \in \mathcal{H}_{J,\mathsf{i}}^{+}\}$ by Theorem 9.18, we find $\mathcal{H}_{J,\mathsf{i}}^{-} \perp_{\mathsf{i}} \mathcal{H}_{J,\mathsf{i}}^{+}$, where $\perp_{\mathsf{i}}$ denotes orthogonality in $\mathcal{H}_{\mathsf{i}}$. Furthermore, we have for $\mathbf{u} \in \mathcal{H}_0 = \ker J$ and $\mathbf{v} \in \mathcal{H}_{J,\mathsf{i}}^{+}$ that

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, J\mathbf{v} \rangle (-\mathsf{i}) = \langle J\mathbf{u}, \mathbf{v} \rangle \mathsf{i} = \langle \mathbf{0}, \mathbf{v} \rangle \mathsf{i} = 0$$

and so $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathsf{i}} = \{\langle \mathbf{u}, \mathbf{v} \rangle\}_{\mathsf{i}} = 0$ and in turn $\mathcal{H}_{J,\mathsf{i}}^{+} \perp \mathcal{H}_0$. Similarly, we see that also $\mathcal{H}_{J,\mathsf{i}}^{-} \perp_{\mathsf{i}} \mathcal{H}_0$. Hence, the direct sum decomposition $\mathcal{H}_{\mathsf{i}} = \mathcal{H}_{J,0} \oplus \mathcal{H}_{J,\mathsf{i}}^{+} \oplus \mathcal{H}_{J,\mathsf{i}}^{-}$ in (9.11) is actually a decomposition into orthogonal subspaces of $\mathcal{H}_{\mathsf{i}}$. The projection E_{+} of $\mathcal{H}$ onto $\mathcal{H}_{J,\mathsf{i}}^{+}$ along $\mathcal{H}_{J,0} \oplus \mathcal{H}_{J,\mathsf{i}}^{-}$ and the projection E_{-} of $\mathcal{H}$ onto $\mathcal{H}_{J,\mathsf{i}}^{-}$ along $\mathcal{H}_{J,0} \oplus \mathcal{H}_{J,\mathsf{i}}^{+}$ are hence orthogonal projections on $\mathcal{H}_{\mathsf{i}}$.

Since the operator $E(\Delta)$ is for $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$ an orthogonal projection on $\mathcal{H}$, it is an orthogonal projection on $\mathcal{H}_{\mathsf{i}}$. A projection is orthogonal if and only if it is self-adjoint. Since E_{+}, E_{-} and E commute mutually, we find for any $\Delta \in \mathsf{B}(\mathbb{C}_{\mathsf{i}})$ and any $\mathbf{u}, \mathbf{v} \in \mathcal{H}_{\mathsf{i}} = \mathcal{H}$ that

$$\begin{aligned}
&\langle \mathbf{u}, E_{\mathsf{i}}(\Delta)\mathbf{v} \rangle_{\mathsf{i}} \\
=&\langle \mathbf{u}, E_{+}E([\Delta \cap \mathbb{C}_{\mathsf{i}}^{+}])\mathbf{v} \rangle_{\mathsf{i}} + \langle \mathbf{u}, E(\Delta \cap \mathbb{R})\mathbf{v} \rangle_{\mathsf{i}} + \langle \mathbf{u}, E_{-}E([\Delta \cap \mathbb{C}_{\mathsf{i}}^{-}])\mathbf{v} \rangle_{\mathsf{i}} \\
=&\langle E_{+}E([\Delta \cap \mathbb{C}_{\mathsf{i}}^{+}])\mathbf{u}, \mathbf{v} \rangle_{\mathsf{i}} + \langle E(\Delta \cap \mathbb{R})\mathbf{u}, \mathbf{v} \rangle_{\mathsf{i}} + \langle E_{-}E([\Delta \cap \mathbb{C}_{\mathsf{i}}^{-}])\mathbf{u}, \mathbf{v} \rangle_{\mathsf{i}} \\
=&\langle E_{\mathsf{i}}(\Delta)\mathbf{u}, \mathbf{v} \rangle_{\mathsf{i}}.
\end{aligned}$$

Hence, $E_{\mathsf{i}}(\Delta)$ is an orthogonal projection on $\mathcal{H}_{\mathsf{i}}$.

□

We present two easy examples of spectral systems that illustrate the intuition behind the concept of a spectral system.

Example 9.28. We consider a compact normal operator T on a quaternionic Hilbert space $\mathcal{H}$. The spectral theorem for compact normal operators in [50] implies that the S-spectrum $\sigma_S(T)$ consists of a (possibly finite) sequence $[s_n] = s_{n,0} + \mathbb{S}s_{n,1}, n \in \Upsilon \subset \mathbb{N}$ of spectral spheres that are (apart from possibly the sphere $[0]$) isolated in $\mathbb{H}$. Moreover it implies the existence of an orthonormal basis of eigenvectors $(\mathbf{b}_\ell)_{\ell\in\Lambda}$ associated with eigenvalues $s_\ell = s_{\ell,0} + \mathbf{i}_{s_\ell} s_{\ell,1}$ with $\mathbf{i}_{s_\ell} = 0$ if $s_\ell \in \mathbb{R}$ such that

$$T\mathbf{v} = \sum_{\ell\in\Lambda} \mathbf{b}_\ell s_\ell \langle \mathbf{b}_\ell, \mathbf{v}\rangle. \tag{9.20}$$

Each eigenvalue s_ℓ obviously belongs to one spectral sphere, namely to $[s_{n(\ell)}]$ with $s_{n(\ell),0} = s_{\ell,0}$ and $s_{n(\ell),1} = s_{\ell,1}$, and for $[s_n] \neq \{0\}$ only finitely many eigenvalues belong to the spectral sphere $[s_n]$. We can hence rewrite (9.20) as

$$T\mathbf{v} = \sum_{[s_n]\in\sigma_S(T)} \sum_{s_\ell\in[s_n]} \mathbf{b}_\ell s_\ell \langle \mathbf{b}_\ell, \mathbf{v}\rangle = \sum_{n\in\Upsilon} \sum_{n(\ell)=n} \mathbf{b}_\ell s_\ell \langle \mathbf{b}_\ell, \mathbf{v}\rangle$$

The spectral measure E of T is then given by

$$E(\Delta)\mathbf{v} = \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\Delta}} \sum_{n(\ell)=n} \mathbf{b}_\ell \langle \mathbf{b}_\ell, \mathbf{v}\rangle \qquad \forall\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$$

If $f \in \mathcal{M}_s^\infty(\mathbb{H}, \mathbb{R})$, then obviously

$$\int_{\mathbb{H}} f(s)\, dE(s)\mathbf{v} = \sum_{n\in\Upsilon} E([s_n])\mathbf{v} f(s_n) = \sum_{n\in\Upsilon} \sum_{n(\ell)=n} \mathbf{b}_\ell \langle \mathbf{b}_\ell, \mathbf{v}\rangle f(s_n). \tag{9.21}$$

In particular

$$\int_{\mathbb{H}} s_0\, dE(s)\mathbf{v} = \sum_{n\in\Upsilon} \sum_{n(\ell)=n} \mathbf{b}_\ell \langle \mathbf{b}_\ell, \mathbf{v}\rangle s_{\ell,0}$$

and

$$\int_{\mathbb{H}} s_1\, dE(s)\mathbf{v} = \sum_{n\in\Upsilon} \sum_{n(\ell)=n} \mathbf{b}_\ell \langle \mathbf{b}_\ell, \mathbf{v}\rangle s_{\ell,1}.$$

If we define

$$\mathsf{J}_0\mathbf{v} := \sum_{n\in\Upsilon} \sum_{n(\ell)=n} \mathbf{b}_\ell \mathbf{i}_{s_\ell} \langle \mathbf{b}_\ell, \mathbf{v}\rangle,$$

then J_0 is an anti-selfadjoint partially unitary operator and (E, J_0) is a spectral system. One can check easily that E and J_0 commute and, as $\mathbf{i}_{s_\ell} = 0$ for $s_\ell \in \mathbb{R}$ and $\mathbf{i}_{s_\ell} \in \mathbb{S}$ with $\mathbf{i}_{s_\ell}^2 = -1$, otherwise one has

$$-\mathsf{J}_0^2\mathbf{v} = -\sum_{n\in\Upsilon} \sum_{n(\ell)=n} \mathbf{b}_\ell \mathbf{i}_{s_\ell}^2 \langle \mathbf{b}_\ell, \mathbf{v}\rangle = \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}} \sum_{n(\ell)=n} \mathbf{b}_\ell \langle \mathbf{b}_\ell, \mathbf{v}\rangle = E(\mathbb{H}\setminus\mathbb{R})\mathbf{v}.$$

In particular $\ker \mathsf{J}_0 = cl(\operatorname{span}_{\mathbb{H}}\{\mathbf{b}_\ell : s_\ell \in \mathbb{R}\}) = E(\mathbb{R})$. Note moreover that J_0 is completely determined by T.

For any function $f = \alpha + \mathbf{i}\beta \in \mathcal{SM}^\infty(\mathbb{H})$, we have because of (9.21) and as $\langle \mathbf{b}_\ell, \mathbf{b}_\kappa\rangle = \delta_{\ell,\kappa}$ that

$$\begin{aligned}
&\int_{\mathbb{H}} f(s)\, dE_{\mathsf{J}_0}(s)\mathbf{v} = \int_{\mathbb{H}} \alpha(s)\, dE(s)\mathbf{v} + \mathsf{J}_0 \int_{\mathbb{H}} f(s)\, dE(s)\mathbf{v}\\
=&\sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell \langle \mathbf{b}_\ell, \mathbf{v}\rangle \alpha(s_{n,0}, s_{n,1})\\
&+ \sum_{m,n\in\Upsilon}\sum_{\substack{n(\ell)=n\\ n(\kappa)=m}} \mathbf{b}_\ell \mathbf{i}_{s_\ell} \langle \mathbf{b}_\ell, \mathbf{b}_\kappa\rangle\langle \mathbf{b}_\kappa, \mathbf{v}\rangle \beta(s_{m,0}, s_{m,1})\\
=&\sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell \alpha(s_{\ell,0}, s_{\ell,1})\langle \mathbf{b}_\ell, \mathbf{v}\rangle\\
&+\sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell \mathbf{i}_{s_\ell}\beta(s_{\ell,0}, s_{\ell,1})\langle \mathbf{b}_\ell, \mathbf{v}\rangle\\
=&\sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell(\alpha(s_{\ell,0}, s_{\ell,1}) + \mathbf{i}_{s_\ell}\beta(s_{\ell,0}, s_{\ell,1}))\langle \mathbf{b}_\ell, \mathbf{v}\rangle
\end{aligned}$$

and so

$$\int_{\mathbb{H}} f(s)\, dE_{\mathsf{J}_0}(s)\mathbf{v} = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell f(s_\ell)\langle \mathbf{b}_\ell, \mathbf{v}\rangle. \tag{9.22}$$

In particular

$$\int_{\mathbb{H}} s\, dE_{\mathsf{J}_0}(s) = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell s_\ell \langle \mathbf{b}_\ell, \mathbf{v}\rangle = T\mathbf{v}.$$

We in particular have $T = A + \mathsf{J}_0 B$ where $A = \int_{\mathbb{H}} s_0\, dE(s)$ is self-adjoint, $B = \int_{\mathbb{H}} s_1\, dE(s)$ is positive and J_0 is anti-selfadjoint and partially unitary as in (2.40). Moreover, E corresponds via Remark 9.9 to the spectral measure obtained from Theorem 2.78.

We choose now $\mathbf{i}, \mathbf{j} \in \mathbb{S}$ with $\mathbf{i} \perp \mathbf{j}$ and for each $\ell \in \Lambda$ with $s_\ell \notin \mathbb{R}$ we choose $h_\ell \in \mathbb{H}$ with $|h_\ell| = 1$ such that $h_\ell^{-1}\mathbf{i}_{s_\ell} h_\ell = \mathbf{i}$ and in turn

$$h_\ell^{-1} s_\ell h_\ell = s_{\ell,0} + h_\ell^{-1}\mathbf{i}_{s_\ell} h_\ell s_1 = s_{\ell,0} + \mathbf{i} s_{\ell,1} =: s_{\ell,\mathbf{i}}.$$

In order to simplify the notation we also set $h_\ell = 1$ and $\mathbf{i}_{s_\ell} = 0$ if $s_\ell \in \mathbb{R}$. Then $\tilde{\mathbf{b}}_\ell := \mathbf{b}_\ell h_\ell, \ell \in \Lambda$ is another orthonormal basis consisting of eigenvector of T and as $h_\ell^{-1} = \overline{h_\ell}/|h_\ell|^2 = \overline{h_\ell}$ we have

$$\begin{aligned}
T\mathbf{v} =&\sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell (h_\ell h_\ell^{-1}) s_\ell (h_\ell h_\ell^{-1})\langle \mathbf{b}_\ell, \mathbf{v}\rangle\\
=&\sum_{n\in\Upsilon}\sum_{n(\ell)=n} (\mathbf{b}_\ell h_\ell)(h_\ell^{-1} s_\ell h_\ell)\langle \mathbf{b}_\ell h_\ell, \mathbf{v}\rangle = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell s_{\ell,\mathbf{i}} \langle \tilde{\mathbf{b}}_\ell, \mathbf{v}\rangle
\end{aligned} \tag{9.23}$$

and similarly

$$\begin{aligned}
J_0\mathbf{v} =& \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell(h_\ell h_\ell^{-1})\mathbf{i}_\ell(h_\ell h_\ell^{-1})\langle\mathbf{b}_\ell,\mathbf{v}\rangle\\
=& \sum_{n\in\Upsilon}\sum_{n(\ell)=n} (\mathbf{b}_\ell h_\ell)(h_\ell^{-1}\mathbf{i}_\ell h_\ell)\langle\mathbf{b}_\ell h_\ell,\mathbf{v}\rangle = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell\mathbf{i}\langle\tilde{\mathbf{b}}_\ell,\mathbf{v}\rangle.
\end{aligned}$$

Recall that $\mathsf{i}\lambda = \lambda\mathsf{i}$ for any $\lambda \in \mathbb{C}_{\mathsf{i}}$ and $\mathsf{ij} = -\mathsf{ji}$. The splitting of $\mathcal{H}$ obtained from Theorem 9.18 is therefore given by

$$\mathcal{H}_{\mathsf{J}_0,0} = \ker J_0 = cl(\operatorname{span}_{\mathbb{H}}\{\tilde{\mathbf{b}}_\ell : s_\ell \in \mathbb{R}\}), \qquad \mathcal{H}^+_{\mathsf{J}_0,\mathsf{i}} := cl(\operatorname{span}_{\mathbb{C}_{\mathsf{i}}}\{\tilde{\mathbf{b}}_\ell : s_\ell \notin \mathbb{R}\})$$

and

$$\mathcal{H}^-_{\mathsf{J}_0,\mathsf{i}} = cl(\operatorname{span}_{\mathbb{C}_{\mathsf{i}}}\{\tilde{\mathbf{b}}_\ell\mathsf{j} : s_\ell \notin \mathbb{R}\}) = \mathcal{H}^+_{\mathsf{J}_0,\mathsf{i}}\mathsf{j}.$$

If $\langle\mathbf{b}_\ell,\mathbf{v}\rangle = a_\ell = a_{\ell,1} + a_{\ell,2}\mathsf{j}$ with $a_{\ell,1}, a_{\ell,2} \in \mathbb{C}_{\mathsf{i}}$, then (9.23) implies

$$\begin{aligned}
T\mathbf{v} =& \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell s_{\ell,\mathsf{i}} a_\ell\\
=& \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{R}}}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell a_\ell s_\ell + \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell a_{\ell,1} s_{\ell,\mathsf{i}} + \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell a_{\ell,2}\mathsf{j}\overline{s_{\ell,\mathsf{i}}}.
\end{aligned} \tag{9.24}$$

If $f \in \mathcal{SM}^\infty(\mathbb{H})$, then the operator $f(T) = \int_{\mathbb{H}} f(s)\,dE_{\mathsf{J}_0}(s)$ can be represented in the basis $\tilde{\mathbf{b}}_\ell$, $\ell \in \Lambda$, using (9.22) as

$$\begin{aligned}
\int f(s)\,dE_{\mathsf{J}_0}(s)\mathbf{v} =& \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell f(s_{\ell,\mathsf{i}}) a_\ell\\
=& \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{R}}}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell a_\ell f(s_\ell)\\
&+ \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell a_{\ell,1} f(s_{\ell,\mathsf{i}}) + \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell a_{\ell,2}\mathsf{j}\overline{f(s_{\ell,\mathsf{i}})}\\
=& \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{R}}}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell a_\ell f(s_\ell)\\
&+ \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell a_{\ell,1} f(s_{\ell,\mathsf{i}}) + \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell a_{\ell,2}\mathsf{j} f(\overline{s_{\ell,\mathsf{i}}}).
\end{aligned} \tag{9.25}$$

as $f(s_\ell) \in \mathbb{R}$ for $s_\ell \in \mathbb{R}$ and $\overline{f(s_{\ell,\mathsf{i}})} = f(\overline{s_{\ell,\mathsf{i}}})$ because f is intrinsic. Note that the representation (9.24) and (9.25) show clearly that $f(T)$ is actually defined by letting f act on the right eigenvalues of T.

Example 9.29. Let us consider the space $L^2(\mathbb{R},\mathbb{H})$ of all quaternion-valued functions on $\mathbb{R}$ that are square-integrable with respect to the Lebesgue measure λ. Endowed with the pointwise multiplication $(fa)(t) = f(t)a$ for $f \in L^2(\mathbb{R},\mathbb{H})$ and $a \in \mathbb{H}$ and with the scalar product

$$\langle g, f\rangle = \int_{\mathbb{R}} \overline{g(t)} f(t)\,d\lambda(t) \qquad \forall f,g \in L^2(\mathbb{R},\mathbb{H}), \tag{9.26}$$

this space is a quaternionic Hilbert space. Let us now consider a bounded measurable function $\varphi : \mathbb{R} \to \mathbb{H}$ and the multiplication operator $(M_{\varphi}f)(s) := \varphi(s)f(s)$. This operator is normal with $(M_{\varphi})^* = M_{\overline{\varphi}}$ and its S-spectrum is the set $\overline{\varphi(\mathbb{R})}$. Indeed, writing $\varphi(t) = \varphi_0(t) + \mathbf{i}_{\varphi(t)}\varphi_1(t)$ with $\varphi_0(t) \in \mathbb{R}$, $\varphi_1(t) > 0$ and $\mathbf{i}_{\varphi(t)} \in \mathbb{S}$ for $\varphi(t) \in \mathbb{H} \setminus \mathbb{R}$ and $\mathbf{i}_{\varphi(t)} = 0$ for $\varphi(t) \in \mathbb{R}$, we find that

$$\begin{aligned}\mathcal{Q}_s(M_{\varphi})f(t) =& M_{\varphi}^2 f(t) - 2s_0 M_{\varphi} f(t) + |s|^2 f(t)\\ =&(\varphi^2(t) - 2s_0\varphi(t) + |s|^2)f(t)\\ =&(\varphi(t) - s_{\mathbf{i}_{\varphi(t)}})(\varphi(t) - \overline{s_{\mathbf{i}_{\varphi(t)}}})f(t)\end{aligned}$$

with $s_{\mathbf{i}_{\varphi(t)}} = s_0 + \mathbf{i}_{\varphi(t)}s_1$ and hence

$$\mathcal{Q}_s(M_{\varphi})^{-1}f(t) = (\varphi(t) - s_{\mathbf{i}_{\varphi(t)}})^{-1}(\varphi(t) - \overline{s_{\mathbf{i}_{\varphi(t)}}})^{-1}f(t)$$

is a bounded operator if $s \notin \overline{\varphi(\mathbb{R})}$. If we define $E(\Delta) = M_{\chi_{\varphi^{-1}(\Delta)}}$ for all $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$ then we obtain a spectral measure on $\mathsf{B}_{\mathsf{S}}(\mathbb{H})$, namely

$$E(\Delta)f(t) = \chi_{\varphi^{-1}(\Delta)}(t)f(t).$$

If we set

$$\mathsf{J}_0 := M_{\mathbf{i}_{\varphi}} \quad \text{i.e.} \quad (\mathsf{J}_0 f)(t) = \mathbf{i}_{\varphi(t)}f(t),$$

then we find that (E, J_0) is a spectral system. Obviously J_0 is anti-selfadjoint and partially unitary and hence an imaginary operator that commutes with E. Since $\mathbf{i}_{\varphi(t)} = 0$ if $\varphi(t) \in \mathbb{R}$ and $\mathbf{i}_{\varphi(t)} \in \mathbb{S}$ otherwise, we have moreover

$$(-\mathsf{J}_0^2 f)(t) = -\mathbf{i}_{\varphi(t)}^2 f(t) = \chi_{\varphi^{-1}(\mathbb{H}\setminus\mathbb{R})}f(t) = (E(\mathbb{H} \setminus \mathbb{R})f)(t).$$

If $g \in \mathcal{M}_s^{\infty}(\mathbb{H}, \mathbb{R})$, then let $g_n(s) = \sum_{\ell=1}^{N_n} a_{n,\ell}\chi_{\Delta_{n,\ell}}(s) \in \mathcal{M}_s^{\infty}(\mathbb{H}, \mathbb{R})$ be a sequence of simple functions that converges uniformly to g. Then

$$\begin{aligned}\int_{\mathbb{H}} g(s)\,dE(s)f(t) &= \lim_{n\to+\infty}\sum_{\ell=1}^{N_n} a_{n,\ell}E(\Delta_{n,\ell})f(t) = \lim_{n\to+\infty}\sum_{\ell=1}^{N_n} a_{n,\ell}\chi_{\varphi^{-1}(\Delta)}(t)f(t)\\ &= \lim_{n\to+\infty}\sum_{\ell=1}^{N_n} a_{n,\ell}\chi_{\Delta}(\varphi(t))f(t) = \lim_{n\to+\infty}(g_n\circ\varphi)(t)f(t) = (g\circ\varphi)(t)f(t).\end{aligned}$$

Hence, if $g(s) = \alpha(s) + \mathbf{i}_s\beta(s) \in \mathcal{SM}^{\infty}(\mathbb{H})$, then

$$\begin{aligned}&\int_{\mathbb{H}} g(s)\,dE_{\mathsf{J}_0}(s)f(t) = \int_{\mathbb{H}} g(s)\,dE(s)f(t)\\ =&\int_{\mathbb{H}} \alpha(s)\,dE(s)f(t) + \mathsf{J}_0\int_{\mathbb{H}} \beta(s)\,dE(s)f(t)\\ =&\alpha(\varphi(t))f(t) + \mathbf{i}_{\varphi(t)}\beta(\varphi(t))f(t)\\ =&(\alpha(\varphi(t)) + \mathbf{i}_{\varphi(t)}\beta(\varphi(t)))f(t) = (g\circ\varphi)(t)f(t)\end{aligned}$$

and so

$$\int_{\mathbb{H}} g(s)\,dE_{\mathsf{J}_0}(s) = M_{g\circ\varphi}.$$

Choosing $g(s) = s$, we in particular find $T = A + \mathsf{J}_0 B$ with $A = \int_{\mathbb{H}} s_0 \, dE(s)$ selfadjoint, $B = \int_{\mathbb{H}} s_1 \, dE(s)$ positive and J_0 anti-selfadjoint and partially unitary as in (2.40). E corresponds via Remark 9.9 to the spectral measure obtained from Theorem 2.78.

9.4 On the Different Approaches to Spectral Integration

Our approach to spectral integration specified some ideas in [87]. We conclude this section with a comparison of this approach with the approaches in [5] and [51], which were explained quickly in Section 2.4. All three approaches are consistent, if things are interpreted correctly. Let us first consider a spectral measure E over $\mathbb{C}_{\mathsf{i}}^{\geq}$ for some $\mathsf{i} \in \mathbb{S}$ in the sense of Definition 2.75, the values of which are orthogonal projections on a quaternionic Hilbert space $\mathcal{H}$. Let furthermore J be a unitary anti-selfadjoint operator on $\mathcal{H}$ that commutes with E and let us interpret the application of J as a multiplication with i from the left as in [5]. By Remark 9.9, we obtain a quaternionic spectral measure on $\mathsf{B}_{\mathsf{S}}(\mathbb{H})$ if we set $\tilde{E}(\Delta) := E\big(\Delta \cap \mathbb{C}_{\mathsf{i}}^{\geq}\big)$ for $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$ and obviously we have

$$\int_{\mathbb{H}} f(s) \, d\tilde{E}(s) = \int_{\mathbb{C}_{\mathsf{i}}} f_{\mathsf{i}}(z) \, dE(z) \qquad \forall f \in \mathcal{M}_s^{\infty}(\mathbb{H}, \mathbb{R}),$$

where $f_{\mathsf{i}} = f|_{\mathbb{C}_{\mathsf{i}}^{\geq}}$. If we set $J := \mathsf{J}\tilde{E}(\mathbb{H} \setminus \mathbb{R}) = \mathsf{J}E(\mathbb{C}_{\mathsf{i}}^{+})$, then J is an imaginary operator and we find that $(\tilde{E}, J)$ is a spectral system on $\mathcal{H}$. Now let $f(s) = \alpha(s) + \mathsf{i}\beta(s) \in \mathcal{SM}^{\infty}(\mathbb{H})$ and let again $f_{\mathsf{i}} = f|_{\mathbb{C}_{\mathsf{i}}^{\geq}}$, $\alpha_{\mathsf{i}} = \alpha|_{\mathbb{C}_{\mathsf{i}}^{\geq}}$ and $\beta_{\mathsf{i}} = \beta|_{\mathbb{C}_{\mathsf{i}}^{\geq}}$. Since $\beta(s) = 0$ if $s \in \mathbb{R}$, we have $\beta(s) = \chi_{\mathbb{H}\setminus\mathbb{R}}(s)\beta(s)$ and in turn

$$\begin{aligned}
\int_{\mathbb{C}_{\mathsf{i}}^{\geq}} f_{\mathsf{i}}(z) \, dE(z) &= \int_{\mathbb{C}_{\mathsf{i}}^{\geq}} \alpha_{\mathsf{i}}(z) \, dE(z) + \mathsf{J} \int_{\mathbb{C}_{\mathsf{i}}^{\geq}} \beta_{\mathsf{i}}(z) \, dE(z) \\
&= \int_{\mathbb{H}} \alpha(s) \, d\tilde{E}(s) + \mathsf{J} \int_{\mathbb{H}} \chi_{\mathbb{H}\setminus\mathbb{R}}(s)\beta(s) \, d\tilde{E}(s) \\
&= \int_{\mathbb{H}} \alpha(s) \, d\tilde{E}(s) + \mathsf{J}E(\mathbb{H} \setminus \mathbb{R}) \int_{\mathbb{H}} \beta(s) \, d\tilde{E}(s) \\
&= \int_{\mathbb{H}} \alpha(s) \, d\tilde{E}(s) + J \int_{\mathbb{H}} \beta(s) \, d\tilde{E}(s) = \int_{\mathbb{H}} f(s) \, d\tilde{E}_J(s).
\end{aligned} \tag{9.27}$$

Hence, for any measurable intrinsic slice function f, the spectral integral of f with respect to the spectral system $(\tilde{E}, J)$ coincides with the spectral integral of $f|_{\mathbb{C}_{\mathsf{i}}^{\geq}}$ with respect to E, where we interpret the application of J as a multiplication with i from the left. Since the mapping $f \mapsto f|_{\mathbb{C}_{\mathsf{i}}^{\geq}}$ is a bijection between the set of all measurable intrinsic slice functions on $\mathbb{H}$ and the set of all measurable $\mathbb{C}_{\mathsf{i}}$-valued functions on $\mathbb{C}_{\mathsf{i}}^{\geq}$ that map the real line into itself, both approaches are equivalent for real slice functions if we identify $\tilde{E}$ with E and f with f_{i}. The same identifications show that the approach in [51] is equivalent to our approach, as long as we only consider intrinsic slice functions. Indeed, if $\mathcal{E} = (E, \mathcal{L})$ is an iqPVM over $\mathbb{C}_{\mathsf{i}}^{\geq}$ on $\mathcal{H}$, then $\mathsf{J}\mathbf{v} := L_{\mathsf{i}}\mathbf{v} = \mathsf{i}\mathbf{v}$ is a unitary and anti-selfadjoint operator on $\mathcal{H}$. As before, we can set $\tilde{E}(\Delta) = E(\Delta \cap \mathbb{C}_{\mathsf{i}}^{\geq})$ and

$J := \mathsf{J}\tilde{E}(\mathbb{H} \setminus \mathbb{R}) = L_{\mathsf{i}}E(\mathbb{C}_{\mathsf{i}}^{+})$. We then find as in (9.27) that

$$\int_{\mathbb{C}_{\mathsf{i}}^{\geq}} f_{\mathsf{i}}(z)\, d\mathcal{E}(z) = \int_{\mathbb{H}} f(s)\, d\tilde{E}_J(s) \qquad \forall f \in \mathcal{SM}^{\infty}(\mathbb{H}). \tag{9.28}$$

For intrinsic slice functions, all three approaches are hence consistent.

Let us continue our discussion of how our approach to spectral integration fits into the existing theory. We recall that any normal operator T on $\mathcal{H}$ can be decomposed as

$$T = A + \mathsf{J}_0 B,$$

with the selfadjoint operator $A = \frac{1}{2}(T + T^*)$, the positive operator $B = \frac{1}{2}|T - T^*|$ and the anti-selfadjoint partially unitary operator J_0 with $\ker J_0 = \ker(T - T^*)$ and $\operatorname{ran} J_0 = \ker(T - T^*)^{\perp}$. Let $\mathcal{E} = (E, \mathcal{L})$ be the iqPVM of T obtained from Theorem 2.81. From [51, Theorem 3.13], we know that $\left(\int_{\mathbb{C}_{\mathsf{i}}^{\geq}} \varphi(z)\, d\mathcal{E}(z)\right)^* = \int_{\mathbb{C}_{\mathsf{i}}^{\geq}} \overline{\varphi(z)}\, d\mathcal{E}(z)$ and $\ker \int_{\mathbb{C}_{\mathsf{i}}^{\geq}} \varphi(z)\, d\mathcal{E}(z) = \operatorname{ran} E(\varphi^{-1}(0))$. Hence

$$T - T^* = \int_{\mathbb{C}_{\mathsf{i}}^{\geq}} z\, d\mathcal{E}(z) - \int_{\mathbb{C}_{\mathsf{i}}^{\geq}} \overline{z}\, d\mathcal{E}(z) = \int_{\mathbb{C}_{\mathsf{i}}^{\geq}} 2\mathbf{i} z_1\, d\mathcal{E}(z).$$

As $z_1 = 0$ if and only if $z \in \mathbb{R}$, we find that $\ker \mathsf{J}_0 = \ker(T - T^*) = \operatorname{ran} E(\mathbb{R})$ and in turn $\operatorname{ran} \mathsf{J}_0 = \ker(T - T^*)^{\perp} = \operatorname{ran} E\big(\mathbb{C}_{\mathsf{i}}^{\geq} \setminus \mathbb{R}\big) = \operatorname{ran} E(\mathbb{C}_{\mathsf{i}}^{+})$.

If we identify E with the spectral measure $\tilde{E}$ on $\mathsf{B}_{\mathsf{S}}(\mathbb{H})$ that is given by $\tilde{E}(\Delta) = E\big(\Delta \cap \mathbb{C}_{\mathsf{i}}^{\geq}\big)$, then $J = L_{\mathsf{i}}E(\mathbb{C}_{\mathsf{i}}^{+})$ is an imaginary operator such that $(\tilde{E}, J)$ is a spectral system as we showed above. The spectral integral of any measurable intrinsic slice function f with respect to $(\tilde{E}, J)$ coincides with the spectral integral of $f|_{\mathbb{C}_{\mathsf{i}}^{\geq}}$ with respect to $\mathcal{E}$. Since $\operatorname{ran} E(\mathbb{C}_{\mathsf{i}}^{+}) = \ker(T - T^*)^{\perp} = \operatorname{ran} \mathsf{J}_0$ and $L_{\mathsf{i}}\mathbf{v} = \mathsf{J}_0\mathbf{v}$ for all $\mathbf{v} \in \ker(T - T^*)^{\perp}$, we moreover find $J = \mathsf{J}_0$. Therefore $(\tilde{E}, \mathsf{J}_0)$ is the spectral system that is for integration of intrinsic slice functions equivalent to $\mathcal{E}$. We can hence rewrite Theorem 2.78 and Theorem 2.81 in the terminology of spectral systems as follows.

Theorem 9.30. *Let $T = A + \mathsf{J}_0 B \in \mathcal{B}(\mathcal{H})$ be a normal operator. There exists a unique quaternionic spectral measure E on $\mathsf{B}_{\mathsf{S}}(\mathbb{H})$ with $E(\mathbb{H} \setminus \sigma_S(T)) = 0$, the values of which are orthogonal projections on $\mathcal{H}$, such that (E, J_0) is a spectral system and such that*

$$T = \int_{\mathbb{H}} s\, dE_{\mathsf{J}_0}(s).$$

We want to point out that the spectral system (E, J_0) is completely determined by T—unlike the unitary anti-selfadjoint operator J that extends J_0 used in [5] and unlike the iqPVM used in [51]. We also want to stress that the proof of the spectral theorem in [5] translates directly into the language of spectral systems: the spectral measure for T is constructed using only real-valued functions, hence the extension J of J_0 is irrelevant in this proof. Indeed, in the article [5], only functions that are restrictions of intrinsic slice functions are integrated so that one can pass to the language of spectral systems by the identification described above without any problems.

Example 9.31. In order to discuss the relations described above let us return to Example 9.28, in which we considered the normal compact operator on a quaternionic Hilbert space $\mathcal{H}$ given by

$$T\mathbf{v} = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell s_\ell \langle \mathbf{b}_\ell, \mathbf{v}\rangle.$$

The spectral system (E, J_0) of T is

$$E(\Delta)\mathbf{v} = \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\Delta}}\sum_{n(\ell)=n} \mathbf{b}_\ell\langle \mathbf{b}_\ell, \mathbf{v}\rangle \qquad \text{and} \qquad \mathsf{J}_0\mathbf{v} = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell \mathbf{i}_{s_\ell}\langle \mathbf{b}_\ell, \mathbf{v}\rangle.$$

The integral of a function $f \in \mathcal{SM}^\infty(\mathbb{H})$ with respect to (E, J_0) is then given by (9.22) as

$$\int_{\mathbb{H}} f(s)\, dE_{\mathsf{J}_0}(s)\mathbf{v} = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell f(s_\ell)\langle \mathbf{b}_\ell, \mathbf{v}\rangle. \tag{9.29}$$

Let $\mathbf{i} \in \mathbb{S}$. If we set $\tilde{E}(\Delta) = E([\Delta])$ for any $\Delta \in \mathsf{B}\big(\mathbb{C}_{\mathbf{i}}^{\geq}\big)$, then $\tilde{E}$ is a quaternionic spectral measure over $\mathbb{C}_{\mathbf{i}}$. In [4] the authors extend J_0 to an anti-selfadjoint and unitary operator J that commutes with T and interpret applying this operator as a multiplication with $\mathbf{i}$ from the left in order to define spectral integrals. One possibility to do this is to define $\imath(\ell) = \mathbf{i}_{s_\ell}$ if $s_\ell \notin \mathbb{R}$ and $\imath(\ell) \in \mathbb{S}$ arbitrary if $s_\ell \in \mathbb{R}$ and to set

$$\mathsf{J}\mathbf{v} = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell \imath(\ell)\langle \mathbf{b}_\ell, \mathbf{v}\rangle$$

and $\mathbf{i}\mathbf{v} = \mathsf{J}\mathbf{v}$

In [51] the authors go even one step further and extend this multiplication with scalars from the left to a full left multiplication $\mathcal{L} = (L_a)_{a\in\mathbb{H}}$ that commutes with E in order obtain an iqPVM $\mathcal{E} = (E, \mathcal{L})$. We can do this by choosing for each $\ell \in \Lambda$ an imaginary unit $\jmath(\ell) \in \mathbb{S}$ with $\jmath(\ell) \perp \imath(\ell)$ and by defining

$$\mathsf{K}\mathbf{v} = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell \jmath(\ell)\langle \mathbf{b}_\ell, \mathbf{v}\rangle.$$

If we choose $\mathbf{j} \in \mathbb{S}$ and define for $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{ij} \in \mathbb{H}$

$$\begin{aligned} a\mathbf{v} =& L_a\mathbf{v} := \mathbf{v}a_0 + \mathsf{J}\mathbf{v}a_1 + \mathsf{K}\mathbf{v}a_2 + \mathsf{JK}\mathbf{v}a_3 \\ =& \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell\big(a_0 + a_1\imath(\ell) + a_2\jmath(\ell) + a_3\imath(\ell)\jmath(\ell)\big)\langle \mathbf{b}_\ell, \mathbf{v}\rangle, \end{aligned}$$

then $\mathcal{L} = (L_a)_{a\in\mathbb{H}}$ is obviously a left multiplication that commutes with E and hence $\mathcal{E} = (\tilde{E}, \mathcal{L})$ is an iqPVM over $\mathbb{C}_{\mathbf{i}}^{\geq}$.

Set $s_{n,\mathsf{i}} = [s_n] \cap \mathbb{C}_{\mathsf{i}}$. For $f_{\mathsf{i}} : \mathbb{C}_{\mathsf{i}}^{\geq} \to \mathbb{H}$, the integral of f_{i} with respect to $\mathcal{E}$ is

$$\begin{aligned}
&\int_{\mathbb{C}_{\mathsf{i}}^{\geq}} f_{\mathsf{i}}(z)\, d\mathcal{E}(z) \\
&= \sum_{n\in\Upsilon} f_{\mathsf{i}}(s_{n,\mathsf{i}})\tilde{E}(\{s_{n,\mathsf{i}}\})\mathbf{v} = \sum_{n\in\Upsilon} f_{\mathsf{i}}(s_{n,\mathsf{i}})E([s_n])\mathbf{v} \\
&= \sum_{n\in\Upsilon} \big(f_0(s_{n,\mathsf{i}}) + f_1(s_{n,\mathsf{i}})\mathsf{J} + f_2(s_{n,\mathsf{i}})\mathsf{K} + f_3(s_{n,\mathsf{i}})\mathsf{JK}\big) \sum_{n(\ell)=n} \mathbf{b}_\ell\langle \mathbf{b}_\ell, \mathbf{v}\rangle \\
&= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \mathbf{b}_\ell\big(f_0(s_{n,\mathsf{i}}) + f_1(s_{n,\mathsf{i}})\imath(\ell) \\
&\qquad\qquad + f_2(s_{n,\mathsf{i}})\jmath(\ell) + f_3(s_{n,\mathsf{i}})\imath(\ell)\jmath(\ell)\big)\langle \mathbf{b}_\ell, \mathbf{v}\rangle,
\end{aligned} \tag{9.30}$$

where $f_0, \ldots, f_3$ are the real-valued component functions such that

$$f_{\mathsf{i}}(z) = f_0(z) + f_1(z)\mathbf{i} + f_2(z)\mathbf{j} + f_3(z)\mathbf{ij}.$$

If f_{i} is the restriction of an intrinsic slice function $f(s) = \alpha(s) + \mathbf{i}_s\beta(s)$, then $f_0\,(s_{n(\ell),\mathsf{i}}) = \alpha(s_{\ell,\mathsf{i}}) = \alpha(s_\ell)$ and $f_1(s_{n(\ell),\mathsf{i}}) = \beta(s_{\ell,\mathsf{i}}) = \beta(s_\ell)$ and $f_2(z) = f_3(z) = 0$. As moreover $f_1(s_{n(\ell),\mathsf{i}}) = \beta(s_\ell) = 0$ if $s_\ell \in \mathbb{R}$ and $\imath(\ell) = \mathbf{i}_{s_\ell}$ if $s_\ell \notin \mathbb{R}$, we find that actually (9.30) equals (9.29) in this case. Note however that for any other function f_{i}, the integral (9.30) depends on the random choice of the functions $\imath(\ell)$ and $\jmath(\ell)$, which are not fully determined by T.

Let us now investigate the relation of (9.30) with the right linear structure of T. Let us therefore change to the eigenbasis $\tilde{\mathbf{b}}_\ell, \ell \in \Lambda$, with $T\tilde{\mathbf{b}}_\ell = \tilde{\mathbf{b}}_\ell s_{\ell,\mathsf{i}}$ defined in Example 9.28. For convenience let us furthermore choose $\imath(\ell)$ and $\jmath(\ell)$ such that

$$\mathsf{J}\mathbf{v} = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell\mathbf{i}\langle\tilde{\mathbf{b}}_\ell, \mathbf{v}\rangle \qquad \text{and} \qquad \mathsf{K}\mathbf{v} = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell\mathbf{j}\langle\tilde{\mathbf{b}}_\ell, \mathbf{v}\rangle.$$

The left-multiplication $\mathcal{L}$ is hence exactly the left-multiplication induced by the basis $\tilde{\mathbf{b}}_\ell, \ell \in \Lambda$ and multiplication of $\mathbf{v}$ with $a \in \mathbb{H}$ from the left exactly corresponds to multiplying the coordinates $\langle\tilde{\mathbf{b}}_\ell, \mathbf{v}\rangle$ with a from the left, i.e. $a\mathbf{v} = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell a\langle\tilde{\mathbf{b}}_\ell, \mathbf{v}\rangle$. (Unlike multiplication with scalars from the right, the multiplication with scalars from the left does however only in this basis correspond to a multiplication of the coordinates. This relation is lost if we change the basis.)

Let us denote $\langle\tilde{\mathbf{b}}_\ell, \mathbf{v}\rangle = a_\ell$ with $a_\ell = a_{\ell,1} + a_{\ell,2}\mathbf{j}$ with $a_{\ell,1}, a_{\ell,2} \in \mathbb{C}_{\mathsf{i}}$ and let $f_{\mathsf{i}} : \mathbb{C}_{\mathsf{i}}^{\geq} \to \mathbb{H}$. If we write $f_{\mathsf{i}}(z) = f_1(z) + f_2(z)\mathbf{j}$, this time with $\mathbb{C}_{\mathsf{i}}$-valued components $f_1, f_2 : \mathbb{C}_{\mathsf{i}}^{\geq} \to \mathbb{C}_{\mathsf{i}}$, then (9.30) yields

$$\begin{aligned}
\int_{\mathbb{C}_{\mathsf{i}}^{\geq}} f_{\mathsf{i}}(z)\, d\mathcal{E}(z) &= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell\big(f_1(s_{n,\mathsf{i}}) + f_2(s_{n,\mathsf{i}})\mathbf{j}\big)(a_1 + a_2\mathbf{j}) \\
&= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell\big(a_1 f_1(s_{n,\mathsf{i}}) + \overline{a_1} f_2(s_{n,\mathsf{i}})\mathbf{j}\big) \\
&\quad + \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{\mathbf{b}}_\ell\big(a_2\mathbf{j}\overline{f_2(s_{n,\mathsf{i}})} - \overline{a_2} f_2(s_{n,\mathsf{i}})\big).
\end{aligned} \tag{9.31}$$

If we compare this with (9.24), then we find that $\int_{\mathbb{C}_{\mathsf{i}}^{\geq}} f_{\mathsf{i}}(z)\, d\mathcal{E}(z)$ does only correspond to an application of f_{i} to the right eigenvalues of T if $f_2 \equiv 0$ and f_1 can be extended to a function on all of $\mathbb{C}_{\mathsf{i}}$ such that $f_1(\overline{s_{\ell,\mathsf{i}}}) = \overline{f_1(s_{\ell,\mathsf{i}})}$. This is however the case if and only if $f_{\mathsf{i}} = f_1$ is the restriction of an intrinsic slice function to $\mathbb{C}_{\mathsf{i}}^{\geq}$.

As pointed out above, spectral integrals of intrinsic slice functions defined in the sense of [5] or [51] can be considered as spectral integrals with respect to a suitably chosen spectral system. The other two approaches—in particular the approach using iqPVMs in [51]—allow however the integration of a larger class of functions.

Ghiloni, Moretti, and Perotti argue in the introduction of [51] that the approach of spectral integration in [5] is complex in nature as it only allows to integrate $\mathbb{C}_{\mathsf{i}}$-valued functions defined on $\mathbb{C}_{\mathsf{i}}^{\geq}$ for some $\mathsf{i} \in \mathbb{S}$. They argue that their approach using iqPVMs on the other hand is quaternionic in nature as it allows to integrate functions that are defined on a complex halfplane and take arbitrary values in the quaternions. We believe that it is rather the other way around. It is the approach to spectral integration using spectral systems that is quaternionic in nature although they only allow to integrate intrinsic slice functions, and we have three main arguments in favour of this point of view:

(i) **Spectral integration with respect to a spectral system does not require the random introduction of any undetermined structure.**

If we consider a normal operator $T = A + \mathsf{J}_0 B$ on a quaternionic Hilbert space, then only its spectral system J_0 is uniquely defined. The extension of J_0 to a unitary anti-selfadjoint operator J that can be interpreted as a multiplication $L_{\mathsf{i}} = J$ with some $\mathsf{i} \in \mathbb{S}$ from the left is not determined by T. Also the multiplication L_{j} with some $\mathsf{j} \in \mathbb{S}$ with $\mathsf{j} \perp \mathsf{i}$ that extends L_{i} to the left multiplication $\mathcal{L}$ in a iqPVM $\mathcal{E} = (E, \mathcal{L})$ associated with T is not determined by T. Their construction in [49] and [51] is based on the spectral theorems for quaternionic selfadjoint operators and for complex linear normal operators.

As we shall see in Chapter 10, the spectral orientation J of a spectral operator T—that is the imaginary operator in the spectral system (E, J) associated with T—on a right Banach space can be constructed once the spectral measure E associated with T is known. Since the spectral theorems for selfadjoint operators and for complex linear operators are not available on Banach spaces, it is however not clear how to extend J to a fully imaginary operator or even further to something that generalizes an iqPVM and whether this is possible at all.

(ii) **Spectral integration with respect to a spectral system has a clear interpretation in terms of the right linear structure on the space.**

The natural domain of a right linear operator is a right Banach space. If a left multiplication is defined on the Banach space, then the operator's spectral properties should be independent of this left multiplication. Integration with respect to a spectral system (E, J) has a clear and intuitive interpretation with respect to the right linear structure of the space: the spectral measure E associates (right) linear subspaces to spectral spheres and the spectral orientation determines how to multiply the spectral values in the corresponding spectral spheres (from the right) onto the vectors in these subspaces.

The role of the left multiplication in an iqPVM in terms of the right linear structure is less clear. Indeed, we doubt that there exists a similarly clear and intuitive interpretation in view of the fact that no relation between left and right eigenvalues has been discovered so far.

(iii) **Extending the class of integrable functions towards non-intrinsic slice functions does not seem to bring any benefit and might not even be meaningful.**

Extending the class of admissible functions for spectral integration beyond the class of measurable intrinsic slice functions seems to add little value to the theory. As pointed out above, the proof of the spectral theorem in [5] translates directly into the language of spectral systems and hence spectral systems offer a framework that is sufficient in order to prove the most powerful result of spectral theory.

Even more, as we discussed in Section 8.3, spectral integrals of functions that are not intrinsic slice functions cannot follow the basic intuition of spectral integration. We also observed this in Example 9.31. In particular, if we define a measurable functional calculus via spectral integration, then this functional calculus does only follow the fundamental intuition of a functional calculus, namely that $f(T)$ should be defined by action of f on the spectral values of T, if the underlying class of functions consists of intrinsic slice functions.

CHAPTER 10

Bounded Quaternionic Spectral Operators

We turn our attention now to the study of quaternionic linear spectral operators, in which we generalise the complex linear theory in [38]. The results presented in this chapter can once more be found in [47].

10.1 The Spectral Decomposition of a Spectral Operator

A complex spectral operator is a bounded operator A on a complex Banach space that has a spectral resolution, i.e. there exists a spectral measure E defined on the Borel sets $\mathsf{B}(\mathbb{C})$ on $\mathbb{C}$ such that $\sigma_S(A|_\Delta) \subset cl(\Delta)$ with $A_\Delta = A|_{\operatorname{ran} E(\Delta)}$ for all $\Delta \in \mathsf{B}(\mathbb{C})$. Chapter 9 showed that spectral systems take over the role of spectral measures in the quaternionic setting. If E is a spectral measure that reduces an operator $T \in \mathcal{B}(V_R)$, then there will in general exist infinitely many imaginary operators J such that (E, J) is a spectral system. We thus have to find a criterion for identifying the one among them that fits the operator T and that can hence serve as its spectral orientation. A first and quite obvious requirement is that T and J commute. This is however not sufficient. Indeed, if J and T commute, then also $-J$ and T commute. More general, any operator that is of the form $\tilde{J} := -E(\Delta)J + E(\mathbb{H} \setminus \Delta)J$ with $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$ is an imaginary operator such that $(E, \tilde{J})$ is a spectral system that commutes with T.

We develop a second criterion by analogy with the finite-dimensional case. Let $T \in \mathcal{B}(\mathbb{H}^n)$ be the operator on $\mathbb{H}^n$ that is given by the diagonal matrix $T= \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and let us assume $\lambda_\ell \notin \mathbb{R}$ for $\ell = 1, \ldots, n$. We intuitively identify the operator $J = \operatorname{diag}(\mathsf{i}_{\lambda_1}, \ldots, \mathsf{i}_{\lambda_n})$ as the spectral orientation for T, cf. also Example 9.28. Obviously J commutes with T. Moreover, if $s_0 \in \mathbb{R}$ and $s_1 > 0$ are arbitrary, then the operator $(s_0\mathcal{I} - s_1 J) - T$ is invertible. Indeed, one has

$$(s_0\mathcal{I} - s_1 J) - T = \operatorname{diag}(\overline{s_{\mathsf{i}_{\lambda_1}}} - \lambda_1, \ldots, \overline{s_{\mathsf{i}_{\lambda_n}}} - \lambda_n),$$

where $s_{\mathsf{i}_{\lambda_\ell}} = s_0 + \mathsf{i}_{\lambda_\ell} s_1$. Since $\overline{s_{\mathsf{i}_{\lambda_\ell}}} - \lambda_\ell = (s_0 - \lambda_{\ell,0}) + \mathsf{i}_{\lambda_\ell}(-s_1 - \lambda_{\ell,1})$ and both $s_1 > 0$ and $\lambda_{\ell,1} > 0$ for all $\ell = 1, \dots, n$, each of the diagonal elements has an inverse and so

$$((s_0\mathcal{I} - s_1 J) - T)^{-1} = \operatorname{diag}\left((\overline{s_{\mathsf{i}_{\lambda_1}}} - \lambda_1)^{-1}, \dots, (\overline{s_{\mathsf{i}_{\lambda_n}}} - \lambda_n)^{-1}\right).$$

This invertibility is the criterion that uniquely identifies J.

Definition 10.1. An operator $T \in \mathcal{B}(V_R)$ is called a spectral operator if there exists a spectral decomposition of T, i.e. a spectral system (E, J) on V_R such that the following three conditions hold:

(i) The spectral system (E, J) commutes with T , i.e. $E(\Delta)T = TE(\Delta)$ for all $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$ and $TJ = JT$,

(ii) If we set $T_\Delta := T|_{V_\Delta}$ with $V_\Delta = E(\Delta)V_R$ for $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$, then
$$\sigma_S(T_\Delta) \subset cl(\Delta) \quad \text{for all } \Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H}).$$

(iii) For any $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, the operator $((s_0\mathcal{I} - s_1 J) - T)|_{V_1}$ has a bounded inverse on $V_1 := E(\mathbb{H} \setminus \mathbb{R})V_R = \operatorname{ran} J$.

The spectral measure E is called a spectral resolution for T and the imaginary operator J is called a spectral orientation of T.

A first easy result, which we shall use frequently, is that the restriction of a spectral operator to an invariant subspace $E(\Delta)V_R$ is again a spectral operator.

Lemma 10.2. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator on V_R and let (E, J) be a spectral decomposition of T. For any $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$, the operator $T_\Delta = T|_{V_\Delta}$ with $V_\Delta = \operatorname{ran} E(\Delta)$ is a spectral operator on V_Δ. A spectral decomposition of T_Δ is (E_Δ, J_Δ) with $E_\Delta(\sigma) = E(\sigma)|_{V_\Delta}$ and $J_\Delta = J|_{V_\Delta}$.*

Proof. Since $E(\Delta)$ commutes with $E(\sigma)$ for any $\sigma \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$ and with J, the restrictions $E_\Delta(\sigma) = E(\sigma)|_{V_\Delta}$ and $J_\Delta = J|_{V_\Delta}$ are right linear operators on V_Δ. It is immediate that E_Δ is a spectral measure on V_Δ. Moreover

$$\ker J_\Delta = \ker J \cap V_\Delta = \operatorname{ran} E(\mathbb{R}) \cap V_\Delta = \operatorname{ran} E_\Delta(\mathbb{R})$$

and
$$\operatorname{ran} J_\Delta = \operatorname{ran} J \cap V_\Delta = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R}) \cap V_\Delta = \operatorname{ran} E_\Delta(\mathbb{H} \setminus \mathbb{R}).$$

Since
$$-J_\Delta^2 = -J^2|_{V_\Delta} = E(\mathbb{H} \setminus \mathbb{R})|_{V_\Delta} = E_\Delta(\mathbb{H} \setminus \mathbb{R}),$$
the operator $-J_\Delta^2$ is the projection of V_Δ onto $\operatorname{ran} J_\Delta$ along $\ker J_\Delta$. Hence J_Δ is an imaginary operator on V_Δ and (E_Δ, J_Δ) is a spectral system. As

$$E_\Delta(\sigma)T_\Delta E(\Delta) = E(\sigma)TE(\Delta) = TE(\sigma)E(\Delta) = T_\Delta E_\Delta(\sigma)E(\Delta)$$

and similar
$$J_\Delta T_\Delta E(\Delta) = JTE(\Delta) = TJE(\Delta) = T_\Delta J_\Delta E(\Delta),$$

this spectral system commutes with T_Δ.

If $\sigma \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$ and we set $V_{\Delta,\sigma} = \operatorname{ran} E_{\Delta}(\sigma)$, then

$$V_{\Delta,\sigma} = \operatorname{ran} E(\sigma)|_{V_\Delta} = \operatorname{ran} E(\sigma)E(\Delta) = \operatorname{ran} E(\sigma \cap \Delta) = V_{\Delta\cap\sigma}.$$

Thus $T_\Delta|_{V_{\Delta,\sigma}} = T|_{V_{\sigma\cap\Delta}}$ and so $\sigma_S(T_{\Delta,\sigma}) = \sigma_S(T_{\Delta\cap\sigma}) \subset \Delta \cup \sigma \subset \sigma$. Hence E_Δ is a spectral resolution for T_Δ. Finally, for $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, the operator $s_0\mathcal{I} - s_1 J - T$ leaves the subspace $V_{\Delta,1} := \operatorname{ran} E_\Delta(\mathbb{H}\setminus\mathbb{R}) = \operatorname{ran} E(\Delta \cap (\mathbb{H}\setminus\mathbb{R}))$ invariant because it commutes with E. Hence, the restriction of $(s_0\mathcal{I} - s_1 J - T)|_{V_1}^{-1}$ to $V_{\Delta,1} \subset V_1 = \operatorname{ran} E(\mathbb{H}\setminus\mathbb{R})$ is a bounded linear operator on $V_{\Delta,1}$. It obviously is the inverse of $(s_0\mathcal{I} - s_1 J_\Delta - T_\Delta)|_{V_{\Delta,1}}$. Therefore (E_Δ, J_Δ) is actually a spectral decomposition of T_Δ, which is in turn a spectral operator.

□

The remainder of this section considers the questions of uniqueness and existence of the spectral decomposition (E, J) of T. We recall the V_R-valued right slice-hyperholomorphic function $\mathcal{R}_s(T;\mathbf{v}) := \mathcal{Q}_s(T)^{-1}\mathbf{v}\overline{s} - T\mathcal{Q}_s(T)^{-1}\mathbf{v}$ on $\rho_S(T)$ for $T \in \mathcal{K}(V_R)$ and $\mathbf{v} \in V_R$, which was defined in Definition 8.6 in order to give a representation of the S-functional calculus for intrinsic functions in terms of the right multiplication on V_R. If T is bounded, then $\mathcal{Q}_s(T)^{-1}$ and T commute and we have

$$\mathcal{R}_s(T;\mathbf{v}) := \mathcal{Q}_s(T)^{-1}(\mathbf{v}\overline{s} - T\mathbf{v}).$$

Definition 10.3. Let $T \in \mathcal{B}(V_R)$ and let $\mathbf{v} \in V_R$. A V_R-valued right slice-hyperholomorphic function $\mathbf{f}$ defined on an axially symmetric open set $\operatorname{dom}(\mathbf{f}) \subset \mathbb{H}$ with $\rho_S(T) \subset \operatorname{dom}(\mathbf{f})$ is called a slice-hyperholomorphic extension of $\mathcal{R}_s(T;\mathbf{v})$ if

$$(T^2 - 2s_0T + |s|^2\mathcal{I})\mathbf{f}(s) = \mathbf{v}\overline{s} - T\mathbf{v} \qquad \forall s \in \operatorname{dom}(\mathbf{f}). \tag{10.1}$$

Obviously such an extension satisfies

$$\mathbf{f}(s) = \mathcal{R}_s(T;\mathbf{v}) \qquad \text{for } s \in \rho_S(T).$$

Definition 10.4. Let $T \in \mathcal{B}(V_R)$ and let $\mathbf{v} \in V_R$. The function $\mathcal{R}_s(T;\mathbf{v})$ is said to have the single valued extension property if any two slice hyperholomorphic extensions $\mathbf{f}$ and $\mathbf{g}$ of $\mathcal{R}_s(T;\mathbf{v})$ satisfy $\mathbf{f}(s) = \mathbf{g}(s)$ for $s \in \operatorname{dom}(\mathbf{f}) \cap \operatorname{dom}(\mathbf{g})$. In this case

$$\rho_S(\mathbf{v}) := \bigcup\{\operatorname{dom}(\mathbf{f}) : \mathbf{f} \text{ is a slice hyperholomorphic extension of } \mathcal{R}_s(T;\mathbf{v})\}$$

is called the S-resolvent set of $\mathbf{v}$ and $\sigma_S(\mathbf{v}) = \mathbb{H}\setminus\rho_S(\mathbf{v})$ is called the S-spectrum of $\mathbf{v}$.

Since it is the union of axially symmetric sets, $\rho_S(\mathbf{v})$ is axially symmetric. Moreover, there exists a unique maximal extension of $\mathcal{R}_s(T;\mathbf{v})$ to $\rho_S(\mathbf{v})$. We shall denote this extension by $\mathbf{v}(s)$.

We shall see soon that the single valued extension property holds for $\mathcal{R}_s(T;\mathbf{v})$ for any $\mathbf{v} \in V_R$ if T is a spectral operator. This is however not true for an arbitrary operator $T \in \mathcal{B}(V_R)$. A counterexample can be constructed analogue to [38, p. 1932].

Lemma 10.5. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let E be a spectral resolution for T. Let $s \in \mathbb{H}$ and let $\Delta \subset \mathbb{H}$ be a closed axially symmetric set such that $s \notin \Delta$. If $\mathbf{v} \in V_R$ satisfies $(T^2 - 2s_0T + |s|^2\mathcal{I})\mathbf{v} = \mathbf{0}$ then*

$$E(\Delta)\mathbf{v} = \mathbf{0} \qquad \textit{and} \qquad E([s])\mathbf{v} = \mathbf{v}.$$

Proof. Assume that $\mathbf{v} \in V_R$ satisfies $\mathcal{Q}_s(T)\mathbf{v} = (T^2 - 2s_0T + |s|^2\mathcal{I})\mathbf{v} = \mathbf{0}$ and let T_Δ be the restriction of T to the subspace $V_\Delta = E(\Delta)V$. As $s \notin \Delta$, we have $s \in \rho_S(T_\Delta)$ and so $\mathcal{Q}_s(T_\Delta)$ is invertible. Since $\mathcal{Q}_s(T_\Delta)^{-1} = \mathcal{Q}_s(T)^{-1}|_{V_\Delta}$, we have

$$\mathcal{Q}_s(T_\Delta)^{-1}(T^2 - 2s_0T + |s|^2\mathcal{I})E(\Delta) = E(\Delta),$$

from which we deduce

$$\begin{aligned} E(\Delta)\mathbf{v} &= \mathcal{Q}_s(T_\Delta)^{-1}(T^2 - 2s_0T + |s|^2\mathcal{I})E(\Delta)\mathbf{v} \\ &= \mathcal{Q}_s(T_\Delta)^{-1}E(\Delta)(T^2 - 2s_0T + |s|^2\mathcal{I})\mathbf{v} = \mathbf{0}. \end{aligned}$$

Now define for $n \in \mathbb{N}$ the closed axially symmetric set

$$\Delta_n = \left\{ p \in \mathbb{H} : \operatorname{dist}(p, [s]) \geq \frac{1}{n} \right\}.$$

By the above, we have $E(\Delta_n)\mathbf{v} = \mathbf{0}$ and in turn

$$(\mathcal{I} - E([s]))\mathbf{v} = \lim_{n \to +\infty} E(\Delta_n)\mathbf{v} = \mathbf{0}$$

so that $\mathbf{v} = E([s])\mathbf{v}$.

□

Lemma 10.6. *If $T \in \mathcal{B}(V_R)$ is a spectral operator, then for any $\mathbf{v} \in V_R$ the function $\mathcal{R}_s(T; \mathbf{v})$ has the single valued extension property.*

Proof. Let $\mathbf{v} \in V_R$ and let $\mathbf{f}$ and $\mathbf{g}$ be two slice hyperholomorphic extensions of $\mathcal{R}_s(T; \mathbf{v})$. We set $\mathbf{h}(s) = \mathbf{f}(s) - \mathbf{g}(s)$ for $s \in \operatorname{dom}(\mathbf{h}) = \operatorname{dom}(\mathbf{f}) \cap \operatorname{dom}(\mathbf{g})$.

If $s \in \operatorname{dom}(\mathbf{h})$ then there exists an axially symmetric neighborhood $U \subset \operatorname{dom}(\mathbf{h})$ of s and for any $p \in U$ we have

$$\begin{aligned} &(T^2 - 2p_0T + |p|^2\mathcal{I})\mathbf{h}(p) \\ =&(T^2 - 2p_0T + |p|^2\mathcal{I})\mathbf{f}(p) - (T^2 - 2p_0T + |p|^2)\mathbf{g}(p) \\ =&(\mathbf{v}\overline{p} - T\mathbf{v}) - (\mathbf{v}\overline{p} - T\mathbf{v}) = \mathbf{0}. \end{aligned}$$

If E is a spectral resolution of T, then we can conclude from the above and Lemma 10.5 that $E([p])\mathbf{h}(p) = \mathbf{h}(p)$ for $p \in U$. We consider now a sequence $s_n \in U$ with $s_n \neq s$ for $n \in U$ such that $s_n \to s$ as $n \to \infty$ and find

$$\mathbf{0} = E([s])E([s_n])\mathbf{h}(s_n) = E([s])\mathbf{h}(s_n) \to E([s])\mathbf{h}(s) = \mathbf{h}(s).$$

Hence, $\mathbf{f}(s) = \mathbf{g}(s)$ and $\mathcal{R}_s(T, \mathbf{v})$ has the single valued extension property.

□

Corollary 10.7. *If $T \in \mathcal{B}(V_R)$ is a spectral operator, then for any $\mathbf{v} \in V_R$ the function $\mathcal{R}_s(T; \mathbf{v})$ has a unique maximal slice hyperholomorphic extension to $\rho_S(\mathbf{v})$. We denote this maximal slice hyperholomorphic extension of $\mathcal{R}_s(T; \mathbf{v})$ by $\mathbf{v}(\cdot)$.*

Corollary 10.8. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let $\mathbf{v} \in V_R$. Then $\sigma_S(\mathbf{v}) = \emptyset$ if and only if $\mathbf{v} = \mathbf{0}$.*

Proof. If $\mathbf{v} = \mathbf{0}$ then $\mathbf{v}(s) = \mathbf{0}$ is the maximal slice hyperholomorphic extension of $\mathcal{R}_s(T;\mathbf{v})$. It is defined on all of $\mathbb{H}$ and hence $\sigma_S(\mathbf{v}) = \emptyset$.

Now assume that $\sigma_S(\mathbf{v}) = \emptyset$ for some $\mathbf{v} \in V_R$ such that the maximal slice hyperholomorphic extension $\mathbf{v}(\cdot)$ of $\mathcal{R}_s(T;\mathbf{v})$ is defined on all of $\mathbb{H}$. For any $\mathbf{w}^* \in V_R^*$, the function $s \to \langle \mathbf{w}^*, \mathbf{v}(s)\rangle$ is an entire right slice-hyperholomorphic function. From the fact that $\mathcal{R}_s(T;\mathbf{v})$ equals the resolvent of T as a bounded operator on $V_{R,\mathbf{i}_s}$, we deduce $\lim_{s\to\infty} \mathcal{R}_s(T;\mathbf{v}) = \mathbf{0}$ and then

$$\lim_{s\to\infty} \langle \mathbf{w}^*, \mathbf{v}(s)\rangle = \lim_{s\to\infty} \langle \mathbf{w}^*, \mathcal{R}_s(T;\mathbf{v})\rangle = 0.$$

Liouville's Theorem for slice hyperholomorphic functions in [35] therefore implies $\langle \mathbf{w}^*, \mathbf{v}(s)\rangle = 0$ for all $s \in \mathbb{H}$. Since $\mathbf{w}^*$ was arbitrary, we obtain $\mathbf{v}(s) = \mathbf{0}$ for all $s \in \mathbb{H}$.

Finally, we can choose $s \in \rho_S(T)$ so that the operator $\mathcal{Q}_s(T) = T^2 - 2s_0T + |s|^2\mathcal{I}$ is invertible and we find because of (10.1) that

$$\begin{aligned}
\mathbf{0} =& \mathbf{v}(s)s - T\mathbf{v}(s) = \mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T)\mathbf{v}(s)s - T\mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T)\mathbf{v}(s)\\
=& \mathcal{Q}_s(T)^{-1}\left(\mathcal{Q}_s(T)\mathbf{v}(s)s - T\mathcal{Q}_s(T)\mathbf{v}(s)\right) =\\
=& \mathcal{Q}_s(T)^{-1}((\mathbf{v}\overline{s} - T\mathbf{v})s - T(\mathbf{v}\overline{s} - T\mathbf{v}))\\
=& \mathcal{Q}_s(T)^{-1}\left(T^2\mathbf{v} - T\mathbf{v}2s_0 + \mathbf{v}|s|^2\right) = \mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T)\mathbf{v} = \mathbf{v}.
\end{aligned}$$

□

Theorem 10.9. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let E be a spectral resolution for T. If $\Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H})$ is closed, then*

$$E(\Delta)V_R = \{\mathbf{v} \in V_R : \sigma_S(\mathbf{v}) \subset \Delta\}.$$

Proof. Let $V_\Delta = E(\Delta)V_R$ and let T_Δ be the restriction of T to V_Δ. Since Δ is closed, Definition 10.1 implies $\sigma_S(T_\Delta) \subset \Delta$. Moreover $\mathcal{Q}_s(T_\Delta) = \mathcal{Q}_s(T)|_{V_\Delta}$ for $s \in \mathbb{H}$. If $\mathbf{v} \in V_\Delta$, then

$$\mathcal{Q}_s(T)\mathcal{R}_s(T;\mathbf{v}) = \mathcal{Q}_s(T_\Delta)\mathcal{Q}_s(T_\Delta)^{-1}(\mathbf{v}\overline{s} - T_\Delta\mathbf{v}) = \mathbf{v}\overline{s} - T\mathbf{v}$$

for $s \in \rho_S(T_\Delta)$. Hence $\mathcal{R}_s(T_\Delta;\mathbf{v})$ is a slice hyperholomorphic extension of $\mathcal{R}_s(T;\mathbf{v})$ to $\rho_S(T_\Delta) \supset \mathbb{H}\setminus\Delta$. Thus $\sigma_S(\mathbf{v}) \subset \Delta$. Since $\mathbf{v} \in V_R$ was arbitrary, we find $E(\Delta)V_R \subset \{\mathbf{v} \in V_R : \sigma_S(\mathbf{v}) \subset \Delta\}$.

In order to show the converse relation, we assume that $\sigma_S(\mathbf{v}) \subset \Delta$. We consider a closed subset $\sigma \in \mathsf{B}_\mathsf{S}(\mathbb{H})$ of the complement of Δ and set $T_\sigma = T|_{V_\sigma}$ with $V_\sigma = E(\sigma)V_R$. As above $\mathcal{R}_s(T_\sigma; E(\sigma)\mathbf{v})$ is then a slice hyperholomorphic extension of $\mathcal{R}_s(T; E(\sigma)\mathbf{v})$ to $\mathbb{H} \setminus \sigma$. If on the other hand $\mathbf{v}(s)$ is the unique maximal slice hyperholomorphic extension of $\mathcal{R}_s(T;\mathbf{v})$, then

$$\begin{aligned}
\mathcal{Q}_s(T)E(\sigma)\mathbf{v}(s) &= E(\sigma)\mathcal{Q}_s(T)\mathbf{v}(s)\\
&= E(\sigma)(\mathbf{v}\overline{s} - T\mathbf{v}) = (E(\sigma)\mathbf{v})\overline{s} - T(E(\sigma)\mathbf{v})
\end{aligned}$$

for $s \in \mathbb{H}\setminus\Delta$ and so $E(\sigma)\mathbf{v}(s)$ is a slice hyperholomorphic extension of $\mathcal{R}_s(T; E(\sigma)\mathbf{v})$ to $\mathbb{H} \setminus \Delta$. Combining these two extensions, we find that $\mathcal{R}_s(T; E(\sigma)\mathbf{v})$ has a slice

hyperholomorphic extension to all of $\mathbb{H}$. Hence, $\sigma_S(E(\sigma)\mathbf{v}) = \emptyset$ so that $E(\Delta)\mathbf{v} = \mathbf{0}$ by Corollary 10.8.

Let us now choose an increasing sequence of closed subsets $\sigma_n \in \mathsf{B}_\mathsf{S}(\mathbb{H})$ of $\mathbb{H} \setminus \Delta$ such that $\bigcup_{n\in\mathbb{N}} \sigma_n = \mathbb{H} \setminus \Delta$. By the above arguments $E(\sigma_n)\mathbf{v} = \mathbf{0}$ for any $n \in \mathbb{N}$. Hence

$$E(\mathbb{H} \setminus \Delta)\mathbf{v} = \lim_{n\to\infty} E(\Delta_n)\mathbf{v} = \mathbf{0},$$

so that in turn $E(\Delta)\mathbf{v} = \mathbf{v}$. We thus obtain $E(\Delta)V_R \supset \{\mathbf{v} \in V_R : \sigma_S(\mathbf{v}) \subset \Delta\}$.

□

The following corollaries are immediate consequences of Theorem 10.9.

Corollary 10.10. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let E be a spectral resolution of T. Then $E(\sigma_S(T)) = \mathcal{I}$.*

Corollary 10.11. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let $\Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H})$ be closed. The set of all $\mathbf{v} \in V_R$ with $\sigma_S(\mathbf{v}) \subset \Delta$ is a closed right subspace of V_R.*

Lemma 10.12. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator. If $A \in \mathcal{B}(V_R)$ commutes with T, then A commutes with every spectral resolution E for T. Moreover, $\sigma_S(A\mathbf{v}) \subset \sigma_S(\mathbf{v})$ for all $\mathbf{v} \in V_R$.*

Proof. For $\mathbf{v} \in V_R$ we have

$$\begin{aligned} &(T^2 - 2s_0T + |s|^2\mathcal{I})A\mathbf{v}(s) \\ =&A(T^2 - 2s_0T + |s|^2\mathcal{I})\mathbf{v}(s) \\ =&A(\mathbf{v}\overline{s} - T\mathbf{v}) = (A\mathbf{v})\overline{s} - T(A\mathbf{v}). \end{aligned}$$

The function $A\mathbf{v}(s)$ is therefore a slice hyperholomorphic extension of $\mathcal{R}_s(T; A\mathbf{v})$ to $\rho_S(\mathbf{v})$ and so $\sigma_S(A\mathbf{v}) \subset \sigma_S(\mathbf{v})$. From Theorem 10.9 we deduce that

$$AE(\Delta)V \subset E(\Delta)V$$

for any closed axially symmetric subset Δ of $\mathbb{H}$.

If σ and Δ are two disjoint closed axially symmetric sets we therefore have

$$E(\Delta)AE(\Delta) = AE(\Delta) \qquad \text{and} \qquad E(\Delta)AE(\sigma) = E(\Delta)E(\sigma)AE(\sigma) = 0.$$

If we choose again an increasing sequence of closed sets $\Delta_n \in \mathsf{B}_\mathsf{S}(\mathbb{H})$ with $\mathbb{H} \setminus \Delta = \bigcup_{n\in\mathbb{N}} \Delta_n$, we therefore have

$$E(\Delta)AE(\mathbb{H} \setminus \Delta)\mathbf{v} = \lim_{n\to\infty} E(\Delta)AE(\Delta_n)\mathbf{v} = \mathbf{0} \qquad \forall \mathbf{v} \in V_R$$

and hence

$$E(\Delta)A = E(\Delta)A[E(\Delta) + E(\mathbb{H} \setminus \Delta)] = E(\Delta)AE(\Delta) = AE(\Delta). \tag{10.2}$$

Since Δ was an arbitrary closed set in $\mathsf{B}_\mathsf{S}(\mathbb{H})$ and since the sigma-algebra $\mathsf{B}_\mathsf{S}(\mathbb{H})$ is generated by sets of this type, we finally conclude that (10.2) holds true for any set $\sigma \in \mathsf{B}_\mathsf{S}(\mathbb{H})$.

□

Lemma 10.13. *The spectral resolution E of a spectral operator $T \in \mathcal{B}(V_R)$ is uniquely determined.*

Proof. Let E and $\tilde{E}$ be two spectral resolutions of T. For any closed set $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$, Theorem 10.9 implies

$$\tilde{E}(\Delta)E(\Delta) = E(\Delta) \qquad \text{and} \qquad E(\Delta)\tilde{E}(\Delta) = \tilde{E}(\Delta)$$

and we deduce from Lemma 10.12 that $E(\Delta) = \tilde{E}(\Delta)$. Since the sigma algebra $\mathsf{B}_{\mathsf{S}}(\mathbb{H})$ is generated by the closed sets in $\mathsf{B}_{\mathsf{S}}(\mathbb{H})$, we obtain $E = \tilde{E}$ and hence the spectral resolution of T is uniquely determined.

□

Before we consider the uniqueness of the spectral orientation, we observe that for certain operators, the existence of a spectral resolution already implies the existence of a spectral orientation and is hence sufficient for them to be a spectral operator.

Proposition 10.14. *Let $T \in \mathcal{B}(V_R)$ and assume that there exists a spectral resolution E for T. If $\sigma_S(T) \cap \mathbb{R} = \emptyset$, then there exists an imaginary operator $J \in \mathcal{B}(V_R)$ that is a spectral orientation for T such that T is a spectral operator with spectral resolution (E, J). Moreover, this spectral orientation is unique.*

Proof. Since $\sigma_S(T)$ is closed with $\sigma_S(T) \cap \mathbb{R} = \emptyset$, we have $\operatorname{dist}(\sigma_S(T), \mathbb{R}) > 0$. We choose $\mathsf{i} \in \mathbb{S}$ and consider T as a complex linear operator on $V_{R,\mathsf{i}}$. Because of Theorem 8.4, the spectrum of T as a $\mathbb{C}_{\mathsf{i}}$-linear operator on $V_{R,\mathsf{i}}$ is $\sigma_{\mathbb{C}_{\mathsf{i}}}(T) = \sigma_S(T) \cap \mathbb{C}_{\mathsf{i}}$. As $\operatorname{dist}(\sigma_S(T), \mathbb{R}) > 0$, the sets

$$\sigma_+ = \sigma_{\mathbb{C}_{\mathsf{i}}}(T) \cap \mathbb{C}_{\mathsf{i}}^+ \qquad \text{and} \qquad \sigma_- = \sigma_{\mathbb{C}_{\mathsf{i}}}(T) \cap \mathbb{C}_{\mathsf{i}}^-$$

are open and closed subsets of $\sigma_{\mathbb{C}_{\mathsf{i}}}(T)$ such that $\sigma_+ \cup \sigma_- = \sigma_{\mathbb{C}_{\mathsf{i}}}(T)$. Via the Riesz-Dunford functional calculus we can hence associate spectral projections E_+ and E_- onto closed invariant $\mathbb{C}_{\mathsf{i}}$-linear subspaces of $V_{R,\mathsf{i}}$ to σ_+ and σ_-. The resolvent of T as a $\mathbb{C}_{\mathsf{i}}$-linear operator on $V_{R,\mathsf{i}}$ at $z \in \rho_{\mathbb{C}_{\mathsf{i}}}(T)$ is $R_z(T)\mathbf{v} := \mathcal{Q}_z(T)^{-1}(\mathbf{v}\overline{z} - T\mathbf{v})$, and hence these projections are given by

$$\begin{aligned} E_+\mathbf{v} &:= \int_{\Gamma_+} \mathcal{Q}_z(T)^{-1}(\mathbf{v}\overline{z} - T\mathbf{v})\, dz \frac{1}{2\pi\mathsf{i}} \\ E_-\mathbf{v} &:= \int_{\Gamma_-} \mathcal{Q}_z(T)^{-1}(\mathbf{v}\overline{z} - T\mathbf{v})\, dz \frac{1}{2\pi\mathsf{i}}, \end{aligned} \tag{10.3}$$

where Γ_+ is a positively oriented Jordan curve that surrounds σ_+ in $\mathbb{C}_{\mathsf{i}}^+$ and Γ_- is a positively oriented Jordan curve that surrounds σ_- in $\mathbb{C}_{\mathsf{i}}^-$. We set

$$J\mathbf{v} := E_-\mathbf{v}(-\mathsf{i}) + E_+\mathbf{v}\mathsf{i}.$$

From Theorem 9.18 we deduce that J is an imaginary operator on V_R if $\Psi : \mathbf{v} \mapsto \mathbf{v}\mathsf{j}$ is a bijection between $V_+ := E_+V_R$ and $V_- := E_-V_R$ for $\mathsf{j} \in \mathbb{S}$ with $\mathsf{j} \perp \mathsf{i}$. This is indeed the case: due to the symmetry of $\sigma_{\mathbb{C}_{\mathsf{i}}}(T) = \sigma_S(T) \cap \mathbb{C}_{\mathsf{i}}$ with respect to the real axis, we find $\sigma_+ = \overline{\sigma_-}$ so that we can choose $\Gamma_-(t) = \overline{\Gamma_+(1-t)}$ for $t \in [0, 1]$ in (10.3).

Because of the relation (8.4) established in Theorem 8.4, the resolvent $R_z(T)$ of T as an operator on $V_{R,\mathbf{i}}$ satisfies $R_{\overline{z}}(T)\mathbf{v} = -\left[R_z(T)(\mathbf{v}\mathbf{j})\right]\mathbf{j}$ and so

$$E_-\mathbf{v} = \int_{\Gamma_-} R_z(T)\mathbf{v}\,dz\frac{1}{2\pi\mathbf{i}} = -\int_{\Gamma_+} R_{\overline{z}}(T)\mathbf{v}\,d\overline{z}\frac{1}{2\pi\mathbf{i}}$$
$$= \int_{\Gamma_+}\left[R_z(T)(\mathbf{v}\mathbf{j})\right]\mathbf{j}\,d\overline{z}\frac{1}{2\pi\mathbf{i}} = \int_{\Gamma_+}\left[R_z(T)(\mathbf{v}\mathbf{j})\right]\,dz\frac{1}{2\pi\mathbf{i}}(-\mathbf{j}) = \left[E_+(\mathbf{v}\mathbf{j})\right](-\mathbf{j}).$$

Hence, we have

$$(E_-\mathbf{v})\,\mathbf{j} = E_+(\mathbf{v}\mathbf{j}) \qquad \forall \mathbf{v} \in V_R. \tag{10.4}$$

If $\mathbf{v} \in V_-$, then $\mathbf{v}\mathbf{j} = (E_-\mathbf{v})\mathbf{j} = E_+(\mathbf{v}\mathbf{j})$ and so $\mathbf{v}\mathbf{j} \in V_+$. Replacing $\mathbf{v}$ by $\mathbf{v}\mathbf{j}$ in (10.4), we find that also $(E_-\mathbf{v}\mathbf{j})\,\mathbf{j} = -E_+(\mathbf{v})$ and in turn $E_-(\mathbf{v}\mathbf{j}) = E_+(\mathbf{v})\mathbf{j}$. For $\mathbf{v} \in V_+$ we thus find $\mathbf{v}\mathbf{j} = E_+(\mathbf{v})\mathbf{j} = E_-(\mathbf{v}\mathbf{j})$ and so $\mathbf{v}\mathbf{j} \in V_-$. Hence, Ψ maps V_+ to V_- and V_- to V_+ and as $\Psi^{-1} = -\Psi$ it is even bijective. We conclude that J is actually an imaginary operator.

Let us now show that (i) in Definition 10.1 holds true. For any $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$ the operator $\mathcal{Q}_z(T)^{-1}$ commutes with $E(\Delta)$. Hence

$$\begin{aligned} E(\Delta)E_+\mathbf{v} &= \int_{\Gamma_+} E(\Delta)\mathcal{Q}_z(T)^{-1}(\mathbf{v}\overline{z} - T\mathbf{v})\,dz\frac{1}{2\pi\mathbf{i}} \\ &= \int_{\Gamma_+} \mathcal{Q}_z(T)^{-1}(E(\Delta)\mathbf{v}\overline{z} - TE(\Delta)\mathbf{v})\,dz\frac{1}{2\pi\mathbf{i}} = E_+E(\Delta)\mathbf{v} \end{aligned} \tag{10.5}$$

for any $\mathbf{v} \in V_{R,\mathbf{i}} = V_R$ and so $E_+E(\Delta) = E(\Delta)E_+$. Similarly, one can show that also $E(\Delta)E_- = E_-E(\Delta)$. By construction, the operator J hence commutes with T and with $E(\Delta)$ for any $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$ as

$$TJ\mathbf{v} = TE_-\mathbf{v}(-\mathbf{i}) + TE_+\mathbf{v}\mathbf{i} = E_-T\mathbf{v}(-\mathbf{i}) + E_+T\mathbf{v}\mathbf{i} = JT\mathbf{v}$$

and

$$\begin{aligned} E(\Delta)J\mathbf{v} &= E(\Delta)E_-\mathbf{v}(-\mathbf{i}) + E(\Delta)E_+\mathbf{v}\mathbf{i} \\ &= E_-E(\Delta)\mathbf{v}(-\mathbf{i}) + E_+E(\Delta)\mathbf{v}\mathbf{i} = JE(\Delta)\mathbf{v}. \end{aligned}$$

Moreover, as $\sigma_S(T) \cap \mathbb{R} = \emptyset$, Corollary 10.10 implies $\operatorname{ran} E(\mathbb{R}) = \{\mathbf{0}\} = \ker J$ and $\operatorname{ran} E(\mathbb{H}\setminus\mathbb{R}) = V_R = \operatorname{ran} J$. Hence, (E, J) is actually a spectral system that moreover commutes with T.

Let us now show condition (iii) of Definition 10.1. If $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, then set $s_{\mathbf{i}} := s_0 + \mathbf{i}s_1$. As $E_+ + E_- = \mathcal{I}$, we then have

$$\begin{aligned} &((s_0\mathcal{I} - s_1 J) - T)\mathbf{v} = \\ =&(E_+ + E_-)\mathbf{v}s_0 - (E_+\mathbf{v})\mathbf{i}s_1 - (E_-\mathbf{v})(-\mathbf{i})s_1 - T(E_+ + E_-)\mathbf{v} \\ =&(E_+\mathbf{v})(s_0 - s_1\mathbf{i}) - T(E_+\mathbf{v}) + (E_-\mathbf{v}))(s_0 + s_1\mathbf{i}) - T(E_-\mathbf{v}) \\ =&(E_+\mathbf{v})\overline{s_{\mathbf{i}}} - T(E_+\mathbf{v}) + (E_-\mathbf{v})s_{\mathbf{i}} - T(E_-\mathbf{v}) \\ =&(\overline{s_{\mathbf{i}}}\mathcal{I}_{V_{R,\mathbf{i}}} - T)E_+\mathbf{v} + (s_{\mathbf{i}}\mathcal{I}_{V_{R,\mathbf{i}}} - T)E_-\mathbf{v}. \end{aligned}$$

Since E_+ and E_- are the Riesz-projectors associated with σ_+ and σ_-, the spectrum $\sigma(T_+)$ of $T_+ := T|_{V_+}$ is $\sigma_+ \subset \mathbb{C}_{\mathbf{i}}^+$ and the spectrum $\sigma(T_-)$ of $T_- := T|_{V_-}$ is $\sigma_- \subset \mathbb{C}_{\mathbf{i}}^-$.

As $s_{\mathbf{i}}$ has positive imaginary part, we find $\overline{s_{\mathbf{i}}} \in \mathbb{C}_{\mathbf{i}}^{-} \subset \rho(T_+)$ and $s_{\mathbf{i}} \in \mathbb{C}_{\mathbf{i}}^{+} \subset \rho(T_-)$ such that $R_{\overline{s_{\mathbf{i}}}}(T_+) := \left(\overline{s_{\mathbf{i}}}\mathcal{I}_{V_+} - T_+\right)^{-1} \in \mathcal{B}(V_+)$ and $R_{s_{\mathbf{i}}}(T)^{-1} := \left(s_{\mathbf{i}}\mathcal{I}_{V_-} - T_-\right)^{-1} \in \mathcal{B}(V_-)$ exist. As $E_+|_{V_+} = \mathcal{I}_{V_+}$ and $E_-|_{V_+} = 0$, they satisfy the relations

$$E_+ R_{\overline{s_{\mathbf{i}}}}(T_+) E_+ = R_{\overline{s_{\mathbf{i}}}}(T_+) E_+ \qquad \text{and} \qquad E_- R_{\overline{s_{\mathbf{i}}}}(T_+) E_+ = 0 \tag{10.6}$$

and similarly also

$$E_- R_{s_{\mathbf{i}}}(T_-) E_- = R_{s_{\mathbf{i}}}(T_-) E_- \qquad \text{and} \qquad E_+ R_{s_{\mathbf{i}}}(T_-) E_- = 0. \tag{10.7}$$

Setting $R(s_0, s_1) := R_{\overline{s_{\mathbf{i}}}}(T_+)E_+ + R_{s_{\mathbf{i}}}(T_-)E_-$, we obtain a bounded $\mathbb{C}_{\mathbf{i}}$-linear operator that is defined on the entire space $V_{R,\mathbf{i}} = V_R$. Because E_+ and E_- commute with T and satisfy $E_+E_- = E_-E_+ = 0$ and because (10.6) and (10.7) hold true, we find for any $\mathbf{v} \in V_R$

$$\begin{aligned}
&R(s_0, s_1)((s_0\mathcal{I} - s_1 J) - T)\mathbf{v} \\
=&\left[R_{\overline{s_{\mathbf{i}}}}(T_+)E_+ + R_{s_{\mathbf{i}}}(T_-)E_-\right]\left[(\overline{s_{\mathbf{i}}}\mathcal{I}_{V_{R,\mathbf{i}}} - T)E_+\mathbf{v} + (s_{\mathbf{i}}\mathcal{I}_{V_{R,\mathbf{i}}} - T)E_-\mathbf{v}\right] \\
=&R_{\overline{s_{\mathbf{i}}}}(T_+)(\overline{s_{\mathbf{i}}}\mathcal{I}_{V_{R,\mathbf{i}}} - T_+)E_+\mathbf{v} + R_{s_{\mathbf{i}}}(T_-)E_-(s_{\mathbf{i}}\mathcal{I}_{V_{R,\mathbf{i}}} - T_-)E_-\mathbf{v} \\
=&E_+\mathbf{v} + E_-\mathbf{v} = \mathbf{v}
\end{aligned}$$

and

$$\begin{aligned}
&((s_0\mathcal{I} - s_1 J) - T)R(s_0, s_1)\mathbf{v} \\
=&\left[(\overline{s_{\mathbf{i}}}\mathcal{I}_{V_{R,\mathbf{i}}} - T)E_+ + (s_{\mathbf{i}}\mathcal{I}_{V_{R,\mathbf{i}}} - T)E_-\right]\left[R_{\overline{s_{\mathbf{i}}}}(T_+)E_+ + R_{s_{\mathbf{i}}}(T_-)E_-\right]\mathbf{v} \\
=&(\overline{s_{\mathbf{i}}}\mathcal{I}_{V_+} - T_+)R_{\overline{s_{\mathbf{i}}}}(T_+)E_+\mathbf{v} + (s_{\mathbf{i}}\mathcal{I}_{V_-} - T_-)R_{s_{\mathbf{i}}}(T_-)E_-\mathbf{v} \\
=&E_+\mathbf{v} + E_-\mathbf{v} = \mathbf{v}.
\end{aligned}$$

Hence, $R(s_0, s_1) \in \mathcal{B}(V_{R,\mathbf{i}})$ is the $\mathbb{C}_{\mathbf{i}}$-linear bounded inverse of $(s_0\mathcal{I} - s_1 J) - T$. Since $(s_0\mathcal{I} - s_1 J) - T$ is quaternionic right linear, its inverse is quaternionic right linear too so that even $((s_0\mathcal{I} - s_1 J) - T)^{-1} \in \mathcal{B}(V_R)$. J is therefore actually a spectral orientation for T and T is in turn a spectral operator with spectral decomposition (E, J).

Finally, we show the uniqueness of the spectral orientation J. Assume that $\widetilde{J}$ is an arbitrary spectral orientation for T. We show that $\widetilde{V_+} := V_{\widetilde{J},\mathbf{i}}^{+}$ equals $V_+ = V_{J,\mathbf{i}}^{+}$. Theorem 9.18 implies then $J = \widetilde{J}$ because $\ker J = \ker \widetilde{J} = \operatorname{ran} E(\mathbb{R}) = \{\mathbf{0}\}$ and $V_{J,\mathbf{i}}^{-} = V_+\mathbf{j} = \widetilde{V_+}\mathbf{j} = V_{\widetilde{J},\mathbf{i}}^{-}$.

Since $\widetilde{J}$ commutes with T, we have $\widetilde{J}E_+ = E_+\widetilde{J}$ as

$$\begin{aligned}
\widetilde{J}E_+\mathbf{v} &= \int_{\Gamma_+} \widetilde{J}\mathcal{Q}_z(T)^{-1}(\mathbf{v}\overline{z} - T\mathbf{v})\, dz \frac{1}{2\pi\mathbf{i}} \\
&= \int_{\Gamma_+} \mathcal{Q}_z(T)^{-1}(\widetilde{J}\mathbf{v}\overline{z} - T\widetilde{J}\mathbf{v})\, dz \frac{1}{2\pi\mathbf{i}} = E_+\widetilde{J}\mathbf{v}.
\end{aligned} \tag{10.8}$$

The projection E_+ therefore leaves $\widetilde{V_+}$ invariant because

$$\widetilde{J}(E_+\mathbf{v}) = E_+\big(\widetilde{J}\mathbf{v}\big) = (E_+\mathbf{v})\mathbf{i} \in \widetilde{V_+}$$

for any $\mathbf{v} \in \widetilde{V_+}$. Hence, $E_+|_{\widetilde{V_+}}$ is a projection on $\widetilde{V_+}$.

We show now that $\ker E_+|_{\widetilde{V_+}} = \{\mathbf{0}\}$ so that $E_+|_{\widetilde{V_+}} = \mathcal{I}_{V_+}$ and hence $\widetilde{V_+} \subset \operatorname{ran} E_+ = V_+$. We do this by constructing a slice hyperholomorphic extension of $\mathcal{R}_s(T;\mathbf{v})$ that is defined on all of $\mathbb{H}$ and applying Corollary 10.8 for any $\mathbf{v} \in \ker E_+|_{\widetilde{V_+}}$.

Let $\mathbf{v} \in \ker E_+|_{\widetilde{V_+}}$. As $\ker E_+|_{\widetilde{V_+}} \subset \ker E_+ = \operatorname{ran} E_- = V_-$, we find $\mathbf{v} \in V_-$. For $z = z_0 + z_1\mathsf{i} \in \mathbb{C}_{\mathsf{i}}$, we define the function

$$\mathbf{f}_{\mathsf{i}}(z;\mathbf{v}) := \begin{cases} R_z(T_-)\mathbf{v}, & z_1 \geq 0 \\ \left(z_0\mathcal{I} + z_1\widetilde{J} - T\right)^{-1}\mathbf{v}, & z_1 < 0. \end{cases}$$

This function is (right) holomorphic on $\mathbb{C}_{\mathsf{i}}$. On $\mathbb{C}_{\mathsf{i}}^{\geq}$ this is obvious because the resolvent of T_- is a holomorphic function. For $z_1 < 0$, we have

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\partial}{\partial z_0}\mathbf{f}_{\mathsf{i}}(z;\mathbf{v}) + \frac{\partial}{\partial z_1}\mathbf{f}_{\mathsf{i}}(z;\mathbf{v})\mathsf{i}\right) \\
=&\frac{1}{2}\left(-\left(z_0\mathcal{I} + z_1\widetilde{J} - T)\right)^{-2}\mathbf{v} - \left(z_0\mathcal{I} + z_1\widetilde{J} - T)\right)^{-2}\widetilde{J}\mathbf{v}\mathsf{i}\right) \\
=&\frac{1}{2}\left(-\left(z_0\mathcal{I} + z_1\widetilde{J} - T)\right)^{-2}\mathbf{v} - \left(z_0\mathcal{I} + z_1\widetilde{J} - T)\right)^{-2}\mathbf{v}\mathsf{i}^2\right) = \mathbf{0},
\end{aligned}$$

as $\widetilde{J}\mathbf{v} = \mathbf{v}\mathsf{i}$ because $\mathbf{v} \in \widetilde{V_+} = V^+_{\widetilde{J},\mathsf{i}}$. The slice extension $\mathbf{f}(s;\mathbf{v})$ of $\mathbf{f}_{\mathsf{i}}(s;\mathbf{v})$ obtained from Corollary 2.10 is a slice hyperholomorphic extension of $\mathcal{R}_s(T;\mathbf{v})$ to all of $\mathbb{H}$ in the sense of Definition 10.3. Indeed, as $\mathcal{Q}_z(T)|_{V_-} = \mathcal{Q}_z(T_-) = (\mathcal{I}_{V_-}\overline{z} - T_-)(\mathcal{I}_{V_-}z - T_-)$, we find for $s \in \mathbb{C}_{\mathsf{i}}^{\geq}$ that

$$\begin{aligned}
&\mathcal{Q}_s(T)\mathbf{f}(s;\mathbf{v}) = \mathcal{Q}_s(T_-)\mathbf{f}_{\mathsf{i}}(s;\mathbf{v}) \\
=&(\overline{s}\mathcal{I}_{V_-} - T_-)(s\mathcal{I}_{V_-} - T_-)R_s(T_-)\mathbf{v} \\
=&(\overline{s}\mathcal{I}_{V_-} - T_-)\mathbf{v} = \mathbf{v}s - T_-\mathbf{v} = \mathbf{v}s - T\mathbf{v}.
\end{aligned}$$

On the other hand, the facts that T and $\widetilde{J}$ commute and that $-\widetilde{J}^2 = \mathcal{I}$ because $\widetilde{J}$ is an imaginary operator with $\operatorname{ran}\widetilde{J} = V_R$ imply

$$\begin{aligned}
&\left(s_0\mathcal{I} + s_1\widetilde{J} - T\right)\left(s_0\mathcal{I} - s_1\widetilde{J} - T\right) \\
=&s_0^2\mathcal{I} - s_0s_1\widetilde{J} - s_0T + s_0s_1\widetilde{J} - s_1^2\widetilde{J}^2 - s_1\widetilde{J}T - s_0T + s_1T\widetilde{J} + T^2 \\
=&|s|^2\mathcal{I} - 2s_0T + T^2 = \mathcal{Q}_s(T).
\end{aligned}$$

For $s = s_0 + (-\mathsf{i})s_1 \in \mathbb{C}_{\mathsf{i}}^-$, we find thus because of $\mathbf{v} \in \widetilde{V_+} = V^+_{\widetilde{J},\mathsf{i}}$ that

$$\begin{aligned}
\mathcal{Q}_s(T)\mathbf{f}(s;\mathbf{v}) &= \left(s_0\mathcal{I} + s_1\widetilde{J} - T\right)\left(s_0\mathcal{I} - s_1\widetilde{J} - T\right)\mathbf{f}_{\mathsf{i}}(s;\mathbf{v}) \\
&= \left(s_0\mathcal{I} + s_1\widetilde{J} - T\right)\left(s_0\mathcal{I} - s_1\widetilde{J} - T\right)\left(s_0\mathcal{I} - s_1\widetilde{J} - T\right)^{-1}\mathbf{v} \\
&= \left(s_0\mathcal{I} + s_1\widetilde{J} - T\right)\mathbf{v} = \mathbf{v}s_0 + \mathbf{v}\mathsf{i}s_1 - T = \mathbf{v}\overline{s} - T\mathbf{v}.
\end{aligned}$$

Finally, for $s \notin \mathbb{C}_{\mathsf{i}}$, Theorem 2.9 yields

$$\begin{aligned}\mathcal{Q}_s(T)\mathbf{f}(s;\mathbf{v}) =& \mathcal{Q}_s(T)\mathbf{f}_{\mathsf{i}}(s_{\mathsf{i}};\mathbf{v})(1-\mathbf{i}\mathbf{i}_s)\frac{1}{2} + \mathcal{Q}_s(T)\mathbf{f}_{\mathsf{i}}\left(\overline{s_{\mathsf{i}}};\mathbf{v}\right)(1+\mathbf{i}\mathbf{i}_s)\frac{1}{2}\\ =& \left(\mathbf{v}\overline{s_{\mathsf{i}}} - T\mathbf{v}\right)(1-\mathbf{i}\mathbf{i}_s)\frac{1}{2} + (\mathbf{v}s_{\mathsf{i}} - T\mathbf{v})(1+\mathbf{i}\mathbf{i}_s)\frac{1}{2}\\ =& \mathbf{v}\left(\overline{s_{\mathsf{i}}}(1-\mathbf{i}\mathbf{i}_s) + s(1+\mathbf{i}\mathbf{i}_s)\right)\frac{1}{2} - T\mathbf{v}\left((1-\mathbf{i}\mathbf{i}_s)+(1+\mathbf{i}\mathbf{i}_s)\right)\frac{1}{2}\\ =& \mathbf{v}(s_{\mathsf{i}} + \overline{s_{\mathsf{i}}} + (s_{\mathsf{i}} - \overline{s_{\mathsf{i}}})\mathbf{i}\mathbf{i}_s)\frac{1}{2} - T\mathbf{v} = \mathbf{v}(s_0 - s_1\mathbf{i}_s) - T\mathbf{v} = \mathbf{v}\overline{s} - T\mathbf{v}.\end{aligned}$$

From Corollary 10.8, we hence deduce that $\mathbf{v} = \mathbf{0}$ and so $\ker E_+|_{\widetilde{V_+}} = \{\mathbf{0}\}$. Since $E_+|_{\widetilde{V_+}}$ is a projection on $\widetilde{V_+}$, we have $\widetilde{V_+} = \ker E_+|_{\widetilde{V_+}} \oplus \operatorname{ran} E_+|_{\widetilde{V_+}} = \{\mathbf{0}\} \oplus \operatorname{ran} E_+|_{\widetilde{V_+}}$. We conclude $\widetilde{V_+} = \operatorname{ran} E_+|_{\widetilde{V_+}} \subset \operatorname{ran} E_+ = V_+$ and therefore have

$$V_R = \widetilde{V_+} \oplus \widetilde{V_+}\mathbf{j} \subset V_+ \oplus V_+\mathbf{j} = V_R.$$

This implies $V_+ = \widetilde{V_+}$ and in turn $J = \widetilde{J}$.

□

Corollary 10.15. *Let $T \in \mathcal{B}(V_R)$ and assume that there exists a spectral resolution for T as in Proposition 10.14. If $\sigma_S(T) = \Delta_1 \cup \Delta_2$ with closed sets $\Delta_1, \Delta_2 \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$ such that $\Delta_1 \subset \mathbb{R}$ and $\Delta_2 \cap \mathbb{R} = \emptyset$, then there exists a unique imaginary operator $J \in \mathcal{B}(V_R)$ that is a spectral orientation for T such that T is a spectral operator with spectral decomposition (E, J).*

Proof. Let $T_2 = T_2|_{V_2}$, where $V_2 = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R}) = \operatorname{ran} E(\Delta_2)$. Then spectral measure $E_2(\Delta) := E(\Delta)|_{V_2}$ for $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$ is by Lemma 10.2 a spectral resolution for T_2. Since $\sigma_S(T_2) \subset \Delta_2$ and $\Delta_2 \cap \mathbb{R} = \emptyset$, Proposition 10.14 implies the existence of a unique spectral orientation J_2 for T_2.

The fact that (E_2, J_2) is a spectral system implies $\operatorname{ran} J_2 = \operatorname{ran} E_2(\mathbb{H} \setminus \mathbb{R})V_2 = V_2$ because $E_2(\mathbb{H} \setminus \mathbb{R}) = E(\mathbb{H} \setminus \mathbb{R})|_{V_2} = \mathcal{I}_{V_2}$. If we set $J = J_2 E(\mathbb{H} \setminus \mathbb{R})$, we find that $\ker J = \operatorname{ran} E(\mathbb{R})$ and $\operatorname{ran} J = V_2 = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$. We also have

$$\begin{aligned}E(\Delta)J =& E(\Delta \cap \mathbb{R})J_2E(\mathbb{H} \setminus \mathbb{R}) + E(\Delta \setminus \mathbb{R})J_2E(\mathbb{H} \setminus \mathbb{R})\\ =& E_2(\Delta \setminus \mathbb{R})J_2E(\mathbb{H} \setminus \mathbb{R}) = J_2E_2(\Delta \setminus \mathbb{R})E(\mathbb{H} \setminus \mathbb{R})\\ =& J_2E(\Delta \setminus \mathbb{R})E(\mathbb{H} \setminus \mathbb{R}) = J_2E(\mathbb{H} \setminus \mathbb{R})E(\Delta \setminus \mathbb{R}) = JE(\Delta),\end{aligned}$$

where the last identity used that $E(\mathbb{H} \setminus \mathbb{R})E(\Delta \cap \mathbb{R}) = 0$. Moreover, we have

$$-J^2 = -J_2E(\mathbb{H} \setminus \mathbb{R})J_2E(\mathbb{H} \setminus \mathbb{R}) = -J_2^2E(\mathbb{H} \setminus \mathbb{R}) = E(\mathbb{H} \setminus \mathbb{R})$$

so that $-J^2$ is a projection onto $\operatorname{ran} J = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$ along $\ker J = \operatorname{ran} E(\mathbb{R})$. Hence, J is an imaginary operator and (E, J) is a spectral system on V_R. Finally, for any $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, we have

$$((s_0\mathcal{I} - s_1J - T)|_{V_2})^{-1} = (s_0\mathcal{I}_{V_2} - s_1J_2 - T_2)^{-1} \in \mathcal{B}(V_2)$$

and hence (E, J) is actually a spectral decomposition of T.

In order to show the uniqueness of J we consider an arbitrary spectral orientation $\widetilde{J}$ for T. Then

$$\ker \widetilde{J} = E(\mathbb{R})V_R = \ker J \quad \text{and} \quad \operatorname{ran} \widetilde{J} = E(\mathbb{H} \setminus \mathbb{R})V_R = \operatorname{ran} J. \tag{10.9}$$

By Lemma 10.2, the operator $\widetilde{J}|_{V_2}$ is a spectral orientation for T_2. The spectral orientation of T_2 is however unique by Proposition 10.14 and hence $\widetilde{J}|_{V_2} = J_2 = J|_{V_2}$. We conclude $\widetilde{J} = J$.

□

Finally we can now show the uniqueness of the spectral orientation of an arbitrary spectral operator.

Theorem 10.16. *The spectral decomposition* (E, J) *of a spectral operator* $T \in \mathcal{B}(V_R)$ *is uniquely determined.*

Proof. The uniqueness of the spectral resolution E has already been shown before in Lemma 10.13. Let J and $\widetilde{J}$ be two spectral orientations for T. Since (10.9) holds true also in this case, we can reduce the problem to showing that $J|_{V_1} = \widetilde{J}|_{V_1}$ with $V_1 := \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$. The operator $T_1 := T|_{V_1}$ is a spectral operator on V_1. By Lemma 10.2, (E_1, J_1) and $(E_1, \widetilde{J}_1)$ with $E_1(\Delta) = E(\Delta)|_{V_1}$ and $J_1 = J|_{V_1}$ and $\widetilde{J}_1 := \widetilde{J}|_{V_1}$ are spectral decompositions of T_1. As $E_0(\mathbb{R}) = 0$, it is hence sufficient to show the uniqueness of the spectral orientation of a spectral operator under the assumption $E(\mathbb{R}) = 0$.

Let hence T be a spectral operator with spectral decomposition (E, J) such that $E(\mathbb{R}) = 0$. If $\operatorname{dist}(\sigma_S(T), \mathbb{R}) > 0$, then we already know that the statement holds true. We have shown this in Proposition 10.14. Otherwise we choose a sequence of pairwise disjoint sets $\Delta_n \in \mathsf{B}_\mathsf{S}(\mathbb{H})$ with $\operatorname{dist}(\Delta_n, \mathbb{R}) > 0$ that cover $\sigma_S(T) \setminus \mathbb{R}$. We can choose for instance

$$\Delta_n := \left\{ s \in \mathbb{H} : -\|T\| \leq s_0 \leq \|T\|, \ \frac{\|T\|}{n+1} < s_1 \leq \frac{\|T\|}{n} \right\}.$$

By Corollary 10.10 and as $E(\mathbb{R}) = 0$, we have

$$E(\sigma_S(T) \setminus \mathbb{R}) = E(\sigma_S(T) \setminus \mathbb{R}) + E(\sigma_S(T) \cap \mathbb{R}) = E(\sigma_S(T)) = \mathcal{I}$$

We therefore find $\sum_{n=1}^{+\infty} E(\Delta_n)\mathbf{v} = E\left(\bigcup_{n\in\mathbb{N}} \Delta_n\right)\mathbf{v} = \mathbf{v}$ for any $\mathbf{v} \in V_R$ because we have $\sigma_S(T) \setminus \mathbb{R} \subset \bigcup_{n\in\mathbb{N}} \Delta_n$.

Since $E(\Delta_n)$ and J commute, the operator J leaves $V_{\Delta_n} := \operatorname{ran} E(\Delta_n)$ invariant. Hence $J_{\Delta_n} = J|_{V_{\Delta_n}}$ is a bounded operator on V_{Δ_n} and we have

$$J\mathbf{v} = J \sum_{n=1}^{+\infty} E(\Delta_n)\mathbf{v} = \sum_{n=1}^{+\infty} JE(\Delta_n)\mathbf{v} = \sum_{n=1}^{+\infty} J_{\Delta_n} E(\Delta_n)\mathbf{v}.$$

Similarly, we see that also $\widetilde{J_{\Delta_n}} := \widetilde{J}|_{V_{\Delta_n}}$ is a bounded operator on V_{Δ_n} and that $\widetilde{J}\mathbf{v} = \sum_{n=1}^{+\infty} \widetilde{J}_{\Delta_n} E(\Delta_n)\mathbf{v}$.

Now observe that T_{Δ_n} is a spectral operator by Lemma 10.2 and (E_n, J_{Δ_n}) with $E_n(\Delta) := E(\Delta)|_{V_{\Delta_n}}$ for $\Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H})$ is a spectral decomposition of T_{Δ_n}. However, also $\left(E_n, \widetilde{J}_{\Delta_n}\right)$ is a spectral decomposition of T_{Δ_n} by Lemma 10.2. Since $\sigma_S\left(T_{\Delta_n}\right) \subset \Delta_n$

and $\operatorname{dist}(\Delta_n, \mathbb{R}) > 0$, Proposition 10.14 implies that the spectral orientation of T_{Δ_n} is unique so that $J_{\Delta_n} = \widetilde{J}_{\Delta_n}$. We thus find

$$J\mathbf{v} = \sum_{n=1}^{+\infty} J_{\Delta_n} E(\Delta_n)\mathbf{v} = \sum_{n=1}^{+\infty} \widetilde{J}_{\Delta_n} E(\Delta_n)\mathbf{v} = \widetilde{J}\mathbf{v}.$$

□

Remark 10.17. In Proposition 10.14 and Corollary 10.15 we showed that under certain assumptions the existence of a spectral resolution E for T already implies the existence of a spectral orientation and is hence a sufficient condition for T to be a spectral operator. One may wonder whether this is true in general. An intuitive approach for showing this follows the idea of the proof of Theorem 10.16. We can cover $\sigma_S(T) \setminus \mathbb{R}$ by pairwise disjoint sets $\Delta_n \in \mathsf{B}_\mathbb{S}(\mathbb{H})$ with $\operatorname{dist}(\Delta_n, \mathbb{R}) > 0$ for each $n \in \mathbb{N}$. On each of the subspaces $V_n := \operatorname{ran} E(\Delta_n)$, the operator T induces the operator $T_n := T|_{V_n}$ with $\sigma_S(T_n) \subset cl(\Delta_n)$. Since $\operatorname{dist}(\Delta_n, \mathbb{R}) > 0$, we can then define $\Delta_{n,+} := \Delta_n \cap \mathbb{C}_{\mathbf{i}}^+$ and $\Delta_{n,-} := \Delta_n \cap \mathbb{C}_{\mathbf{i}}^-$ for an arbitrary imaginary unit $\mathbf{i} \in \mathbb{S}$ and consider the Riesz-projectors $E_{n,+} := \chi_{\Delta_{n,+}}(T_n)$ and $E_{n,-} := \chi_{\Delta_{n,-}}(T_n)$ of T_n on $V_{n,\mathbf{i}}$ associated with $\Delta_{n,+}$ and $\Delta_{n,-}$. Just as we did it in the proof of Proposition 10.14, we can then construct a spectral orientation for T_n by setting $J_n\mathbf{v} = E_{n,+}\mathbf{v}\mathbf{i} + E_{n,-}\mathbf{v}(-\mathbf{i})$ for $\mathbf{v} \in V_n$. The spectral orientation of J must then be

$$J\mathbf{v} = \sum_{n=1}^{+\infty} J_n E(\Delta_n)\mathbf{v} = \sum_{n=1}^{+\infty} E_{n,+}E(\Delta_n)\mathbf{v}\mathbf{i} + E_{n,-}E(\Delta_n)\mathbf{v}(-\mathbf{i}). \tag{10.10}$$

If T is a spectral operator, then $E_{n,+} = E_+|_{V_n}$ and $E_{n,-} = E_-|_{V_n}$, where E_+ and E_- are as usual the projections of V_R onto $V_{J,\mathbf{i}}^+$ and $V_{J,\mathbf{i}}^-$ along $V_0 \oplus V_{J,\mathbf{i}}^-$ resp. $V_0 \oplus V_{J,\mathbf{i}}^+$. Hence the Riesz-projectors $E_{n,+}$ and $E_{n,-}$ are uniformly bounded in $n \in \mathbb{N}$ and the above series converges. The spectral orientation of T can therefore be constructed as described above if T is a spectral operator.

This procedure however fails if the Riesz-projectors $E_{n,+}$ and $E_{n,-}$ are not uniformly bounded because the convergence of the above series is in this case not guaranteed. The next example presents an operator for which the above series does actually not converge for this reason although the operator has a quaternionic spectral resolution. Hence, the existence of a spectral resolution does in general not imply the existence of a spectral orientation.

Example 10.18. Let $\ell^2(\mathbb{H})$ be the space of all square-summable sequences with quaternionic entries and choose $\mathbf{i}, \mathbf{j} \in \mathbb{S}$ with $\mathbf{i} \perp \mathbf{j}$. We define an operator T on $\ell^2(\mathbb{H})$ by the following rule: if $(b_n)_{n\in\mathbb{N}} = T(a_n)_{n\in\mathbb{N}}$, then

$$\begin{pmatrix} b_{2m-1} \\ b_{2m} \end{pmatrix} = \frac{1}{m^2} \begin{pmatrix} \mathbf{i} & 2m\mathbf{i} \\ 0 & -\mathbf{i} \end{pmatrix} \begin{pmatrix} a_{2m-1} \\ a_{2m} \end{pmatrix}. \tag{10.11}$$

For neatness, let us denote the matrix in the above equation by J_m and let us set $T_m := \frac{1}{m^2} J_m$, that is

$$J_m := \begin{pmatrix} \mathbf{i} & 2m\mathbf{i} \\ 0 & -\mathbf{i} \end{pmatrix} \qquad \text{and} \qquad T_m := \frac{1}{m^2} \begin{pmatrix} \mathbf{i} & 2m\mathbf{i} \\ 0 & -\mathbf{i} \end{pmatrix}.$$

Since all matrix norms are equivalent, there exists a constant $C > 0$ such that

$$\|M\| \ \leq C \max_{\ell,\kappa\in\{1,2\}} |m_{\ell,\kappa}| \qquad \forall M = \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{pmatrix} \in \mathbb{H}^{2\times 2} \tag{10.12}$$

such that $\|J_m\| \leq 2Cm$. We thus find for (10.11) that

$$\|(b_{2m-1}, b_{2m})^T\|_2 \leq \frac{2C}{m}\|(a_{2m-1}, a_{2m})^T\|_2 \leq 2C\|(a_{2m-1}, a_{2m})^T\|_2.$$

and in turn

$$\begin{aligned} \|T(a_n)_{n\in\mathbb{N}}\|^2_{\ell^2(\mathbb{H})} &= \sum_{m=1}^{+\infty} |b_{2m-1}|^2 + |b_{2m}|^2 \\ &\leq \sum_{m=1}^{+\infty} 4C^2 \left(|a_{2m-1}|^2 + |a_{2m}|^2\right) = 4C^2 \left\|(a_n)_{n\in\mathbb{N}}\right\|^2_{\ell^2(\mathbb{H})}. \end{aligned} \tag{10.13}$$

Hence T is a bounded right-linear operator on $\ell^2(\mathbb{H})$.

We show now that the S-spectrum of T is the set $\Lambda = \{0\} \cup \cup_{n\in\mathbb{N}} \frac{1}{n^2}\mathbb{S}$. For $s \in \mathbb{H}$, the operator $\mathcal{Q}_s(T) = T^2 - 2s_0 T + |s|^2$ is given by the following relation: if $(c_n)_{n\in\mathbb{N}} = \mathcal{Q}_s(T)(a_n)_{n\in\mathbb{N}}$ then

$$\begin{pmatrix} c_{2m-1} \\ c_{2m} \end{pmatrix} = \begin{pmatrix} -\frac{1}{m^2} - 2\mathbf{i}\frac{s_0}{m^2} + |s|^2 & -4\mathbf{i}\frac{s_0}{m} \\ 0 & -\frac{1}{m^2} - 2\mathbf{i}\frac{s_0}{m^2} + |s|^2 \end{pmatrix} \begin{pmatrix} a_{2m-1} \\ a_{2m} \end{pmatrix}. \tag{10.14}$$

The inverse of the above matrix is

$$\begin{aligned} \mathcal{Q}_s(T_m)^{-1} &= \begin{pmatrix} \frac{m^4}{|s|^2 m^4 - 2is_0 m^2 - 1} & \frac{4im^7 s_0}{|s|^4 m^8 + 2\left(s_0^2 - s_1^2\right)m^4 + 1} \\ 0 & \frac{m^4}{|s|^2 m^4 + 2is_0 m^2 - 1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\left(s_{\mathbf{i}} - \frac{\mathbf{i}}{m^2}\right)\left(\overline{s_{\mathbf{i}}} - \frac{\mathbf{i}}{m^2}\right)} & \frac{4is_0}{m\left(s_{\mathbf{i}} + \frac{\mathbf{i}}{m^2}\right)\left(s_{\mathbf{i}} - \frac{\mathbf{i}}{m^2}\right)\left(\overline{s_{\mathbf{i}}} + \frac{\mathbf{i}}{m^2}\right)\left(\overline{s_{\mathbf{i}}} - \frac{\mathbf{i}}{m^2}\right)} \\ 0 & \frac{1}{\left(s_{\mathbf{i}} + \frac{\mathbf{i}}{m^2}\right)\left(\overline{s_{\mathbf{i}}} + \frac{\mathbf{i}}{m^2}\right)} \end{pmatrix}. \end{aligned}$$

with $s_{\mathbf{i}} = s_0 + \mathbf{i}s_1$. Hence, $\mathcal{Q}_s(T_m)^{-1}$ exists for $s_{\mathbf{i}} \neq \frac{1}{m^2}\mathbf{i}$. We have

$$\left|s_{\mathbf{i}} - \frac{\mathbf{i}}{m^2}\right| \left|\overline{s_{\mathbf{i}}} - \frac{\mathbf{i}}{m^2}\right| = \left|s_{\mathbf{i}} + \frac{\mathbf{i}}{m^2}\right| \left|\overline{s_{\mathbf{i}}} + \frac{\mathbf{i}}{m^2}\right| \geq 2\left|s_{\mathbf{i}} - \frac{\mathbf{i}}{m^2}\right| = 2\mathrm{dist}\left(s, \left[\frac{\mathbf{i}}{m}\right]\right)$$

and so

$$\left\|\mathcal{Q}_s(T_m)^{-1}\right\| \ \leq C \max\left\{\frac{1}{2\mathrm{dist}\left(s, \left[\frac{\mathbf{i}}{m^2}\right]\right)}, \frac{|s_0|}{m\left(\mathrm{dist}\left(s, \left[\frac{\mathbf{i}}{m^2}\right]\right)\right)^2}\right\}, \tag{10.15}$$

where C is the constant in (10.12). If $s \notin \Lambda$, then $0 < \mathrm{dist}(s, \mathit{\Lambda}) \leq \mathrm{dist}\left(s, \left[\frac{\mathbf{i}}{m^2}\right]\right)$ and hence the matrices $\mathcal{Q}_s(T_m)^{-1}$ are for $m \in \mathbb{N}$ uniformly bounded by

$$\left\|\mathcal{Q}_s(T_m)^{-1}\right\| \ \leq C \max\left\{\frac{1}{2\mathrm{dist}\left(s, \Lambda\right)}, \frac{|s_0|}{\mathrm{dist}\left(s, \Lambda\right)^2}\right\}.$$

The operator $\mathcal{Q}_s(T)^{-1}$ is then given by the relation

$$\begin{pmatrix} a_{2m-1} \\ a_{2m} \end{pmatrix} = \mathcal{Q}_s(T_m)^{-1} \begin{pmatrix} c_{2m-1} \\ c_{2m} \end{pmatrix}, \tag{10.16}$$

for $(a_n)_{n\in\mathbb{N}} = Q_s(T)^{-1}(c_n)_{n\in\mathbb{N}}$. A computation similar to the one in (10.13) shows that this operator is bounded on $\ell^2(\mathbb{H})$. Thus $s \in \rho_S(T)$ if $s \notin \Lambda$ and in turn $\sigma_S(T) \subset \Lambda$.

For any $m \in \mathbb{N}$, we set $s_m = \frac{1}{m^2}\mathbf{i}$. The sphere $[s_m] = \frac{1}{m^2}\mathbb{S}$ is an eigensphere of T and the associated eigenspace V_m is the right-linear span of $\mathbf{e}_{2m-1}$ and $\mathbf{e}_{2m}$, where $\mathbf{e}_n = (\delta_{n,\ell})_{\ell\in\mathbb{N}}$, as one can see easily from (10.14). A straightforward computation moreover shows that the vectors $\mathbf{v}_{2m-1} := \mathbf{e}_{2m-1}$ and $\mathbf{v}_{2m} := -\mathbf{e}_{2m-1}\mathbf{j} + \frac{1}{m}\mathbf{e}_{2m}\mathbf{j}$ are eigenvectors of T with respect to the eigenvalue s_m. Hence $[s_m] \subset \sigma_S(T)$. Since $\sigma_S(T)$ is closed, we finally find $\Lambda = cl(\bigcup_{m\in\mathbb{N}}[s_m]) \subset \sigma_S(T)$ and in turn $\sigma_S(T) = \Lambda$.

Let E_m for $m \in \mathbb{N}$ be the orthogonal projection of $\ell^2(\mathbb{H})$ onto the subspace $V_m := \operatorname{span}_{\mathbb{H}}\{\mathbf{e}_{2m-1}, \mathbf{e}_{2m}\}$, that is $E_m(a_n)_{n\in\mathbb{N}} = \mathbf{e}_{2m-1}a_{2m-1} + \mathbf{e}_{2m}a_{2m}$. We define for any set $\Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H})$ the operator

$$E(\Delta) = \sum_{m\in I_\Delta} E_m \qquad \text{with} \qquad I_\Delta := \left\{ m \in \mathbb{N} : \frac{1}{m^2}\mathbb{S} \subset \Delta \right\}.$$

It is immediate that E is a spectral measure on $\ell^2(\mathbb{H})$, that $\|E(\Delta)\| \le 1$ for any $\Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H})$ and that $E(\Delta)$ commutes with T for any $\Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H})$. Moreover, if $s \notin cl(\Delta)$, then the pseudo-resolvent $\mathcal{Q}_s(T_\Delta)^{-1}$ of $T_\Delta = T|_{V_\Delta}$ with $V_\Delta = \operatorname{ran} E(\Delta)$ is given by

$$\mathcal{Q}_s(T_\Delta)^{-1} = \left(\sum_{m\in I_\Delta} \mathcal{Q}_s(T_m)^{-1} E_m \right) \Bigg|_{\operatorname{ran} E(\Delta)}.$$

Since $0 < \operatorname{dist}\left(s, \bigcup_{m\in I_\Delta}\left[\frac{\mathbf{i}}{m^2}\right]\right) = \inf_{m\in I_\Delta} \operatorname{dist}\left(s, \left[\frac{\mathbf{i}}{m^2}\right]\right)$, the operators $\mathcal{Q}_s(T_m)^{-1}$ are uniformly bounded for $m \in I_\Delta$. Computations similar to (10.13) show that $\mathcal{Q}_s(T_\Delta)^{-1}$ is a bounded operator on V_Δ. Hence, $s \in \rho_S(T_\Delta)$ and in turn $\sigma_S(T_\Delta) \subset cl(\Delta)$. Altogether we obtain that E is a spectral resolution for T.

In order to construct a spectral orientation for T, we first observe that J_m is a spectral orientation for T_m. For $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, we have

$$s_0\mathcal{I}_{\mathbb{H}^2} - s_1 J_m - T_m = \begin{pmatrix} s_0 - \left(s_1 + \frac{1}{m^2}\right)\mathbf{i} & -\left(s_1 + \frac{1}{m^2}\right)2m\mathbf{i} \\ 0 & s_0 + \left(s_1 + \frac{1}{m^2}\right)\mathbf{i} \end{pmatrix},$$

the inverse of which is given by the matrix

$$(s_0\mathcal{I}_{\mathbb{H}^2} - s_1 J_m - T_m)^{-1} = \begin{pmatrix} \frac{1}{s_0 - \left(s_1 + \frac{1}{m^2}\right)\mathbf{i}} & \frac{2\mathbf{i}m\left(\frac{1}{m^2}+s_1\right)}{s_0^2 + \left(\frac{1}{m^2}+s_1\right)^2} \\ 0 & \frac{1}{s_0 + \left(\frac{1}{m^2}+s_1\right)\mathbf{i}} \end{pmatrix}.$$

Since $s_1 > 0$, each entry has non-zero denominator and hence we have that the operator $(s_0\mathcal{I}_{\mathbb{H}^2} - s_1 J_m - T_m)^{-1}$ belongs to $\mathcal{B}(\mathbb{H}^2)$.

If $J \in \mathcal{B}(\ell^2(\mathbb{H}))$ is a spectral orientation for T, then the restriction $J|_{V_m}$ of J to $V_m = \operatorname{span}_{\mathbb{H}}\{\mathbf{e}_{2m-1}, \mathbf{e}_{2m}\}$ is also a spectral orientation for T_m. The uniqueness of the spectral orientation implies $J|_{V_m} = J_m$ and hence $J = \sum_{m=1}^{+\infty} J|_{V_m} E\left(\frac{1}{m^2}\mathbb{S}\right) = \sum_{m=1}^{+\infty} J_m E_m$.

This series does however not converge because the operators J_{V_m} are not uniformly bounded. Hence, it does not define a bounded operator on $\ell^2(\mathbb{H})$. Indeed, the sequence $a_{2m-1} = 0$, $a_{2m} = m^{-\frac{3}{2}}$ for instance belongs to $\ell^2(\mathbb{H})$, but

$$\begin{aligned}\left\| \sum_{m=1}^{+\infty} J_m E_m (a_n)_{n\in\mathbb{N}} \right\|^2_{\ell^2(\mathbb{H})} &= \sum_{m=1}^{+\infty} \left\| \begin{pmatrix} \mathsf{i} & 2m\mathsf{i} \\ 0 & -\mathsf{i} \end{pmatrix} \begin{pmatrix} 0 \\ m^{-\frac{3}{2}} \end{pmatrix} \right\|_2^2 \\ &= 2\sum_{m=1}^{+\infty} 4\frac{1}{m} + \frac{1}{m^3} = +\infty.\end{aligned}$$

Hence there cannot exist a spectral orientation for T and in turn T is not a spectral operator on $\ell^2(\mathbb{H})$.

We conclude this example with a remark on its geometric intuition. Let us identify $\mathbb{H}^2 \cong \mathbb{C}_{\mathsf{i}}^4$, which is for any $\mathsf{j} \in \mathbb{S}$ with $\mathsf{j} \perp \mathsf{i}$ spanned by the basis vectors

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} \mathsf{j} \\ 0 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b}_4 = \begin{pmatrix} 0 \\ \mathsf{j} \end{pmatrix}.$$

The vectors $\mathbf{v}_{m,1} = \mathbf{b}_1$ and $\mathbf{v}_{m,2} = -\mathbf{b}_2 + \frac{1}{m}\mathbf{b}_4$ are eigenvectors of J_m with respect to i and the vectors $\mathbf{v}_1\mathsf{j} = \mathbf{b}_2$ and $\mathbf{v}_{m,2} = \mathbf{b}_1 - \frac{1}{m}\mathbf{b}_3$ are eigenvectors of J_m with respect to $-\mathsf{i}$. We thus find $V_{J_m,\mathsf{i}}^+ = \operatorname{span}_{\mathbb{C}_\mathsf{i}}\{\mathbf{b}_1, -\mathbf{b}_2 + \frac{1}{m}\mathbf{b}_4\}$ and $V_{J_m,\mathsf{i}}^- = V_{J_m,\mathsf{i}}^+\mathsf{j}$. However, as m tends to infinity, the vector $\mathbf{v}_2$ tends to $\mathbf{v}_1\mathsf{j}$ and $\mathbf{v}_2\mathsf{j}$ tends to $\mathbf{v}_1$. Hence, intuitively, in the limit $V_{J_m,\mathsf{i}}^- = V_{J_m,\mathsf{i}}^+\mathsf{j} = V_{J_m,\mathsf{i}}^+$ and consequently the projections of $\mathbb{H}^2 = \mathbb{C}_\mathsf{i}^4$ onto $V_{J_m,\mathsf{i}}^+$ along $V_{J_m,\mathsf{i}}^-$ become unbounded.

Finally, the notion of quaternionic spectral operator is again backwards compatible with complex theory on $V_{R,\mathsf{i}}$.

Theorem 10.19. *An operator $T \in \mathcal{B}(V_R)$ is a quaternionic spectral operator if and only if it is a spectral operator on $V_{R,\mathsf{i}}$ for some (and hence any) $\mathsf{i} \in \mathbb{S}$. (See [38] for the complex theory.) If furthermore (E, J) is the quaternionic spectral decomposition of T and E_i is the spectral resolution of T as a complex $\mathbb{C}_\mathsf{i}$-linear operator on $V_{R,\mathsf{i}}$, then*

$$\begin{aligned} &E(\Delta) = E_\mathsf{i}(\Delta \cap \mathbb{C}_\mathsf{i}) \quad \forall \Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H}) \\ &J\mathbf{v} = E_\mathsf{i}(\mathbb{C}_\mathsf{i}^+)\mathbf{v}\mathsf{i} + E_\mathsf{i}(\mathbb{C}_\mathsf{i}^-)\mathbf{v}(-\mathsf{i}) \quad \forall \mathbf{v} \in V_R. \end{aligned} \tag{10.17}$$

Conversely, E_i is the spectral measure on V_R determined by (E, J) that was constructed in Lemma 9.26.

Proof. Let us first assume that $T \in \mathcal{B}(V_R)$ is a quaternionic spectral operator with spectral decomposition (E, J) in the sense of Definition 10.1 and let $\mathsf{i} \in \mathbb{S}$. Let E_+ be the projection of $\operatorname{ran} J = V_{J,\mathsf{i}}^+ \oplus V_{J,\mathsf{i}}^-$ onto $V_{J,\mathsf{i}}^+$ along $V_{J,\mathsf{i}}^-$ and let E_- be the projection of $\operatorname{ran} J$ onto $V_{J,\mathsf{i}}^-$ along $V_{J,\mathsf{i}}^+$, cf. Theorem 9.18. Since T and $E(\Delta)$ for $\Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H})$ commute with J, they leave the spaces $V_{J,\mathsf{i}}^+$ and $V_{J,\mathsf{i}}^-$ invariant and hence they commute with E_+ and E_-. By Lemma 9.26 the set function E_i on $\mathbb{C}_\mathsf{i}$ defined in (9.15), which is given by

$$E_\mathsf{i}(\Delta) = E_+E\left(\left[\Delta \cap \mathbb{C}_\mathsf{i}^+\right]\right) + E(\Delta \cap \mathbb{R}) + E_-E\left(\left[\Delta \cap \mathbb{C}_\mathsf{i}^-\right]\right), \tag{10.18}$$

for $\Delta \in \mathsf{B}(\mathbb{C}_{\mathsf{i}})$ is a spectral measure on $V_{R,\mathsf{i}}$. Since the spectral measure E and the projections E_+ and E_- commute with T, the spectral measure E_{i} commutes with T too.

If $\Delta \in \mathsf{B}(\mathbb{C}_{\mathsf{i}})$ is a subset of $\mathbb{C}_{\mathsf{i}}^+$, then $J\mathbf{v} = \mathbf{v}\mathsf{i}$ for $\mathbf{v} \in V_{\mathsf{i},\Delta} := \operatorname{ran} E_{\mathsf{i}}(\Delta)$ as $\operatorname{ran} E_{\mathsf{i}}(\Delta) = \operatorname{ran}(E_+ E([\Delta])) \subset V_{J,\mathsf{i}}^+$. For $z = z_0 + \mathsf{i} z_1 \in \mathbb{C}_{\mathsf{i}}$ and $\mathbf{v} \in V_{\mathsf{i},\Delta}$, we thus have

$$\begin{aligned}(z\mathcal{I}_{V_{\mathsf{i},\Delta}} - T)\mathbf{v} &= \mathbf{v}z_0 + \mathbf{v}\mathsf{i}z_1 - T\mathbf{v} \\ &= \mathbf{v}z_0 + J\mathbf{v}z_1 - T\mathbf{v} = (z_0\mathcal{I}_{V_{\mathsf{i},\Delta}} + z_1 J - T)\mathbf{v}.\end{aligned}$$

If $z \in \mathbb{C}_{\mathsf{i}}^-$, then the inverse of $(z_0\mathcal{I}_{V_{R,\mathsf{i}}} + z_1 J - T)|_{\operatorname{ran} J}$ exists because J is the spectral orientation of T. We thus have $R_z(T_\Delta) = (z_0\mathcal{I}_{V_{R,\mathsf{i}}} + z_1 J - T)^{-1}|_{V_{\mathsf{i},\Delta}}$ and so $\mathbb{C}_{\mathsf{i}}^- \subset \rho(T_\Delta)$. If on the other hand $z \in \mathbb{C}_{\mathsf{i}}^{\geq} \setminus cl(\Delta)$, then $z \in \rho_S(T_{[\Delta]})$ where $T_{[\Delta]} = T|_{V_{[\Delta]}}$ with $V_{[\Delta]} = \operatorname{ran} E([\Delta])$. Hence, $\mathcal{Q}_z(T_{[\Delta]})$ has a bounded inverse on $V_{[\Delta]}$. By the construction of E_{i} we have $V_{\mathsf{i},\Delta} = E_+ V_{[\Delta]}$ and since $T_{[\Delta]}$ and E_+ commute $\mathcal{Q}_z(T_{[\Delta]})^{-1}$ leaves $V_{\mathsf{i},\Delta}$ invariant so that $\mathcal{Q}_z(T_{[\Delta]})^{-1}\big|_{V_{\mathsf{i},\Delta}}$ defines a bounded $\mathbb{C}_{\mathsf{i}}$-linear operator on $V_{\mathsf{i},\Delta}$. Because of Theorem 8.4, the resolvent of T_Δ at z is therefore given by

$$R_z(T)\mathbf{v} = \mathcal{Q}_s(T_{[\Delta]})^{-1}(\mathbf{v}\overline{z} - T_\Delta \mathbf{v}) \qquad \forall \mathbf{v} \in V_{\mathsf{i},\Delta}.$$

Altogether, we conclude $\rho(T_\Delta) \supset \mathbb{C}_{\mathsf{i}}^- \cup \left(\mathbb{C}_{\mathsf{i}}^{\geq} \setminus cl(\Delta)\right) = \mathbb{C}_{\mathsf{i}} \setminus cl(\Delta)$ and in turn $\sigma(T_\Delta) \subset cl(\Delta)$. Similarly, we see that $\sigma(T_\Delta) \subset cl(\Delta)$ if $\Delta \subset \mathbb{C}_{\mathsf{i}}^-$. If on the other hand $\Delta \subset \mathbb{R}$, then $E_{\mathsf{i}}(\Delta) = E(\Delta)$ so that T_Δ is a quaternionic linear operator with $\sigma_S(T_\Delta) \subset cl(\Delta)$. By Theorem 8.4, we have $\sigma(T_\Delta) = \sigma_{\mathbb{C}_{\mathsf{i}}}(T_\Delta) = \sigma_S(T) \subset cl(\Delta)$. Finally, if $\Delta \in \mathsf{B}(\mathbb{C}_{\mathsf{i}})$ is arbitrary and $z \notin cl(\Delta)$, we can set $\Delta_+ := \Delta \cap \mathbb{C}_{\mathsf{i}}^+$, $\Delta_- := \Delta \cap \mathbb{C}_{\mathsf{i}}^-$ and $\Delta_{\mathbb{R}} := \Delta \cap \mathbb{R}$. Then z belong to the resolvent sets of each of the operators T_{Δ_+}, T_{Δ_-} and $T_{\Delta_{\mathbb{R}}}$ and we find

$$R_z(T) = R_z(T_{\Delta_+})E_{\mathsf{i}}(\Delta_+) + R_z(T_{\Delta_{\mathbb{R}}})E(\Delta_{\mathbb{R}}) + R_z(T_{\Delta_-})E_{\mathsf{i}}(\Delta_-).$$

We thus have $\sigma(T_\Delta) \subset cl(\Delta)$. Hence, T is a spectral operator on $V_{R,\mathsf{i}}$ and E_{i} is its ($\mathbb{C}_{\mathsf{i}}$-complex) spectral resolution on $V_{R,\mathsf{i}}$.

Now assume that T is a bounded quaternionic linear operator on V_R and that for some $\mathsf{i} \in \mathbb{S}$ there exists a $\mathbb{C}_{\mathsf{i}}$-linear spectral resolution E_{i} for T as a $\mathbb{C}_{\mathsf{i}}$-linear operator on $V_{R,\mathsf{i}}$. Following Definition 6 of [38, Chapter XV.2], an analytic extension of $R_z(T)\mathbf{v}$ with $\mathbf{v} \in V_{R,\mathsf{i}} = V_R$ is a holomorphic function $\mathbf{f}$ defined on a set $\mathcal{D}(\mathbf{f})$ such that $(z\mathcal{I}_{V_{R,\mathsf{i}}} - T)\mathbf{f}(z) = \mathbf{v}$ for $z \in \mathcal{D}(\mathbf{f})$. The resolvent $\rho(\mathbf{v})$ is the domain of the unique maximal analytic extension of $R_z(T)\mathbf{v}$ and the spectrum $\sigma(\mathbf{v})$ is the complement of $\rho(\mathbf{v})$ in $\mathbb{C}_{\mathsf{i}}$. (We defined the quaternionic counterparts of these concepts in Definition 10.3 and Definition 10.4.) Analogue to Theorem 10.9, we have

$$E_{\mathsf{i}}(\Delta)V_{R,\mathsf{i}} = \{\mathbf{v} \in V_{R,\mathsf{i}} = V_R : \sigma(\mathbf{v}) \subset \Delta\}, \qquad \forall \Delta \in \mathsf{B}(\mathbb{C}_{\mathsf{i}}). \tag{10.19}$$

Let $\mathbf{v} \in V_{R,\mathsf{i}}$, let $\mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$ and let $\mathbf{f}$ be the unique maximal analytic extension of $R_z(T)\mathbf{v}$ defined on $\rho(\mathbf{v})$. The mapping $z \mapsto \mathbf{f}\,(\overline{z})\,\mathsf{j}$ is then holomorphic on $\overline{\rho(\mathbf{v})}$: for any $z \in \overline{\rho(\mathbf{v})}$, we have $\overline{z} \in \rho(\mathbf{v})$ and in turn

$$\lim_{h\to 0}\left(\mathbf{f}\left(\overline{z+h}\right)\mathsf{j} - \mathbf{f}\,(\overline{z})\,\mathsf{j}\right)h^{-1} = \lim_{h\to 0}\left(\mathbf{f}(\overline{z}+\overline{h}) - \mathbf{f}(\overline{z})\right)\overline{h}^{-1}\mathsf{j} = \mathbf{f}'\,(\overline{z})\,\mathsf{j}.$$

Since T is quaternionic linear, we moreover have for $z \in \overline{\rho(\mathbf{v})}$ that

$$\left(z\mathcal{I}_{V_{R,\mathsf{i}}} - T\right)\left(\mathbf{f}\left(\overline{z}\right)\mathbf{j}\right) = \mathbf{f}\left(\overline{z}\right)\mathbf{j}z - T\left(\mathbf{f}\left(\overline{z}\right)\mathbf{j}\right) = \Big(\mathbf{f}\left(\overline{z}\right)\overline{z} - T\left(\mathbf{f}\left(\overline{z}\right)\right)\Big)\mathbf{j} = \mathbf{v}\mathbf{j}.$$

Hence $z \mapsto \mathbf{f}\left(\overline{z}\right)\mathbf{j}$ is an analytic extension of $R_z(T)(\mathbf{v}\mathbf{j})$ that is defined on $\overline{\rho(\mathbf{v})}$. Consequently $\rho(\mathbf{v}\mathbf{j}) \supset \overline{\rho(\mathbf{v})}$ and in turn $\sigma(\mathbf{v}\mathbf{j}) \subset \overline{\sigma(\mathbf{v})}$. If $\tilde{\mathbf{f}}$ is the maximal analytic extension of $R_z(T)(\mathbf{v}\mathbf{j})$, then similar arguments show that $z \mapsto \tilde{\mathbf{f}}\left(\overline{z}\right)(-\mathbf{j})$ is an analytic extension of $R_z(T)\mathbf{v}$. Since this function is defined on $\overline{\rho(\mathbf{v}\mathbf{j})}$, we find $\rho(\mathbf{v}) \supset \overline{\rho(\mathbf{v}\mathbf{j})}$ and in turn $\sigma(\mathbf{v}) \subset \overline{\sigma(\mathbf{v}\mathbf{j})}$. Altogether, we obtain $\sigma(\mathbf{v}) = \overline{\sigma(\mathbf{v}\mathbf{j})}$ and $\tilde{\mathbf{f}}(z) = \mathbf{f}\left(\overline{z}\right)\mathbf{j}$. From (10.19) we deduce

$$\begin{aligned} \operatorname{ran} E_{\mathsf{i}}\left(\overline{\Delta}\right) &= \left\{\mathbf{v} \in V_{R,\mathsf{i}} = V_R : \sigma(\mathbf{v}) \subset \overline{\Delta}\right\} \\ &= \left\{\mathbf{v}\mathbf{j} \in V_{R,\mathsf{i}} = V_R : \sigma(\mathbf{v}) \subset \Delta\right\} = \left(\operatorname{ran} E_{\mathsf{i}}(\Delta)\right)\mathbf{j}. \end{aligned} \tag{10.20}$$

In order to construct the quaternionic spectral resolution of T, we define now

$$E(\Delta) := E_{\mathsf{i}}(\Delta \cap \mathbb{C}_{\mathsf{i}}), \qquad \forall \Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H}).$$

Obviously this operator is a bounded $\mathbb{C}_{\mathsf{i}}$-linear projection on $V_R = V_{R,\mathsf{i}}$. We show now that it is also quaternionic linear. Due to the axial symmetry of Δ, the identity (10.20) implies

$$(\operatorname{ran} E(\Delta))\mathbf{j} = \left(\operatorname{ran} E_{\mathsf{i}}(\Delta \cap \mathbb{C}_{\mathsf{i}})\right)\mathbf{j} = \operatorname{ran} E_{\mathsf{i}}\left(\overline{\Delta \cap \mathbb{C}_{\mathsf{i}}}\right) = \operatorname{ran} E_{\mathsf{i}}\left(\Delta \cap \mathbb{C}_{\mathsf{i}}\right) = \operatorname{ran} E(\Delta).$$

Similarly we find

$$\begin{aligned} (\ker E(\Delta))\mathbf{j} &= (\ker E_{\mathsf{i}}(\Delta \cap \mathbb{C}_{\mathsf{i}}))\mathbf{j} = (\operatorname{ran} E_{\mathsf{i}}(\mathbb{C}_{\mathsf{i}} \setminus \Delta))\mathbf{j} = \operatorname{ran} E_{\mathsf{i}}\left(\overline{\mathbb{C}_{\mathsf{i}} \setminus \Delta}\right) \\ &= \operatorname{ran} E_{\mathsf{i}}\left(\mathbb{C}_{\mathsf{i}} \setminus \Delta\right) = \ker E_{\mathsf{i}}(\Delta \cap \mathbb{C}_{\mathsf{i}}) = \ker E(\Delta). \end{aligned}$$

If we write $\mathbf{v} \in V_R$ as $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ with $\mathbf{v}_0 \in \ker E(\Delta)$ and $\mathbf{v}_1 \in \operatorname{ran} E(\Delta)$, we thus have

$$E(\Delta)(\mathbf{v}\mathbf{j}) = E(\Delta)(\mathbf{v}_0\mathbf{j}) + E(\Delta)(\mathbf{v}_1\mathbf{j}) = \mathbf{v}_1\mathbf{j} = (E(\Delta)\mathbf{v})\mathbf{j}.$$

Writing $a \in \mathbb{H}$ as $a = a_1 + \mathbf{j}a_2$ with $a_1, a_2 \in \mathbb{C}_{\mathsf{i}}$, we find due to the $\mathbb{C}_{\mathsf{i}}$-linearity of $E(\Delta)$ that even

$$\begin{aligned} E(\Delta)(\mathbf{v}a) =&(E(\Delta)\mathbf{v})a_1 + (E(\Delta)\mathbf{v}\mathbf{j})a_2 \\ =&(E(\Delta)\mathbf{v})a_1 + (E(\Delta)\mathbf{v})\mathbf{j}a_2 = (E(\Delta)\mathbf{v})a. \end{aligned}$$

Hence, the set function $\Delta \mapsto E(\Delta)$ defined in (10.18) takes values that are bounded quaternionic linear projections on V_R. It is immediate that it moreover satisfies (i) to (iv) in Definition 9.7 because E_{i} is a spectral measure on $V_{R,\mathsf{i}}$ and hence has the respective properties. Consequently, E is a quaternionic spectral measure. Since E_{i} commutes with T, also E commutes with T. From Theorem 8.4 and the fact that $\sigma(T|_{\operatorname{ran} E_{\mathsf{i}}(\Delta_{\mathsf{i}})}) \subset cl(\Delta_{\mathsf{i}})$ for $\Delta_{\mathsf{i}} \in \mathsf{B}(\mathbb{C}_{\mathsf{i}})$, we deduce for $T_\Delta = T|_{\operatorname{ran} E(\Delta)} = T|_{\operatorname{ran} E_{\mathsf{i}}(\Delta \cap \mathbb{C}_{\mathsf{i}})}$ that

$$\sigma_S(T_\Delta) = [\sigma_{\mathbb{C}_{\mathsf{i}}}(T_\Delta)] \subset [cl(\Delta \cap \mathbb{C}_{\mathsf{i}})] = cl([\Delta \cap \mathbb{C}_{\mathsf{i}}]) = cl(\Delta).$$

Therefore E is a spectral resolution for T.

Let us now set $V_0 = \operatorname{ran} E_{\mathsf{i}}(\mathbb{R})$, $V_+ := \operatorname{ran} E_{\mathsf{i}}\left(\mathbb{C}_{\mathsf{i}}^+\right)$ and $V_- := \operatorname{ran} E_{\mathsf{i}}\left(\mathbb{C}_{\mathsf{i}}^-\right)$. Then $V_{R,\mathsf{i}} = V_0 \oplus V_+ \oplus V_-$ is a decomposition of V_R into closed $\mathbb{C}_{\mathsf{i}}$-linear subspaces. The

space $V_0 = \operatorname{ran} E_{\mathsf{i}}(\mathbb{R}) = \operatorname{ran} E(\mathbb{R})$ is even a quaternionic right linear subspace of V_R because $E(\mathbb{R})$ is a quaternionic right linear operator. Moreover (10.20) shows that $\mathbf{v} \mapsto \mathbf{v}\mathbf{j}$ is a bijection from V_+ to V_-. By Theorem 9.18 the operator

$$J\mathbf{v} = E_{\mathsf{i}}\left(\mathbb{C}_{\mathsf{i}}^{+}\right)\mathbf{v}\mathbf{i} + E_{\mathsf{i}}\left(\mathbb{C}_{\mathsf{i}}^{-}\right)\mathbf{v}(-\mathbf{i})$$

is an imaginary operator on V_R. Since E_{i} commutes with T and $E(\Delta)$ for $\Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$, also J commutes with T and $E(\Delta)$. Moreover $\ker J = V_0 = \operatorname{ran} E(\mathbb{R})$ and $\operatorname{ran} J = \operatorname{ran} E_{\mathsf{i}}(\mathbb{C}_{\mathsf{i}}^{+}) \oplus \operatorname{ran} E_{\mathsf{i}}(\mathbb{C}_{\mathsf{i}}^{-}) = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$ and hence (E, J) is a spectral system that commutes with T. Finally, we have $\sigma(T_+) \subset \mathbb{C}_{\mathsf{i}}^{\geq}$ for $T_+ = T|_{V_+} = T|_{\operatorname{ran} E_{\mathsf{i}}(\mathbb{C}_{\mathsf{i}}^{+})}$ and hence the resolvent $R_z(T_+)$ of T_+ exists for any $z \in \mathbb{C}_{\mathsf{i}}^{-}$. Similarly, the resolvent $R_z(T_-)$ of $T_- = T|_{V_-} = T|_{\operatorname{ran} E_{\mathsf{i}}(\mathbb{C}_{\mathsf{i}}^{-})}$ exists for any $z \in \mathbb{C}_{\mathsf{i}}^{+}$. For $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$ we can hence set $s_{\mathsf{i}} = s_0 + \mathbf{i}s_1$ and define by

$$R(s_0, s_1) := \left(R_{\overline{s_{\mathsf{i}}}}(T_+)E_+ + R_{s_{\mathsf{i}}}(T_-)E_-\right)|_{V_+ \oplus V_-}$$

with $E_+ = E_{\mathsf{i}}(\mathbb{C}_{\mathsf{i}}^{+})$ and $E_- = E_{\mathsf{i}}(\mathbb{C}_{\mathsf{i}}^{-})$ a bounded operator on $V_+ \oplus V_- = \operatorname{ran} E(\mathbb{H}\setminus\mathbb{R})$. Since T leaves V_+ and V_- invariant, we then have for $\mathbf{v} = \mathbf{v}_+ + \mathbf{v}_- \in V_+ \oplus V_-$ that

$$\begin{aligned}
&R(s_0, s_1)(s_0\mathcal{I} - s_1 J - T)\mathbf{v} \\
=&R(s_0, s_1)\left(\mathbf{v}_+ s_0 - J\mathbf{v}_+ s_1 - T\mathbf{v}_+ + \mathbf{v}_- s_0 - J\mathbf{v}_- s_1 - T\mathbf{v}_-\right) \\
=&R(s_0, s_1)\left(\mathbf{v}_+\overline{s_{\mathsf{i}}} - T\mathbf{v}_+\right) + R(s_0, s_1)\left(\mathbf{v}_- s_{\mathsf{i}} - T\mathbf{v}_-\right) \\
=&R_{\overline{s_{\mathsf{i}}}}(T_+)\left(\mathbf{v}_+\overline{s_{\mathsf{i}}} - T_+\mathbf{v}_+\right) + R_{s_{\mathsf{i}}}(T_-)\left(\mathbf{v}_- s_{\mathsf{i}} - T_-\mathbf{v}_-\right) = \mathbf{v}_+ + \mathbf{v}_- = \mathbf{v}.
\end{aligned}$$

Similarly we find that

$$\begin{aligned}
&(s_0\mathcal{I} - s_1 J - T)R(s_0, s_1)\mathbf{v} = \\
=&(s_0\mathcal{I} - s_1 J - T)R_{\overline{s_{\mathsf{i}}}}(T_+)\mathbf{v}_+ + (s_0\mathcal{I} - s_1 J - T)R_{s_{\mathsf{i}}}(T_-)\mathbf{v}_- \\
=&R_{\overline{s_{\mathsf{i}}}}(T_+)\mathbf{v}_+ s_0 - J(R_{\overline{s_{\mathsf{i}}}}(T_+)\mathbf{v}_+)s_1 - TR_{\overline{s_{\mathsf{i}}}}(T_+)\mathbf{v}_+ \\
&+ R_{s_{\mathsf{i}}}(T_-)\mathbf{v}_- s_0 - J(R_{s_{\mathsf{i}}}(T_-)\mathbf{v}_-)s_1 - TR_{s_{\mathsf{i}}}(T_-)\mathbf{v}_- \\
=&R_{\overline{s_{\mathsf{i}}}}(T_+)\mathbf{v}_+(s_0 - \mathbf{i}s_1) - R_{\overline{s_{\mathsf{i}}}}(T_+)T_+\mathbf{v}_+ \\
&+ R_{s_{\mathsf{i}}}(T_-)\mathbf{v}_-(s_0 + \mathbf{i}s_1) - R_{s_{\mathsf{i}}}(T_-)T_-\mathbf{v}_- \\
=&R_{\overline{s_{\mathsf{i}}}}(T_+)\left(\mathbf{v}_+\overline{s} - T_+\mathbf{v}_+\right) + R_{s_{\mathsf{i}}}(T_-)\left(\mathbf{v}_- s - T_-\mathbf{v}_-\right) = \mathbf{v}_+ + \mathbf{v}_- = \mathbf{v}.
\end{aligned}$$

Hence $R(s_0, s_1)$ is the bounded inverse of $(s_0\mathcal{I} - s_1 J - T)|_{\operatorname{ran} E(\mathbb{H}\setminus\mathbb{R})}$ and so J is actually a spectral orientation for T. Consequently, T is a quaternionic spectral operator and the relation (10.17) holds true.

□

Remark 10.20. We want to stress that Theorem 10.19 showed a one-to-one relation between quaternionic spectral operators on V_R and $\mathbb{C}_{\mathsf{i}}$-complex spectral operators on $V_{R,\mathsf{i}}$ that are furthermore compatible with the quaternionic scalar multiplication. It did not show a one-to-one relation between quaternionic spectral operators on V_R and $\mathbb{C}_{\mathsf{i}}$-complex spectral operators on $V_{R,\mathsf{i}}$. There exist $\mathbb{C}_{\mathsf{i}}$-complex spectral operators on $V_{R,\mathsf{i}}$ that are not quaternionic linear and can hence not be quaternionic spectral operators.

10.2 Canonical Reduction and Intrinsic S-Functional Calculus

As in the complex case any bounded quaternionic spectral operator T can be decomposed into the sum $T = S + N$ of a scalar operator S and a quasi-nilpotent operator N. The intrinsic S-functional calculus for a spectral operator can then be expressed as a Taylor series similar to the one in [22] that involves functions of S obtained via spectral integration and powers of N. Analogue to the complex case in [38], the operator $f(T)$ is therefore already determined by the values of f on $\sigma_S(T)$ and not only by its values on a neighborhood of $\sigma_S(T)$.

Definition 10.21. An operator $S \in \mathcal{B}(V_R)$ is said to be of scalar type if it is a spectral operator and satisfies the identity

$$S = \int s \, dE_J(s), \tag{10.21}$$

where (E, J) is the spectral decomposition of S.

Remark 10.22. If we start from a spectral system (E, J) and S is the operator defined by (10.21), then S is an operator of scalar type and (E, J) is its spectral decomposition. This can easily be checked by direct calculations or indirectly via the following argument: by Lemma 9.26, we can choose $\mathsf{i} \in \mathbb{S}$ and find

$$S = \int_{\mathbb{H}} s \, dE_J(s) = \int_{\mathbb{C}_{\mathsf{i}}} z \, dE_{\mathsf{i}}(z),$$

where E_{i} is the spectral measure constructed in (9.15). From the complex theory in [38], we deduce that S is a spectral operator on $V_{R,\mathsf{i}}$ with spectral decomposition E_{i} that is furthermore quaternionic linear. By Theorem 10.19 this is equivalent to S being a quaternionic spectral operator on V_R with spectral decomposition (E, J).

Lemma 10.23. *Let S be an operator of scalar type with spectral decomposition (E, J). An operator $A \in \mathcal{B}(V_R)$ commutes with S if and only if it commutes with the spectral system (E, J).*

Proof. If $A \in \mathcal{B}(V_R)$ commutes with (E, J) then it commutes with $S = \int_{\mathbb{H}} s \, dE_J(s)$ because of Lemma 9.24. If on the other hand A commutes with S, then it also commutes with E by Lemma 10.12. By Lemma 9.10 it commutes in turn with the operator $f(T) = \int_{\mathbb{H}} f(s) \, dE(s)$ for any $f \in \mathcal{M}_s^{\infty}(\mathbb{H}, \mathbb{R})$. If we define

$$S_0 := \int_{\mathbb{H}} \operatorname{Re}(s) \, dE(s) \qquad \text{and} \qquad S_1 := \int_{\mathbb{H}} \underline{s} \, dE_J(S) = J \int_{\mathbb{H}} |\underline{s}| \, dE(s),$$

where $\underline{s} = \mathsf{i}_s s_1$ denotes the imaginary part of a quaternion s, then $AS = SA$ and $AS_0 = S_0 A$ and in turn

$$AS_1 = A(S - S_0) = AS - AS_0 = SA - S_0 A = (S - S_0)A = S_1 A.$$

We can now choose pairwise disjoint sets $\Delta_n \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$, $n \in \mathbb{N}$, such that $\sigma_S(T) \setminus \mathbb{R} = \bigcup_{n \in \mathbb{N}} \Delta_n$ and such that $\operatorname{dist}(\Delta_n, \mathbb{R}) > 0$ for any $n \in \mathbb{N}$. Then $s \mapsto |\underline{s}|^{-1} \chi_{\Delta_n}(s)$ belongs

to $\mathcal{M}_s^\infty(\mathbb{H},\mathbb{R})$ for any $n \in \mathbb{N}$ and in turn

$$\begin{aligned}
AJE(\Delta_n) =& AJ\left(\int_{\mathbb{H}} |\underline{s}||\underline{s}|^{-1}\chi_{\Delta_n}(s)\,dE(s)\right)E(\Delta_n)\\
=& AJ\left(\int_{\mathbb{H}} |\underline{s}|\,dE(s)\right)\left(\int_{\mathbb{H}} |\underline{s}|^{-1}\chi_{\Delta_n}(s)\,dE(s)\right)E(\Delta_n)\\
=& AS_1\left(\int_{\mathbb{H}} |\underline{s}|^{-1}\chi_{\Delta_n}(s)\,dE(s)\right)E(\Delta_n)\\
=& S_1\left(\int_{\mathbb{H}} |\underline{s}|^{-1}\chi_{\Delta_n}(s)\,dE(s)\right)E(\Delta_n)A\\
=& J\left(\int_{\mathbb{H}} |\underline{s}|\,dE(s)\right)\left(\int_{\mathbb{H}} |\underline{s}|^{-1}\chi_{\Delta_n}(s)\,dE(s)\right)E(\Delta_n)A\\
=& J\left(\int_{\mathbb{H}} |\underline{s}||\underline{s}|^{-1}\chi_{\Delta_n}(s)\,dE(s)\right)E(\Delta_n)A = JE(\Delta_n)A.
\end{aligned}$$

Since $\sigma_S(S)\setminus\mathbb{R} \subset \bigcup_{n\in\mathbb{N}}\Delta_n$, we have $\sum_{n=0}^{+\infty} E(\Delta_n)\mathbf{v} = E(\sigma_S(T)\setminus\mathbb{R})\mathbf{v} = E(\mathbb{H}\setminus\mathbb{R})\mathbf{v}$ for all $\mathbf{v} \in V_R$ by Corollary 10.10. As $J = JE(\mathbb{H}\setminus\mathbb{R})$, we hence find

$$\begin{aligned}
AJ\mathbf{v} =& AJE(\mathbb{H}\setminus\mathbb{R})\mathbf{v} = \sum_{n=1}^{+\infty} AJE(\Delta_n)\mathbf{v}\\
=& \sum_{n=1}^{+\infty} JE(\Delta_n)A\mathbf{v} = JE(\mathbb{H}\setminus\mathbb{R})A\mathbf{v} = JA\mathbf{v},
\end{aligned}$$

which finishes the proof.

□

Definition 10.24. An operator $N \in \mathcal{B}(V_R)$ is called quasi-nilpotent if

$$\lim_{n\to\infty} \|N^n\|^{\frac{1}{n}} = 0. \tag{10.22}$$

The following corollaries are immediate consequences of Gelfand's formula

$$r(T) = \lim_{n\to+\infty} \|T^n\|^{\frac{1}{n}},$$

for the spectral radius $r(T) = \max_{s\in\sigma_S(T)} |s|$ of T and of (v) of Lemma 8.14.

Corollary 10.25. *An operator $N \in \mathcal{B}(V_R)$ is quasi-nilpotent if and only if $\sigma_S(T) = \{0\}$.*

Corollary 10.26. *Let $S, N \in \mathcal{B}(V_R)$ be commuting operators and let N be quasi-nilpotent. Then $\sigma_S(S+N) = \sigma_S(S)$.*

We are now ready to show the main result of this section: the canonical reduction of a spectral operator, the quaternionic analogue of Theorem 5 in [38, Chapter XV.4.3].

Theorem 10.27. *An operator $T \in \mathcal{B}(V_R)$ is a spectral operator if and only if it is the sum $T = S + N$ of a bounded operator S of scalar type and a quasi-nilpotent operator N that commutes with S. Furthermore, this decomposition is unique and T and S have the same S-spectrum and the same spectral decomposition (E, J).*

Proof. Let us first show that any operator $T \in \mathcal{B}(V_R)$ that is the sum $T = S + N$ of an operator S of scalar type and a quasi-nilpotent operator N that commutes with S is a spectral operator. If (E, J) is the spectral decomposition of S, then Lemma 10.23 implies $E(\Delta)N = NE(\Delta)$ for all $\Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H})$ and $JN = NJ$. Since $T = S + N$, we find that also T commutes with (E, J).

Let now $\Delta \in \mathsf{B}_\mathsf{S}(\mathbb{H})$. Then $T_\Delta = S_\Delta + N_\Delta$, where as usual the subscript Δ denotes the restriction of an operator to $V_\Delta = E(\Delta)V_R$. Since N_Δ inherits the property of being quasi-nilpotent from N and commutes with S_Δ, we deduce from Corollary 10.26 that

$$\sigma_S(T_\Delta) = \sigma_S(S_\Delta + N_\Delta) = \sigma_S(S_\Delta) \subset cl(\Delta).$$

Thus (E, J) satisfies (i) and (ii) of Definition 10.1. It remains to show that also (iii) holds true. Therefore let $V_0 = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$ and set $T_0 = T|_{V_0}$, $S_0 = S|_{V_0}$, $N_0 = N|_{V_0}$ and $J_0 = J|_{V_0}$ and choose $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$. Since (E, J) is the spectral resolution of S, the operator $s_0\mathcal{I}_{V_0} - s_1 J_0 - S_0$ has a bounded inverse $R(s_0, s_1) = (s_0\mathcal{I}_{V_0} - s_1 J_0 - S_0)^{-1} \in \mathcal{B}(V_0)$. The operator N_0 is quasi-nilpotent because N is quasi-nilpotent and hence it satisfies (10.22). The root test thus shows the convergence of the series $\sum_{n=0}^{+\infty} N_0^n R(s_0, s_1)^{n+1}$ in $\mathcal{B}(V_0)$. Since T_0, N_0, S_0 and J_0 commute mutually, we have

$$\begin{aligned}
&(s_0\mathcal{I}_{V_0} - s_1 J_0 - T_0) \sum_{n=0}^{+\infty} N_0^n R(s_0, s_1)^{n+1} \\
&= \sum_{n=0}^{+\infty} N_0^n R(s_0, s_1)^{n+1} (s_0\mathcal{I}_{V_0} - s_1 J_0 - S_0 - N_0) \\
&= \sum_{n=0}^{+\infty} N_0^n R(s_0, s_1)^{n+1} (s_0\mathcal{I}_{V_0} - s_1 J_0 - S_0) - \sum_{n=0}^{+\infty} N_0^n R(s_0, s_1)^{n+1} N_0 \\
&= \sum_{n=0}^{+\infty} N_0^n R(s_0, s_1)^{n} - \sum_{n=0}^{+\infty} N_0^{n+1} R(s_0, s_1)^{n+1} = \mathcal{I}_{V_0}.
\end{aligned}$$

We find that $s_0\mathcal{I}_0 - s_1 J_0 - T_0$ has a bounded inverse for $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$ such that J is a spectral orientation for T. Hence, T is a spectral operator and T and S have the same spectral decomposition (E, J).

Since the spectral decomposition of T is uniquely determined, $S = \int_{\mathbb{H}} s \, dE_J(s)$ and in turn also $N = T - S$ are uniquely determined. Moreover, Corollary 10.26 implies that $\sigma_S(T) = \sigma_S(S)$.

Now assume that T is a spectral operator and let (E, J) be its spectral decomposition. We set

$$S := \int_{\mathbb{H}} s \, dE_J(s) \qquad \text{and} \qquad N := T - S.$$

By Remark 10.22 the operator S is of scalar type and its spectral decomposition is (E, J). Since T commutes with (E, J), it commutes with S by Lemma 10.23. Consequently, $N = T - S$ also commutes with S and with T. What remains to show is that N is quasi-nilpotent. In view of Corollary 10.25, it is sufficient to show that $\sigma_S(N)$ is for any $\varepsilon > 0$ contained in the open ball $B_\varepsilon(0)$ of radius ε centered at 0 .

For arbitrary $\varepsilon > 0$, we choose $\alpha > 0$ such that $0 < (1 + C_{E,J})\alpha < \varepsilon$, where $C_{E,J} > 0$ is the constant in (9.14). We decompose $\sigma_S(T)$ into the union of disjoint

axially symmetric Borel sets $\Delta_1, \dots, \Delta_n \in \mathsf{B}_{\mathsf{S}}(\mathbb{H})$ such that for each $\ell \in \{1, \dots, n\}$ the set Δ_ℓ is contained in a closed axially symmetric set, whose intersection with any complex halfplane is a half-disk of diameter α. More precisely, we assume that there exist points $s_1, \dots, s_n \in \mathbb{H}$ such that for all $\ell = 1, \dots, n$

$$\Delta_\ell \subset B_\alpha^+([s_\ell]) = \{p \in \mathbb{H} : \operatorname{dist}(p, [s_\ell]) \leq \alpha \text{ and } p_1 \geq s_{\ell,1}\}.$$

Observe that we have either $s_\ell \in \mathbb{R}$ or $B_\alpha^+([s_\ell]) \cap \mathbb{R} = \emptyset$.

We set $V_{\Delta_\ell} = E(\Delta_\ell)V_R$. As T and S commute with $E(\Delta_\ell)$, also $N = T - S$ does and so $NV_{\Delta_\ell} \subset V_{\Delta_\ell}$. Hence, $N_{\Delta_\ell} = N|_{V_{\Delta_\ell}} \in \mathcal{B}(V_{\Delta_\ell})$. If s belongs to $\rho_S(N_{\Delta_\ell})$ for all $\ell \in \{1, \dots, n\}$, we can set

$$\mathcal{Q}(s)^{-1} := \sum_{\ell=1}^{n} \mathcal{Q}_s(N_{\Delta_\ell})^{-1} E(\Delta_\ell),$$

where

$$\mathcal{Q}_s(N_{\Delta_\ell})^{-1} = \left(N_{\Delta_\ell}^2 - 2s_0 N_{\Delta_\ell} + |s|^2 \mathcal{I}_{V_{\Delta_\ell}}\right)^{-1} \in \mathcal{B}(V_{\Delta_\ell})$$

is the pseudo-resolvent of N_{Δ_ℓ} at s. The operator $\mathcal{Q}(s)^{-1}$ commutes with $E(\Delta_\ell)$ for any $\ell \in \{1, \dots, n\}$ such that

$$\begin{aligned}&(N^2 - 2s_0 N + |s|^2 \mathcal{I}_{V_R})\mathcal{Q}(s)^{-1}\\ =& \sum_{\ell=1}^{n} (N_{\Delta_\ell}^2 - 2s_0 N_{\Delta_\ell} + |s|^2 \mathcal{I}_{V_{\Delta_\ell}}) \mathcal{Q}_s(N_{\Delta_\ell})^{-1} E(\Delta_\ell) = \sum_{\ell=1}^{n} E(\Delta_\ell) = \mathcal{I}_{V_R}\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{Q}(s)^{-1}(N^2 - 2s_0 N + |s|^2 \mathcal{I}_{V_R}) = \\
=& \sum_{\ell=1}^{n} \mathcal{Q}_s(N_{\Delta_\ell})^{-1} E(\Delta_\ell)(N^2 - 2s_0 N + |s|^2 \mathcal{I}_{V_R}) \\
=& \sum_{\ell=1}^{n} \mathcal{Q}_s(N_{\Delta_\ell})^{-1} (N_{\Delta_\ell}^2 - 2s_0 N_{\Delta_\ell} + |s|^2 \mathcal{I}_{V_{\Delta_\ell}}) E(\Delta_\ell) \\
=& \sum_{\ell=1}^{n} E(\Delta_\ell) = \mathcal{I}_{V_R}.
\end{aligned}$$

Therefore, we find $s \in \rho_S(N)$ such so we have $\bigcap_{\ell=1}^{n} \rho_S(N_{\Delta_\ell}) \subset \rho_S(N)$ and in turn $\sigma_S(N) \subset \bigcup_{\ell=1}^{n} \sigma_S(N_{\Delta_\ell})$. It is hence sufficient to show that $\sigma_S(N_{\Delta_\ell}) \subset B_\varepsilon(0)$ for all $\ell = 1, \dots, n$.

We distinguish two cases: if $s_\ell \in \mathbb{R}$, then we write

$$N_{\Delta_\ell} = (T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}) + (s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell}).$$

Since $s_\ell \in \mathbb{R}$, we have for $p \in \mathbb{H}$ that

$$\begin{aligned}
&\mathcal{Q}_p(T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}) = \\
=&(T_{\Delta_\ell}^2 - 2s_\ell T_{\Delta_\ell} + s_\ell^2 \mathcal{I}_{V_{\Delta_\ell}} - 2p_0(T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}) + (p_0^2 + p_1^2)\mathcal{I}_{V_{\Delta_\ell}} \\
=&T_{\Delta_\ell}^2 - 2(p_0 - s_\ell)T_{\Delta_\ell} + \left((p_0 - s_\ell)^2 + p_1^2\right) \mathcal{I}_{V_{\Delta_\ell}} = \mathcal{Q}_{p - s_\ell}(T_{\Delta_\ell})
\end{aligned}$$

and thus

$$\begin{aligned}\sigma_S(T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_\ell}) =& \{p - s_\ell \in \mathbb{H} : p \in \sigma_S(T_{\Delta_\ell})\}\\ \subset& \{p - s_\ell \in \mathbb{H} : p \in B_\alpha^+(s_\ell)\} = B_\alpha(0).\end{aligned} \tag{10.23}$$

Moreover, the function $f(s) = (s_\ell - s)\chi_{\Delta_\ell}(s)$ is an intrinsic slice function because $s_\ell \in \mathbb{R}$. As it is bounded, its integral with respect to (E, J) is defined and

$$s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell} = \left(\int_{\mathbb{H}} (s_\ell - s)\chi_{\Delta_\ell}(s)\, dE_J(s) \right)\Big|_{V_{\Delta_\ell}}.$$

We thus have

$$\|s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell}\| \leq C_{E,J} \,\|(s_\ell - s)\chi_{\Delta_\ell}(s)\|_\infty \leq C_{E,J}\alpha \tag{10.24}$$

because $\Delta_\ell \subset B_\alpha^+([s_\ell]) = cl(B_\alpha(s_\ell))$. Since the operator $T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}$ and the operator $s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell}$ commute, we conclude from (v) in Lemma 8.14 together with (10.23) and (10.24) that

$$\begin{aligned}&\sigma_S(T_{\Delta\ell}) = \sigma_S\Big((T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}) + (s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell}) \Big)\\ &\subset \Big\{ s \in \mathbb{H} : \operatorname{dist}\Big(s, \sigma_S\big(T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}\big)\Big) \leq C_{E,J}\alpha \Big\} \subset B_{\alpha(1+C_{E,J})}(0) \subset B_\varepsilon(0).\end{aligned}$$

If $s_\ell \notin \mathbb{R}$, then let us write

$$N_{\Delta_\ell} = (T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1} J_{\Delta_\ell}) + (s_\ell \mathcal{I}_{V_{\Delta_\ell}} + s_{\ell,1} J_{\Delta_\ell} - S_{\Delta_\ell}) \tag{10.25}$$

with $J_{\Delta_\ell} = J|_{V_{\Delta_\ell}}$. Since $E(\Delta_\ell)$ and J commute, J_{Δ_ℓ} is an imaginary operator on V_{Δ_ℓ} and it moreover commutes with T_{Δ_ℓ}. Since $-J_{\Delta_\ell}^2 = -J^2|_{V_\Delta} = E(\mathbb{H} \setminus \mathbb{R})|_{V_{\Delta_\ell}} = \mathcal{I}_{V_{\Delta_\ell}}$ as $\Delta_\ell \subset \mathbb{H} \setminus \mathbb{R}$, we find for $s = s_0 + \mathbf{i}_s s_1 \in \mathbb{H}$ with $s_1 \geq 0$ that

$$\begin{aligned}&\big(s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T_{\Delta_\ell}\big)\big(s_0 \mathcal{I}_{V_{\Delta_\ell}} - s_1 J_{\Delta_\ell} - T_{\Delta_\ell}\big) =\\ &= s_0^2 - s_1^2 J_{\Delta_\ell}^2 - 2 s_0 T_{\Delta_\ell} + T_{\Delta_\ell}^2 = \mathcal{Q}_s(T_{\Delta_\ell}).\end{aligned} \tag{10.26}$$

Because of condition (iii) in Definition 10.1, the operator $(s_0\mathcal{I} - s_1 J - T)|_{\operatorname{ran} E(\mathbb{H}\setminus\mathbb{R})}$ is invertible if $s_1 > 0$. Since this operator commutes with $E(\Delta_\ell)$, the restriction of its inverse to V_{Δ_ℓ} is the inverse of $(s_0 \mathcal{I}_{V_{\Delta_\ell}} - s_1 J_{\Delta_\ell} - T_{\Delta_\ell})$ in $\mathcal{B}(V_{\Delta_\ell})$. Hence, if $s_1 > 0$, then $(s_0 \mathcal{I}_{V_{\Delta_\ell}} - s_1 J_{\Delta_\ell} - T_{\Delta_\ell})^{-1} \in \mathcal{B}(V_{\Delta_\ell})$ and we conclude from (10.26) that

$$\big(s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T_{\Delta_\ell}\big)^{-1} \in \mathcal{B}(V_{\Delta_\ell}) \quad \Longleftrightarrow \quad \mathcal{Q}_s(T_{\Delta_\ell})^{-1} \in \mathcal{B}(V_{\Delta_\ell}). \tag{10.27}$$

If on the other hand $s_1 = 0$, then both factors on the left-hand side of (10.26) agree and so (10.27) holds true also in this case. Hence, $s \in \rho_S(T_{\Delta_\ell})$ if and only if the operator $(s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T)$ has an inverse in $\mathcal{B}(V_{\Delta_\ell})$. Since

$$\sigma_S(T_{\Delta_\ell}) \subset \overline{\Delta_\ell} \subset B_\alpha^+([s_\ell]) \subset \{s = s_0 + \mathbf{i}_s s_1 \in \mathbb{H} : s_1 \geq s_{\ell,1}\},$$

the operator $s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T_{\Delta_\ell}$ is in particular invertible for any quaternion $s \in \mathbb{H}$ with $0 \leq s_1 < s_{\ell,1}$. As J_{Δ_ℓ} is a spectral orientation for T_{Δ_ℓ}, this operator is also invertible if $s_1 < 0$ and hence we even find

$$(s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T_{\Delta_\ell})^{-1} \in \mathcal{B}(V_{\Delta_\ell}) \quad \forall s_0, s_1 \in \mathbb{R} : s_1 < s_{\ell,1}. \tag{10.28}$$

We can use these observations to deduce a spectral mapping property: a straight forward computation using the facts that T_{Δ_ℓ} and J_{Δ_ℓ} commute and that $J_{\Delta_\ell}^2 = -\mathcal{I}_{V_{\Delta_\ell}}$ shows

$$\begin{aligned}
&\mathcal{Q}_s(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell}) \\
&= \Big((s_0 + s_{\ell,0})\mathcal{I}_{V_{\Delta_\ell}} + (s_1 + s_{\ell,1})J_{\Delta_\ell} - T_{\Delta_\ell}\Big) \\
&\quad \cdot \Big((s_0 + s_{\ell,0})\mathcal{I}_{V_{\Delta_\ell}} + (s_{\ell,1} - s_1)J_{\Delta_\ell} - T_{\Delta_\ell}\Big).
\end{aligned} \tag{10.29}$$

If $s_1 > 0$ then the second factor is invertible because of (10.28). Hence, we have $s \in \rho_S(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell})$ if and only if the first factor in (10.29) is invertible, i.e. if and only if

$$\big((s_0 + s_{\ell,0})\mathcal{I}_{V_{\Delta_\ell}} + (s_1 + s_{\ell,1})J_{\Delta_\ell} - T_{\Delta_\ell}\big)^{-1} \in \mathcal{B}(V_{\Delta_\ell}). \tag{10.30}$$

If on the other hand $s_1 = 0$, then both factors in (10.29) agree. Hence, also in this case, s belongs to $\rho_S(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell})$ if and only if the operator in (10.30) exists. By (10.27), the existence of (10.30) is however equivalent to

$$s_0 + s_{\ell,0} + (s_1 + s_{\ell,1})\mathbb{S} \subset \rho_S(T_\Delta)$$

so that

$$\rho_S(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell}) = \{s \in \mathbb{H} : s_0 + s_{\ell,0} + (s_1 + s_{\ell,1})\mathbf{i}_s \in \rho_S(T_{\Delta_\ell})\}$$

and in turn

$$\begin{aligned}
\sigma_S(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell}) &= \{s \in \mathbb{H} : s_0 + s_{\ell,1} + (s_1 + s_{\ell,1})\mathbf{i}_s \in \sigma_S(T_{\Delta_\ell})\} \\
&\subset \{s \in \mathbb{H} : s_0 + s_{\ell,0} + (s_1 + s_{\ell,1})\mathbf{i}_s \in B_\alpha^+(s_\ell)\} = B_\alpha(0).
\end{aligned}$$

For the second operator in (10.25), we have again

$$s_\ell\mathcal{I}_{V_{\Delta_\ell}} + s_{\ell,1}J_{\Delta_\ell} - S_{\Delta_\ell} = \left(\int_{\mathbb{H}} (s_{\ell,0} + i_s s_{\ell,1} - s)\chi_{\Delta_\ell}(s)\, dE_J(s)\right)\Bigg|_{V_{\Delta_\ell}}$$

and so

$$\|s_\ell\mathcal{I}_{V_{\Delta_\ell}} + s_{\ell,1}J_{\Delta_\ell} - S_{\Delta_\ell}\| \leq C_{E,J}\|(s_{\ell,0} + i_s s_{\ell,1} - s)\chi_{\Delta_\ell}(s)\|_\infty \leq C_{E,J}\,\alpha.$$

Since the operators $T_{\Delta_\ell} - s_\ell\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell}$ and $s_\ell\mathcal{I}_{V_{\Delta_\ell}} + s_{\ell,1}J_{\Delta_\ell} - S_{\Delta_\ell}$ commute, we conclude as before from (v) in Lemma 8.14 that $\sigma_S(T_{\Delta_\ell}) \subset B_{\alpha(1+C_{E,J})}(0) \subset B_\varepsilon(0)$.

Altogether, we obtain that N is quasi-nilpotent, which concludes the proof.

□

Definition 10.28. Let $T \in \mathcal{B}(V_R)$ be a spectral operator and decompose $T = S + N$ as in Theorem 10.27. The scalar operator S is called the scalar part of T and the quasi-nilpotent operator N is called the radical part of T.

Remark 10.29. Let $T \in \mathcal{B}(V_R)$ be a spectral operator. The canonical decomposition of T into its scalar part and its radical part obviously coincides for any $\mathbf{i} \in \mathbb{S}$ with the canonical decomposition of T as a $\mathbb{C}_{\mathbf{i}}$-linear spectral operator on $V_{\mathbf{i}}$.

The remainder of this section discusses the S-functional calculus for spectral operators. Similar to the complex case, one can express $f(T)$ for any intrinsic function f as a formal Taylor series in the radical part N of T. The Taylor coefficients are spectral integrals of f with respect to the spectral decomposition of T. Hence these coefficients, and in turn also $f(T)$, do only depend on the values of f on the S-spectrum $\sigma_S(T)$ of T and not on the values of f on an entire neighborhood of $\sigma_S(T)$. The operator $f(T)$ is again a spectral operator and its spectral decomposition can easily be constructed from the spectral decomposition of T.

Proposition 10.30. *Let $S \in \mathcal{B}(V_R)$ be an operator of scalar type. If $f \in \mathcal{SH}(\sigma_S(S))$, then*

$$f(S) = \int_{\mathbb{H}} f(s)\, dE_J(s), \tag{10.31}$$

where $f(S)$ is intended in the sense of the S-functional calculus introduced in Section 8.2.

Proof. Since $1(T) = \mathcal{I} = \int_{\mathbb{H}} 1\, dE_J(s)$ and $s(S) = S = \int_{\mathbb{H}} s\, dE_J(s)$, the product rule and the $\mathbb{R}$-linearity of both the S-functional calculus and the spectral integration imply that (10.31) holds true for any intrinsic polynomial. It in turn also holds true for any intrinsic rational function in $\mathcal{SH}(\sigma_S(S))$, i.e. for any function r of the form $r(s) = p(s)q(s)^{-1}$ with intrinsic polynomials p and q such that $q(s) \neq 0$ for any $s \in \sigma_S(S)$.

Let now $f \in \mathcal{SH}(\sigma_S(S))$ be arbitrary and let U be a bounded axially symmetric open set such that $\sigma_S(T) \subset U$ and $cl(U) \subset \mathcal{D}(f)$. Runge's theorem for slice hyperholomorphic functions in [37] implies the existence of a sequence of intrinsic rational functions $r_n \in \mathcal{SH}\big(cl(U)\big)$ such that $r_n \to f$ uniformly on $cl(U)$. Because of Lemma 9.24, we thus have

$$\int_{\mathbb{H}} f(s)\, dE_J(s) \;= \lim_{n\to+\infty} \int_{\mathbb{H}} r_n(s)\, dE_J(s) = \lim_{n\to+\infty} r_n(S) = f(S).$$

□

Theorem 10.31. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator with spectral decomposition (E, J) and let $T = S + N$ be the decomposition of T into scalar and radical part. If $f \in \mathcal{SH}(\sigma_S(T))$, then*

$$f(T) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!} \int_{\mathbb{H}} ({\partial_S}^n f)(s)\, dE_J(s), \tag{10.32}$$

where $f(T)$ is intended in the sense of the the S-functional calculus in Chapter 8 and the series converges in the operator norm.

Proof. Since $T = S + N$ with $SN = NS$ and $\sigma_S(N) = \{0\}$, it follows from (v) in Lemma 8.14 that

$$f(T) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!} \left({\partial_S}^n f\right)(S).$$

What remains to show is that

$$({\partial_S}^n f)(S) = \int_{\mathbb{H}} ({\partial_S}^n f)(s)\, dE_J(s), \tag{10.33}$$

but this follows immediately from Proposition 10.30.

□

The operator $f(T)$ is again a spectral operator and its radical part can be easily obtained from the above series expansion.

Definition 10.32. A spectral operator $T \in \mathcal{B}(V_R)$ is called of type $m \in \mathbb{N}$ if and only if its radical part satisfies $N^{m+1} = 0$.

Lemma 10.33. *A spectral operator $T \in \mathcal{B}(V_R)$ with spectral resolution (E, J) and radical part N is of type m if and only if*

$$f(T) = \sum_{n=0}^{m} N^n \frac{1}{n!} \int_{\mathbb{H}} (\partial_S{}^n f)(s)\, dE_J(s) \quad \forall f \in \mathcal{SH}(\sigma_S(T)). \tag{10.34}$$

In particular T is a scalar operator if and only if it is of type 0.

Proof. If T is of type m then the above formula follows immediately from Theorem 10.31 and $N^{m+1} = 0$. If on the other hand (10.34) holds true, then we choose $f(s) = \frac{1}{m!} s^m$ in (10.32) and (10.34) and subtract these two expressions. We obtain

$$0 = N^{m+1} \int_{\mathbb{H}} dE_J(s) = N^{m+1}.$$

□

Theorem 10.34. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator with spectral decomposition (E, J). If $f \in \mathcal{SH}(\sigma_S(T))$, then $f(T)$ is a spectral operator and the spectral decomposition $(\tilde{E}, \tilde{J})$ of $f(T)$ is given by*

$$\tilde{E}(\Delta) = E\big(f^{-1}(\Delta)\big) \quad \forall \Delta \in \mathsf{B}_{\mathsf{S}}(\mathbb{H}) \qquad \textit{and} \qquad \tilde{J} = \int_{\mathbb{H}} \mathbf{i}_{f(s)}\, dE_J(s),$$

where $\mathbf{i}_{f(s)} = 0$ if $f(s) \in \mathbb{R}$ and $\mathbf{i}_{f(s)} = \underline{f(s)}/|\underline{f(s)}|$ if $f(s) \in \mathbb{H} \setminus \mathbb{R}$. For any $g \in \mathcal{SM}^{\infty}(\mathbb{H})$ we have

$$\int_{\mathbb{H}} g(s)\, d\tilde{E}_{\tilde{J}}(s) = \int_{\mathbb{H}} (g \circ f)(s)\, dE_J(s) \tag{10.35}$$

and if S is the scalar part of T, then $f(S)$ is the scalar part of $f(T)$.

Proof. We first show that $f(S)$ is a scalar operator with spectral decomposition $(\tilde{E}, \tilde{J})$. By Corollary 9.22 the function f is $\mathsf{B}_{\mathsf{S}}(\mathbb{H})$-$\mathsf{B}_{\mathsf{S}}(\mathbb{H})$-measurable, such that $\tilde{E}$ is a well-defined spectral measure on $\mathsf{B}_{\mathsf{S}}(\mathbb{H})$.

The operator $\tilde{J}$ obviously commutes with E. Writing $f(s) = \alpha(s) + \mathbf{i}_s\beta(s)$ as in Lemma 9.21, we moreover have $\mathbf{i}_{f(s)} = \mathbf{i}_s \operatorname{sgn}(\beta(s))$. If we set

$$\Delta_+ = \{\, s \in \mathbb{H} : \beta(s) > 0\}, \qquad \Delta_- = \{s \in \mathbb{H} : \beta(s) < 0\}$$

and

$$\Delta_0 = \{s \in \mathbb{H} : \beta(s) = 0\},$$

we therefore have

$$\tilde{J} = JE(\Delta_+) - JE(\Delta_-).$$

As $\beta(s) = 0$ for any $s \in \mathbb{R}$, we have $\mathbb{R} \subset \Delta_0$ and hence $V_+ = \operatorname{ran} E(\Delta_+) \subset \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R}) = \operatorname{ran} J$ and similarly also $V_- = \operatorname{ran} E(\Delta_-) \subset \operatorname{ran} J$. Since J and E commute, V_+ and V_- are invariant subspaces of J contained in $\operatorname{ran} J$ such that J_+ and J_- define bounded surjective operators on V_+ resp. V_-. Moreover $\ker J = \operatorname{ran} E(\mathbb{R})$ and hence $\ker J|_{V_+} = V_+ \cap \ker J = \{\mathbf{0}\}$ and $\ker J|_{V_-} = V_- \cap \ker J = \{\mathbf{0}\}$, such that $\ker \tilde{J} = \operatorname{ran} E(\Delta_0)$ and $\operatorname{ran} \tilde{J} = \operatorname{ran} E(\Delta_+) \oplus \operatorname{ran} E(\Delta_-) = \operatorname{ran} E(\Delta_+ \cup \Delta_-)$.

Now observe that $f(s) \in \mathbb{R}$ if and only if $\beta(s) = 0$. Hence, $f^{-1}(\mathbb{R}) = \Delta_0$ and $f^{-1}(\mathbb{H} \setminus \mathbb{R}) = \Delta_+ \cup \Delta_-$ and we find

$$\operatorname{ran} \tilde{J} = \operatorname{ran} E(\Delta_+ \cup \Delta_-) = \operatorname{ran} E\left(f^{-1}(\mathbb{H} \setminus \mathbb{R})\right) = \operatorname{ran} \tilde{E}(\mathbb{H} \setminus \mathbb{R})$$

and

$$\ker \tilde{J} = \operatorname{ran} E(\Delta_0) = \operatorname{ran} E\left(f^{-1}(\mathbb{R})\right) = \operatorname{ran} \tilde{E}(\mathbb{R}).$$

Moreover, as $E(\Delta_+)E(\Delta_-) = E(\Delta_-)E(\Delta_+) = 0$ and $-J^2 = E(\mathbb{H} \setminus \mathbb{R})$, we have

$$\begin{aligned} -\tilde{J}^2 &= -\, J^2 E(\Delta_+)^2 - (-J^2)E(\Delta_-)^2 \\ &= E(\mathbb{H} \setminus \mathbb{R})E(\Delta_+) + E(\mathbb{H} \setminus \mathbb{R})E(\Delta_-) \\ &= E(\Delta_+ \cup \Delta_-) = \tilde{E}(\mathbb{H} \setminus \mathbb{R}), \end{aligned}$$

where we used that $\Delta_+ \subset \mathbb{H} \setminus \mathbb{R}$ and $\Delta_- \subset \mathbb{H} \setminus \mathbb{R}$ as $\mathbb{R} \subset \Delta_0$. Hence $-\tilde{J}^2$ is the projection onto $\operatorname{ran} \tilde{J}$ along $\ker \tilde{J}$ and so $\tilde{J}$ is actually an imaginary operator and so $(\tilde{E}, \tilde{J})$ is a spectral system.

Let $g = \sum_{\ell=0}^{n} a_\ell \chi_{\Delta_\ell} \in \mathcal{M}_s^\infty(\mathbb{H}, \mathbb{R})$ be a simple function. Then $(g \circ f)(s) = \sum_{\ell=0}^{n} a_\ell\, \chi_{f^{-1}(\Delta_\ell)}(s)$ is also a simple function in $\mathcal{M}_s^\infty(\mathbb{H}, \mathbb{R})$ and

$$\int_{\mathbb{H}} g(s)\, d\tilde{E}(s) = \sum_{\ell=0}^{n} a_\ell \tilde{E}(\Delta_\ell) = \sum_{\ell=0}^{n} a_\ell E\big(f^{-1}(\Delta_\ell)\big) = \int_{\mathbb{H}} (g \circ f)(s)\, dE(s).$$

Due to the density of simple functions in $(\mathcal{M}_s^\infty(\mathbb{H}, \mathbb{R}), \|.\|_\infty)$, we hence find

$$\int_{\mathbb{H}} g(s)\, d\tilde{E}(s) = \int_{\mathbb{H}} (g \circ f)(s)\, dE(s), \qquad \forall g \in \mathcal{M}_s^\infty(\mathbb{H}, \mathbb{R}).$$

If $g \in \mathcal{SM}^\infty(\mathbb{H})$ then we deduce from Lemma 9.21 that $g(s) = \gamma(s) + \mathsf{i}_s \delta(s)$ with $\gamma, \delta \in \mathcal{M}_s^\infty(\mathbb{H}, \mathbb{R})$ and $\mathsf{i}_s = \underline{s}/|\underline{s}|$ if $s \notin \mathbb{R}$ and $\mathsf{i}_s = \delta(s) = 0$ if $s \in \mathbb{R}$. We then have $(g \circ f)(s) = \gamma(f(s)) + \mathsf{i}_{f(s)} \delta(f(s))$ and find

$$\begin{aligned} \int_{\mathbb{H}} g(s)\, d\tilde{E}_{\tilde{J}}(s) &= \int_{\mathbb{H}} \gamma(s)\, d\tilde{E}(s) + \tilde{J} \int_{\mathbb{H}} \delta(s)\, d\tilde{E}(s) \\ &= \int_{\mathbb{H}} (\gamma \circ f)(s)\, dE(s) + \tilde{J} \int_{\mathbb{H}} (\delta \circ f)(s)\, dE(s) \\ &= \int_{\mathbb{H}} (\gamma \circ f)(s)\, dE_J(s) + \int_{\mathbb{H}} \mathsf{i}_{f(s)}\, dE_J(s) \int_{\mathbb{H}} (\delta \circ f)(s)\, dE(s) \\ &= \int_{\mathbb{H}} (\gamma \circ f)(s) + \mathsf{i}_{f(s)} (\delta \circ f)(s)\, dE_J(s) = \int_{\mathbb{H}} (g \circ f)(s)\, dE_J(s) \end{aligned}$$

and hence (10.35) holds true. Choosing in particular $g(s) = s$, we deduce from Proposition 10.30 that

$$f(S) = \int_{\mathbb{H}} f(s)\, dE_J(s) = \int_{\mathbb{H}} s\, d\tilde{E}_{\tilde{J}}(s).$$

By Remark 10.22, $f(S)$ is a scalar operator with spectral decomposition $(\tilde{E}, \tilde{J})$.

Theorem 10.31 implies $f(T) = f(S) + \Theta$ with

$$\Theta := \sum_{n=1}^{+\infty} N^n \frac{1}{n!} (\partial_S{}^n f)(S).$$

If we can show that Θ is a quasi-nilpotent operator, then the statement of the theorem follows from Theorem 10.27. We first observe that each term in the sum is a quasi-nilpotent operator because N^n and $(\partial_S{}^n f)(S)$ commute due to (ii) in Lemma 8.14 so that

$$0 \leq \lim_{k\to+\infty} \left\| \left(N^n \frac{1}{n!} (\partial_S^n f)(S) \right)^k \right\|^{\frac{1}{k}} \leq \left\| \frac{1}{n!} (\partial_S^n f)(S) \right\| \left(\lim_{k\to\infty} \left\| N^{nk} \right\|^{\frac{1}{nk}} \right)^n = 0.$$

Corollary 10.25 thus implies $\sigma_S\left(N^n \frac{1}{n!}(\partial_S{}^n f(S))\right) = \{0\}$. By induction we conclude from (v) in Lemma 8.14 and Corollary 10.25 that for each $m \in \mathbb{N}$ the finite sum $\Theta_1(m) := \sum_{n=1}^{m} N^n \frac{1}{n!}(\partial_S{}^n f)(S)$ is quasi-nilpotent and satisfies $\sigma_S(\Theta(m)) = \{0\}$.

Since the series Θ converges in the operator norm, for any $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that $\Theta_2(m_\varepsilon) := \sum_{n=m_\varepsilon+1}^{+\infty} N^n \frac{1}{n!}(\partial_S{}^n f)(S)$ satisfies $\|\Theta_2(m_\varepsilon)\| < \varepsilon$. Hence $\sigma_S(\Theta_2(m_\varepsilon)) \subset B_\varepsilon(0)$ and as $\Theta = \Theta_1(m_\varepsilon) + \Theta_2(m_\varepsilon)$ and $\Theta_1(m_\varepsilon)$ and $\Theta_2(m_\varepsilon)$ commute, we conclude from (v) in Lemma 8.14 that $\sigma_S(\Theta) \subset B_\varepsilon(0)$. As $\varepsilon > 0$ was arbitrary, we find $\sigma_S(\Theta) = \{0\}$. By Corollary 10.25, Θ is quasi-nilpotent.

We have shown that $f(T) = f(S) + \Theta$, that $f(S)$ is a scalar operator with spectral decomposition $(\tilde{E}, \tilde{J})$ and that Θ is quasi-nilpotent. From Theorem 10.27 we therefore deduce that $f(T)$ is a spectral operator with spectral decomposition $(\tilde{E}, \tilde{J})$, that $f(S)$ is its scalar part and that Θ is its radical part. This concludes the proof.

□

Corollary 10.35. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let $f \in \mathcal{SH}(\sigma_S(T))$. If T is of type $m \in \mathbb{N}$, then $f(T)$ is of type m too.*

Proof. If $T = S + N$ is the decomposition of T into its scalar and its radical part and T is of type m such that $N^{m+1} = 0$, then the radical part Θ of $f(T)$ is due to Lemma 10.33 and Theorem 10.34 given by

$$\Theta \ = f(T) - f(S) = \sum_{n=1}^{+\infty} N^n \frac{1}{n!} (\partial_S{}^n f)(S) = \sum_{n=1}^{m} N^n \frac{1}{n!} (\partial_S{}^n f)(S).$$

Obviously also $\Theta^{m+1} = 0$.

□

Part III

Applications

CHAPTER 11

Spectral Theory of the Nabla Operator and Fractional Evolution Processes

If $u(\mathbf{x}, t)$ is the temperature at the point $\mathbf{x} \in \mathbb{R}^3$ and the time $t > 0$ and κ is the thermal diffusivity of the considered material, then the heat equation

$$\partial_t u(\mathbf{x}, t) - \kappa \Delta u(\mathbf{x}, t) = 0, \tag{11.1}$$

where $\Delta = \sum_{\ell=1}^{3} \partial_{x_\ell}^2$ with $\mathbf{x} = (x_1, x_2, x_3)^T$, describes the evolution of the temperature distribution in space and time. (For mathematical treatment, one usually sets $\kappa = 1$ and we shall also do this in the following.) As explained in the introduction, this model has however several unphysical properties, so that scientists tried to modify it and one approach to do this lead to the definition of the fractional heat equation. In order to modify the properties of the equation, one replaced the negative Laplacian in (11.1) by its fractional power of exponent α and considered the evolution equation

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) + (-\Delta)^{\alpha} u(\mathbf{x}, t) = 0. \tag{11.2}$$

There are different approaches for defining the fractional Laplace operator, but each approach leads to a global integral operator, which is in contrast to the local differential operator Δ able to take long distance effects into account.

We wanted to develop a similar approach for defining fractional evolution equations. We explained in the introduction that we wanted to replace the gradient in Fourier's law of conductivity (1.16) by its fractional power instead of directly replacing the negative Laplacian by its fractional power in (11.1). This would lead to the equation

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) - \operatorname{div}(\nabla^{\alpha} u(\mathbf{x}, t)) = 0.$$

We wanted to do this by identifying the gradient with the quaternionic nabla operator and applying the theory developed in Chapter 7.

In this chapter, we develop the spectral theory of the quaternionic nabla operator on $L^2(\mathbb{R}^3, \mathbb{H})$. We find that the theory in Chapter 7 is not directly applicable because it turns out that the nabla operator does not belong to the class of sectorial operators. We therefore present a slightly modified approach and show that this allows us to reproduce the fractional heat equation (11.2) using quaternionic techniques. Finally, we give an example for a more general operator with non-constant coefficients that can be treated with the developed methods. The results presented in this chapter can be found in [19].

11.1 Spectral Properties of the Nabla Operator

The gradient of a function $v : \mathbb{R}^3 \to \mathbb{R}$ is the vector-valued function

$$\nabla v(\mathbf{x}) = \begin{pmatrix} \partial_{x_1} v(\mathbf{x}) \\ \partial_{x_2} v(\mathbf{x}) \\ \partial_{x_3} v(\mathbf{x}) \end{pmatrix}, \qquad \text{for } \mathbf{x} = (x_1, x_2, x_3)^T.$$

If we identify $\mathbb{R}$ with the set of real quaternions and $\mathbb{R}^3$ with the set of purely imaginary quaternions, this corresponds to the quaternionic nabla operator

$$\nabla = \partial_{x_1} e_1 + \partial_{x_2} e_2 + \partial_{x_3} e_3.$$

In the following we shall often denote the standard basis of the quaternions by $\mathsf{I} := e_1$, $\mathsf{J} := e_2$ and $\mathsf{K} := e_3 = \mathsf{IJ} = -\mathsf{JI}$. This suggests a relation with the complex theory, which we shall use intensively. With this notation, we have

$$\nabla = \partial_{x_1} \mathsf{I} + \partial_{x_2} \mathsf{J} + \partial_{x_3} \mathsf{K}.$$

We study the properties of a quaternionic nabla operator on the space $L^2(\mathbb{R}^3, \mathbb{H})$ of all square-integrable quaternion-valued functions on $\mathbb{R}^3$, which is a quaternionic right Hilbert space when endowed with the scalar product

$$\langle w, v \rangle = \int_{\mathbb{R}^3} \overline{w(\mathbf{x})}\, v(\mathbf{x})\, d\mathbf{x}.$$

On this space, the nabla operator is closed and has dense domain. This follows immediately from its representation (11.4) in the Fourier space that we derive in the proof of Theorem 11.1.

Let $v \in L^2(\mathbb{R}^3, \mathbb{H})$ and write $v(\mathbf{x}) = v_1(\mathbf{x}) + v_2(\mathbf{x})\mathsf{J}$ with two $\mathbb{C}_{\mathsf{I}}$-valued functions v_1 and v_2. As $|v(\mathbf{x})|^2 = |v_1(\mathbf{x})|^2 + |v_2(\mathbf{x})|^2$, we have

$$\|v\|^2_{L^2(\mathbb{R}^3,\mathbb{H})} = \|v_1\|^2_{L^2(\mathbb{R}^3,\mathbb{C}_{\mathsf{I}})} + \|v_2\|^2_{L_2(\mathbb{R},^3,\mathbb{C}_{\mathsf{I}})}, \tag{11.3}$$

where $L^2(\mathbb{R}^3, \mathbb{H})$ denotes the complex Hilbert space over $\mathbb{C}_{\mathsf{I}}$ of all square-integrable $\mathbb{C}_{\mathsf{I}}$-valued functions on $\mathbb{R}^3$. Hence, $v \in L^2(\mathbb{R}^3, \mathbb{H})$ if and only if $v_1, v_2 \in L^2(\mathbb{H}, \mathbb{C}_{\mathsf{I}})$.

Theorem 11.1. *The S-spectrum of ∇ as an operator on $L^2(\mathbb{R}^3, \mathbb{H})$ is*

$$\sigma_S(\nabla) = \mathbb{R}.$$

Proof. Let us consider $L^2(\mathbb{R}^3, \mathbb{H})$ as a Hilbert space over $\mathbb{C}_{\mathbf{I}}$ by restricting the right scalar multiplication to $\mathbb{C}_{\mathbf{I}}$ and setting

$$\langle w, v\rangle_{\mathbf{I}} := \{\langle w, v\rangle_{L^2(\mathbb{R}^3,\mathbb{H})}\}_{\mathbf{I}}.$$

Here $\{\cdot\}_{\mathbf{I}}$ denotes the $\mathbb{C}_{\mathbf{I}}$-part of a quaternion: if $a = a_1 + a_2\mathbf{J} = a_1 + \mathbf{J}\overline{a_2}$ with $a_1, a_2 \in \mathbb{C}_{\mathbf{I}}$, then $\{a\}_{\mathbf{I}} := a_1$. If we write $v, w \in L^2(\mathbb{R}^3, \mathbb{H})$ as $v = v_1 + \mathbf{J}v_2$ and $w = w_1 + \mathbf{J}w_2$ with $v_1, v_2, w_1, w_2 \in L^2(\mathbb{R}^3, \mathbb{C}_{\mathbf{I}})$, then

$$\begin{aligned}
&\langle w, v\rangle_{L^2(\mathbb{R}^3,\mathbb{H})} = \\
&= \int_{\mathbb{R}^3} \overline{(w_1(\mathbf{x}) + \mathbf{J}w_2(\mathbf{x}))}(v_1(\mathbf{x}) + \mathbf{J}v_2(\mathbf{x}))\, d\mathbf{x} \\
&= \int_{\mathbb{R}^3} \overline{w_1(\mathbf{x})}v_1(\mathbf{x})\, d\mathbf{x} + \int_{\mathbb{R}^3} \overline{w_2(\mathbf{x})}(-\mathbf{J})v_1(\mathbf{x})\, d\mathbf{x} \\
&\quad + \int_{\mathbb{R}^3} \overline{w_1(\mathbf{x})}\mathbf{J}v_2(\mathbf{x})\, d\mathbf{x} + \int_{\mathbb{R}^3} \overline{w_2(\mathbf{x})}(-\mathbf{J}^2)v_2(\mathbf{x})\, d\mathbf{x} \\
&= \int_{\mathbb{R}^3} \overline{w_1(\mathbf{x})}v_1(\mathbf{x})\, d\mathbf{x} + \int_{\mathbb{R}^3} \overline{w_2(\mathbf{x})}v_2(\mathbf{x})\, d\mathbf{x} \\
&\quad + \mathbf{J}\left(-\int_{\mathbb{R}^3} w_2(\mathbf{x})v_1(\mathbf{x})\, d\mathbf{x} + \int_{\mathbb{R}^3} w_1(\mathbf{x})v_2(\mathbf{x})\, d\mathbf{x}\right).
\end{aligned}$$

Therefore we have

$$\langle w, v\rangle_{\mathbf{I}} := \langle w_1, v_1\rangle_{L^2(\mathbb{R}^3,\mathbb{C}_{\mathbf{I}})} + \langle w_2, v_2\rangle_{L^2(\mathbb{R}^3,\mathbb{C}_{\mathbf{I}})}$$

and hence $L^2(\mathbb{R}^3, \mathbb{H})$ considered as a $\mathbb{C}_{\mathbf{I}}$-complex Hilbert space with the scalar product $\langle\cdot,\cdot\rangle_{\mathbf{I}}$ equals $L^2(\mathbb{R}^3, \mathbb{C}_{\mathbf{I}}) \oplus L^2(\mathbb{R}^3, \mathbb{C}_{\mathbf{I}})$. Moreover, because of (11.3), the quaternionic scalar product $\langle\cdot,\cdot\rangle$ and the $\mathbb{C}_{\mathbf{I}}$-complex scalar product $\langle\cdot,\cdot\rangle_{\mathbf{I}}$ induce the same norm on $L^2(\mathbb{R}^3, \mathbb{H})$. Applying the nabla operator to $v = v_1 + \mathbf{J}v_2$, we find

$$\begin{aligned}
\nabla v(\mathbf{x}) =& (\mathbf{I}\partial_{x_1} + \mathbf{J}\partial_{x_2} + \mathbf{K}\partial_{x_3})(v_1(\mathbf{x}) + \mathbf{J}v_2(\mathbf{x})) \\
=& \mathbf{I}\partial_{x_1}v_1(\mathbf{x}) + \mathbf{J}\partial_{x_2}v_1(\mathbf{x}) + \mathbf{K}\partial_{x_3}v_1(\mathbf{x}) \\
& + \mathbf{I}\partial_{x_1}\mathbf{J}v_2(\mathbf{x}) + \mathbf{J}\partial_{x_2}\mathbf{J}v_2(\mathbf{x}) + \mathbf{K}\partial_{x_3}\mathbf{J}v_2(\mathbf{x}) \\
=& \mathbf{I}\partial_{x_1}v_1(\mathbf{x}) - \partial_{x_2}v_2(\mathbf{x}) - \mathbf{I}\partial_{x_3}v_2(\mathbf{x}) \\
& + \mathbf{J}\left(-\mathbf{I}\partial_{x_1}v_2(\mathbf{x}) + \partial_{x_2}v_1(\mathbf{x}) - \mathbf{I}\partial_{x_3}v_1(\mathbf{x})\right).
\end{aligned}$$

Writing this in terms of the components $L^2(\mathbb{R}^3, \mathbb{H}) \cong L^2(\mathbb{R}^3, \mathbb{C}_{\mathbf{I}}) \oplus L^2(\mathbb{R}^3, \mathbb{C}_{\mathbf{I}})$, we obtain

$$\nabla \begin{pmatrix} v_1(\mathbf{x}) \\ v_2(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \mathbf{I}\partial_{x_1}v_1(\mathbf{x}) - \partial_{x_2}v_2(\mathbf{x}) - \mathbf{I}\partial_{x_3}v_2(\mathbf{x}) \\ -\mathbf{I}\partial_{x_1}v_2(\mathbf{x}) + \partial_{x_2}v_1(\mathbf{x}) - \mathbf{I}\partial_{x_3}v_1(\mathbf{x}) \end{pmatrix}.$$

If we apply the Fourier transform on $L^2(\mathbb{R}^3, \mathbb{C}_{\mathbf{I}})$ componentwise, this turns into

$$\widehat{\nabla} \begin{pmatrix} \widehat{v_1}(\mathbf{x}) \\ \widehat{v_2}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} -\xi_1 & -\mathbf{I}\xi_2 + \xi_3 \\ \mathbf{I}\xi_2 + \xi_3 & \xi_1 \end{pmatrix} \begin{pmatrix} \widehat{v_1}(\boldsymbol{\xi}) \\ \widehat{v_2}(\boldsymbol{\xi}) \end{pmatrix}. \tag{11.4}$$

Hence, in the Fourier space, the Nabla operator corresponds to the multiplication operator $M_G : \widehat{v} \mapsto G\widehat{v}$ on $\widehat{V} := L^2(\mathbb{R}^3, \mathbb{C}_{\mathbf{I}}) \oplus L^2(\mathbb{R}^3, \mathbb{C}_{\mathbf{I}})$ that is generated by the matrix

valued function

$$G(\boldsymbol{\xi}) := \begin{pmatrix} -\xi_1 & -\mathsf{I}\xi_2 + \xi_3 \\ \mathsf{I}\xi_2 + \xi_3 & \xi_1 \end{pmatrix}. \tag{11.5}$$

For $s \in \mathbb{C}_{\mathsf{I}}$, we find

$$s\mathcal{I}_{\widehat{V}} - G(\boldsymbol{\xi}) = \begin{pmatrix} s + \xi_1 & \mathsf{I}\xi_2 - \xi_3 \\ -\mathsf{I}\xi_2 - \xi_3 & s - \xi_1 \end{pmatrix}.$$

For $s \in \mathbb{C}_{\mathsf{I}}$, the inverse of $s\mathcal{I}_{\widehat{V}} - M_G$ is hence given by the multiplication operator $M_{(s\mathcal{I}-G)^{-1}}$ determined the matrix-valued function

$$(s\mathcal{I}_{\widehat{V}} - G(\boldsymbol{\xi}))^{-1} = \frac{1}{s^2 - \xi_1^2 - \xi_2^2 - \xi_3^2} \begin{pmatrix} s - \xi_1 & -\mathsf{I}\xi_2 + \xi_3 \\ \mathsf{I}\xi_2 + \xi_3 & s + \xi_1 \end{pmatrix}.$$

This operator is bounded if and only if the function $\boldsymbol{\xi} \mapsto (s\mathcal{I} - G(\boldsymbol{\xi}))^{-1}$ is bounded on $\mathbb{R}^3$, that is if and only $s \notin \mathbb{R}$. Hence, $\sigma(M_G) = \mathbb{R}$.

The componentwise Fourier transform Ψ is a unitary $\mathbb{C}_{\mathsf{I}}$-linear operator from the space $L^2(\mathbb{R}^3, \mathbb{H}) \cong L^2(\mathbb{R}^3, \mathbb{C}_{\mathsf{I}}) \oplus L^2(\mathbb{R}^2, \mathbb{C}_{\mathsf{I}})$ to $\widehat{V}$ under which ∇ corresponds to M_G, that is $\nabla = \Psi^{-1} M_G \Psi$. The spectrum $\sigma_{\mathbb{C}_{\mathsf{I}}}(\nabla)$ of ∇ considered as a $\mathbb{C}_{\mathsf{I}}$-linear operator on $L^2(\mathbb{R}^3, \mathbb{H})$ therefore equals $\sigma_{\mathbb{C}_{\mathsf{I}}}(\nabla) = \sigma(M_G) = \mathbb{R}$. By Theorem 8.4, we however have $\sigma_{\mathbb{C}_{\mathsf{I}}}(\nabla) = \sigma_S(\nabla) \cap \mathbb{C}_{\mathsf{I}}$ and so $\sigma_S(\nabla) = \mathbb{R}$.

□

11.2 A Different Characterization of $\sigma_S(\nabla)$

The above result shows that the gradient does not belong to the class of sectorial operators as $(-\infty, 0) \not\subset \rho_S(T)$ so that the theory developed in Chapter 7 is not directly applicable. Even worse, we cannot find any other slice hyperholomorphic functional calculus that allows us to define fractional powers ∇^α of ∇ because the scalar function s^α is not slice hyperholomorphic on $(-\infty, 0]$ and hence not slice hyperholomorphic on $\sigma_S(\nabla)$.

In order to write the fractional heat equation nevertheless using quaternionic techniques, we need to introduce a new characterization of the S-spectrum and a new way of writing the S-resolvents that apply only to operators with commuting components on a two-sided quaternionic Banach space V. They were introduced in [31] for bounded operators. However, the following proof of the characterization of the S-spectrum is original. It can be found in [19] and it also applies to unbounded operators.

Definition 11.2. Let V be a two-sided quaternionic Banach space. For a closed operator $T = T_0 + \sum_{\ell=1}^3 T_\ell e_\ell \in \mathcal{KC}(V)$ with commuting commuting components, we define $\overline{T} = T - \sum_{\ell=1}^3 T_\ell e_\ell$ with $\operatorname{dom}\left(\overline{T}\right) = \cap_{\ell=0}^3 \operatorname{dom}(T_\ell) = \operatorname{dom}(T)$.

McIntosh showed in [68, Theorem 3.3] that an operator $T \in \mathcal{B}(V)$ with commuting components is invertible if and only if $T\overline{T} = \overline{T}T = \sum_{\ell=0}^3 T_\ell^2$ is invertible. This holds true also for an unbounded operator with commuting components as the next lemma shows.

Lemma 11.3. *Let $T \in \mathcal{KC}(V)$. Then the following statements are equivalent.*

(i) The operator T has a bounded inverse.

(ii) The operator $\overline{T}$ has a bounded inverse.

(iii) The operator $\overline{T}T$ has a bounded inverse.

Proof. First of all, we observe that, due to $\operatorname{dom}(T) = \operatorname{dom}\left(\overline{T}\right)$, we have

$$\operatorname{dom}\left(\overline{T}T\right) = \{\, \mathbf{v} \in V : T\mathbf{v} \in \operatorname{dom}(T)\} = \operatorname{dom}\left(T^2\right).$$

Since $\operatorname{dom}(T^2) = \bigcap_{\ell,\kappa=0}^{3} \operatorname{dom}(T_\ell T_\kappa) = \bigcap_{\ell=0}^{3} \operatorname{dom}(T_\ell^2)$ and

$$\overline{T}T\mathbf{v} = T_0^2\mathbf{v} + \sum_{\ell=1}^{3} e_\ell T_0 T_\ell \mathbf{v} - \sum_{\ell=1}^{3} e_\ell T_\ell T_0 \mathbf{v} - \sum_{\ell,\kappa=1}^{3} e_\ell e_\kappa T_\ell T_\kappa \mathbf{v} = \sum_{\ell=0}^{3} T_\ell^2 \mathbf{v}$$

because $e_\ell e_\kappa = -e_\kappa e_\ell$ and $e_\ell^2 = -1$ for $1 \leq \ell, \kappa \leq 3$ with $\ell \neq \kappa$, we thus have $\overline{T}T = \sum_{\ell=0}^{3} T_\ell^2$. In particular, $\overline{T}T$ is a scalar operator and hence commutes with any quaternion.

If $T\overline{T}$ is invertible, then $\left(T\overline{T}\right)^{-1} = \left(\sum_{\ell=0}^{3} T_\ell^2\right)^{-1}$ commutes with each of the components T_ℓ and it also commutes with the imaginary units e_ℓ. Hence, it commutes with T and so the inverse T^{-1} is given by $T^{-1} = \overline{T}\left(T\overline{T}\right)^{-1}$ because

$$\left(\overline{T}\left(T\overline{T}\right)^{-1}\right) T\mathbf{v} = \overline{T}T\left(\overline{T}T\right)^{-1}\mathbf{v} \qquad \forall \mathbf{v} \in \operatorname{dom}(T)$$

and

$$T\left(\overline{T}\left(T\overline{T}\right)^{-1}\right)\mathbf{v} = \left(T\overline{T}\right)\left(T\overline{T}\right)^{-1}\mathbf{v} = \mathbf{v} \quad \forall \mathbf{v} \in V.$$

Consequently, the invertibility of $T\overline{T}$ implies the invertibility of T.

If on the other hand T is invertible and $T^{-1} = S_0 + \sum_{\kappa=1}^{3} S_\kappa e_\kappa \in \mathcal{B}(V)$, then

$$\begin{aligned}
\mathcal{I}|_{\operatorname{dom}(T)} =& T^{-1}T = \left(S_0 + \sum_{\kappa=1}^{3} S_\kappa e_\kappa\right)\left(T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell\right) \\
=& S_0T_0 - \sum_{\ell=1}^{3} S_\ell T_\ell + (S_2T_3 - S_3T_2)e_1 \\
&+ (S_3T_1 - S_1T_3)e_2 + (S_1T_2 - S_2T_1)e_3,
\end{aligned}$$

from which we conclude that

$$\mathcal{I}|_{\operatorname{dom}(T)} = S_0T_0 - \sum_{\ell=1}^{3} S_\ell T_\ell \qquad \text{and} \qquad S_\ell T_\kappa - S_\kappa T_\ell = 0 \quad 1 \leq \ell < \kappa \leq 3.$$

Therefore

$$\begin{aligned}
\overline{S}\,\overline{T} =& \left(S_0 - \sum_{\ell=1}^{3} S_\ell e_\ell\right)\left(T_0 - \sum_{\ell=1}^{3} T_\ell e_\ell\right) \\
=& S_0T_0 - \sum_{\ell=1}^{3} S_\ell T_\ell + (S_2T_3 - S_3T_2)e_1 \\
&+ (S_3T_1 - S_1T_3)e_2 + (S_1T_2 - S_2T_1)e_3 = \mathcal{I}|_{\operatorname{dom}(T)}.
\end{aligned}$$

Similarly, we see that $TS = \mathcal{I}$ also implies $\overline{T}\,\overline{S} = \mathcal{I}$. Hence, the invertibility of T implies the invertibility of $\overline{T}$ and $\overline{T}^{-1} = \overline{T^{-1}}$. Thus, if T is invertible, we have $\left(T\overline{T}\right)^{-1} = \overline{T}^{-1}T^{-1} \in \mathcal{B}(V)$. Altogether, we find that T is invertible if and only if $T\overline{T} = \overline{T}T$ is invertible.

□

Theorem 11.4. *Let $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{KC}(V)$ with dense domain. If we set*

$$\mathcal{Q}_{c,s}(T) = s^2\mathcal{I} - 2sT_0 + T\overline{T},$$

then

$$\rho_S(T) = \left\{\ s \in \mathbb{H} : \mathcal{Q}_{c,s}(T)^{-1} \in \mathcal{B}(V)\right\} \tag{11.6}$$

and

$$\begin{aligned} S_L^{-1}(s,T) =&(s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T) \\ S_R^{-1}(s,T) =&\mathcal{Q}_{c,s}(T)^{-1}s - \sum_{\ell=0}^{3} T_\ell \mathcal{Q}_{c,s}(T)^{-1}e_\ell. \end{aligned} \tag{11.7}$$

Proof. Since T and $\overline{T}$ commute, we have $\overline{\mathcal{Q}_s(T)} = \mathcal{Q}_s(\overline{T})$ and $\overline{\mathcal{Q}_{c,s}(T)} = \mathcal{Q}_{c,\overline{s}}(T)$. For $\mathbf{v} \in \operatorname{dom}(T^4) = \operatorname{dom}\left(\mathcal{Q}_{c,s}(T)\mathcal{Q}_{c,\overline{s}}(T)\right)$, we thus find

$$\begin{aligned} \mathcal{Q}_{c,s}(T)\overline{\mathcal{Q}_{c,s}(T)}\mathbf{v} =&(s^2\mathcal{I} - 2sT_0 + T\overline{T})(\overline{s}^2\mathcal{I} - 2\overline{s}T_0 + T\overline{T})\mathbf{v} \\ =&|s|^4\mathcal{I}\mathbf{v} - 2s|s|^2T_0\mathbf{v} + s^2T\overline{T}\mathbf{v} \\ &- 2|s|^2T_0\overline{s}\mathbf{v} + 4|s|^2T_0^2\mathbf{v} - 2sT_0T\overline{T}\mathbf{v} \\ &+ \overline{s}^2T\overline{T}\mathbf{v} - 2\overline{s}T_0T\overline{T}\mathbf{v} + (T\overline{T})^2\mathbf{v} \\ =&|s|^4\mathcal{I}\mathbf{v} - 2s_0|s|^2T\mathbf{v} - 2s_0|s|^2\overline{T}\mathbf{v} + 2\mathrm{Re}(s^2)T\overline{T}\mathbf{v} \\ &+ 4|s|^2T_0^2\mathbf{v} - 2s_0T^2\overline{T}\mathbf{v} - 2s_0T\overline{T}^2\mathbf{v} + T^2\overline{T}^2\mathbf{v}, \end{aligned}$$

where we used in the last identity that $2s_0 = s + \overline{s}$, that $|s|^2 = s\overline{s}$, and that $2T_0\mathbf{v} = T\mathbf{v} + \overline{T}\mathbf{v}$. As

$$2\mathrm{Re}(s^2)T\overline{T}\mathbf{v} = 2s_0^2T\overline{T}\mathbf{v} - 2s_1^2T\overline{T}\mathbf{v}$$

and

$$4|s|^2T_0^2\mathbf{v} = |s|^2(T + \overline{T})^2\mathbf{v} = |s|^2T^2\mathbf{v} + 2s_0^2T\overline{T}\mathbf{v} + s_1^2T\overline{T}\mathbf{v}\ + |s|^2\overline{T}^2\mathbf{v}$$

we further find

$$\begin{aligned} \mathcal{Q}_{c,s}(T)\overline{\mathcal{Q}_{c,s}(T)}\mathbf{v} =&|s|^2(|s|^2\mathcal{I} - 2s_0T + T^2)\mathbf{v} \\ &- 2s_0\overline{T}(|s|^2\mathcal{I} - 2s_0T + T^2)\mathbf{v} \\ &+ \overline{T}^2(|s|^2\mathcal{I} - 2s_0T + T^2)\mathbf{v} = \mathcal{Q}_s(T)\overline{\mathcal{Q}_s(T)}\mathbf{v}. \end{aligned}$$

By the above arguments, we hence have

$$\begin{aligned} &\mathcal{Q}_{c,s}(T)^{-1} \in \mathcal{B}(V) \Longleftrightarrow \left(\mathcal{Q}_{c,s}(T)\overline{\mathcal{Q}_{c,s}(T)}\right)^{-1} \in \mathcal{B}(V) \\ \Longleftrightarrow &\left(\mathcal{Q}_s(T)\overline{\mathcal{Q}_s(T)}\right)^{-1} \in \mathcal{B}(V) \Longleftrightarrow \mathcal{Q}_s(T)^{-1} \in \mathcal{B}(V) \end{aligned}$$

and hence (11.6) holds true.

If $\mathbf{v} \in \operatorname{dom}(T^2) = \operatorname{dom}(\mathcal{Q}_{c,s}(T))$ with $\mathcal{Q}_{c,s}(T) \in \operatorname{dom}(T)$, we have

$$
\begin{aligned}
&(s\mathcal{I} - T)\mathcal{Q}_{c,s}(T)\mathbf{v} = (\overline{s}\mathcal{I} - T)\left(s^2\mathcal{I} - 2sT_0 + T\overline{T}\right)\mathbf{v} \\
=&|s|^2 s\mathcal{I}\mathbf{v} - Ts^2\mathbf{v} - 2|s|^2 T_0\mathbf{v} + 2TT_0 s\mathbf{v} + \overline{s}T\overline{T}\mathbf{v} - T^2\overline{T}\mathbf{v} \\
=&|s|^2 s\mathcal{I}\mathbf{v} - Ts^2\mathbf{v} - |s|^2 T\mathbf{v} - |s|^2\overline{T}\mathbf{v} + T^2 s\mathbf{v} + T\overline{T}s\mathbf{v} + \overline{s}T\overline{T}\mathbf{v} - T^2\overline{T}\mathbf{v} \\
=&|s|^2\left(s\mathcal{I} - \overline{T}\right)\mathbf{v} - 2s_0T\left(s\mathcal{I} - \overline{T}\right)\mathbf{v} + T^2\left(s\mathcal{I} - \overline{T}\right)\mathbf{v} \\
=&\left(T^2 - 2s_0T + |s|^2\mathcal{I}\right)\left(s\mathcal{I} - \overline{T}\right)\mathbf{v} = \mathcal{Q}_s(T)\left(s\mathcal{I} - \overline{T}\right)\mathbf{v}.
\end{aligned}
$$

For any $\mathbf{u} \in \operatorname{dom}(T)$, we can set $\mathbf{v} = \mathcal{Q}_{c,s}(T)^{-1}\mathbf{u} \in \operatorname{dom}(T^2)$. If we apply the operator $\mathcal{Q}_s(T)^{-1}$ to the above identity from the right, we then obtain

$$
S_L^{-1}(s,T)\mathbf{u} = \mathcal{Q}_s(T)^{-1}(s\mathcal{I} - T)\mathbf{u} = \left(s\mathcal{I} - \overline{T}\right)\mathcal{Q}_{c,s}(T)^{-1}\mathbf{u}
$$

and a density argument shows that (11.7) holds true for the left S-resolvent. Similar computations show also the identity for the right S-resolvent equation.

□

Let us now turn back to the nabla operator on the quaternionic right Hilbert space $L^2(\mathbb{R}^3,\mathbb{H})$. If $\mathsf{i} \in \mathbb{S}$ is an arbitrary imaginary unit and $\mathsf{j} \in \mathbb{S}$ with $\mathsf{j} \perp \mathsf{i}$, then any $v \in L^2(\mathbb{R}^3,\mathbb{H})$ can be written as $v = v_1 + v_2\mathsf{j}$ with components v_1, v_2 in $L^2(\mathbb{R}^3,\mathbb{C}_\mathsf{i})$, i.e. $L^2(\mathbb{R}^3,\mathbb{H}) = L^2(\mathbb{R}^3,\mathbb{C}_\mathsf{i}) \oplus L^2(\mathbb{R}^3,\mathbb{C}_\mathsf{i})\mathsf{j}$. Contrary to the decomposition $v = v_1 + \mathsf{j}v_1$, which we used in the proof of Theorem 11.1 with $\mathsf{i} = \mathsf{I}$ and $\mathsf{j} = \mathsf{J}$, this decomposition is not compatible with the $\mathbb{C}_\mathsf{i}$-right vector space structure of $L^2(\mathbb{R}^3,\mathbb{H})$ as $va = v_1a + v_2\overline{a}\mathsf{j}$ for any $a \in \mathbb{C}_\mathsf{i}$. However, this identification has a different advantage: any closed $\mathbb{C}_\mathsf{i}$-linear operator $A : \operatorname{dom}(A) \subset L^2(\mathbb{R}^3,\mathbb{C}_\mathsf{I}) \to L^2(\mathbb{R}^3,\mathbb{C}_\mathsf{I})$ extends to a closed $\mathbb{H}$-linear operator on $L^2(\mathbb{R}^3,\mathbb{H})$ with domain $\operatorname{dom}(A) \oplus \operatorname{dom}(A)\mathsf{j}$, namely to the operator $A(v_1 + v_2\mathsf{j}) := A(v_1) + A(v_2)\mathsf{j}$. Moreover, if A is bounded, then its extension to $L^2(\mathbb{R}^3,\mathbb{H})$ has the same norm as A. We shall denote an operator on $L^2(\mathbb{R}^3,\mathbb{C}_\mathsf{i})$ and its extension to $L^2(\mathbb{R}^3,\mathbb{H}) = L^2(\mathbb{R}^3,\mathbb{C}_\mathsf{i}) \oplus L^2(\mathbb{R}^3,\mathbb{C}_\mathsf{i})\mathsf{j}$ via componentwise application by the same symbol. This will not cause any confusion as it will be clear from the context to which we refer.

Theorem 11.5. *Let Δ be the Laplace operator on $L^2(\mathbb{H},\mathbb{C}_\mathsf{i})$ and let $R_z(-\Delta)$ be the resolvent of $-\Delta$ at $z \in \mathbb{C}_\mathsf{i}$. We have*

$$
\sigma_S(\nabla)^2 = \left\{s^2 \in \mathbb{H} : s \in \sigma_S(T)\right\} = \sigma(-\Delta) \tag{11.8}
$$

and

$$
\mathcal{Q}_{c,s}(\nabla)^{-1} = R_{s^2}(-\Delta) \qquad \forall s \in \mathbb{C}_\mathsf{i} \setminus \mathbb{R}. \tag{11.9}
$$

Proof. Since the components of ∇ commute and $e_\kappa e_\ell = -e_\ell e_\kappa$ for $1 \leq \kappa, \ell \leq 3$ with

$\kappa \neq \ell$, we have

$$\begin{aligned}\nabla^2 &= \sum_{\ell,\kappa=1}^{3} \partial_{x_\ell}\partial_{x_\kappa} e_\ell e_\kappa \\ &= \sum_{\ell=1}^{3} -\partial_{x_\ell}^2 + \sum_{1\leq \ell<\kappa\leq 3} \left(\partial_{x_\ell}\partial_{x_\kappa} - \partial_{x_\kappa}\partial_{x_\ell}\right) e_\ell e_\kappa \\ &= \sum_{\ell=1}^{3} -\partial_{x_\ell}^2 = -\Delta.\end{aligned}$$

As $\nabla_0 = 0$, we have $\overline{\nabla} = -\nabla$ and in turn

$$\mathcal{Q}_{c,s}(\nabla) = s^2\mathcal{I} - 2s\nabla_0 + \nabla\overline{\nabla} = s^2\mathcal{I} - \nabla^2 = s^2\mathcal{I} - (-\Delta)$$

Hence, $\mathcal{Q}_{c,s}(\nabla)$ is invertible if and only if $s^2\mathcal{I} - (-\Delta)$ is invertible. In this case

$$\mathcal{Q}_{c,s}(\nabla) = (s^2\mathcal{I} - (-\Delta))^{-1} = R_{s^2}(-\Delta).$$

□

11.3 A Relation with the Fractional Heat Equation

As one can easily verify, the nabla operator is selfadjoint on $L^2(\mathbb{R}^3, \mathbb{H})$. From the spectral theorem for unbounded normal quaternionic linear operators in [5], we hence deduce the existence of a unique spectral measure E on $\sigma_S(\nabla) = \mathbb{R}$, the values of which are orthogonal quaternionic linear projections on $L^2(\mathbb{R}^3, \mathbb{H})$, such that

$$\nabla = \int_{\mathbb{R}} s\, dE(s).$$

Via the measurable functional calculus for intrinsic slice functions, it is now possible to define $f_\alpha(s) = s^\alpha \chi_{[0,+\infty)}(s)$ of T as

$$f_\alpha(\nabla) = \int_{\mathbb{R}} s^\alpha \chi_{[0,+\infty)}(s)\, dE(s),$$

where $\chi_{[0,+\infty)}$ denotes the characteristic function of the set $[0, +\infty)$. This corresponds to defining ∇^α at least on the subspace associated with the spectral values $[0, +\infty)$, on which s^α is defined. (Observe that even with the measurable functional calculus the operator ∇^α cannot be defined, because s^α is not defined on $(-\infty, 0)$.)

We shall now give an integral representation for this operator via an approach similar to the one of the slice hyperholomorphic H^∞-functional calculus. Surprisingly, this yields a possibility to obtain the fractional heat equation via quaternionic operator techniques applied to the nabla operator.

For $\alpha \in (0, 1)$, we define

$$f_\alpha(\nabla)v := \frac{1}{2\pi}\int_{-\mathsf{I}\mathbb{R}} S_L^{-1}(s,\nabla)\, ds_{\mathsf{I}}\, s^{\alpha-1}\nabla v \qquad \forall v \in \operatorname{dom}(\nabla). \tag{11.10}$$

Intuitively, this corresponds to Balakrishnan's formula for ∇^α, where only spectral values on the positive real axis are taken into account, i.e. points where s^α is actually defined, because the path of integration surrounds only the positive real axis.

Theorem 11.6. *The integral* (11.10) *converges for any* $v \in \mathrm{dom}(\nabla)$ *and hence defines a quaternionic linear operator on* $L^2(\mathbb{R}^3, \mathbb{H})$.

Proof. If we write the integral (11.10) explicitly, we have

$$\begin{aligned} f_\alpha(\nabla)v =& \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_L^{-1}(-\mathsf{I}t, \nabla)\,(-\mathsf{I})^2\,(-\mathsf{I}t)^{\alpha-1}\nabla v \\ =& -\frac{1}{2\pi}\int_0^{+\infty} S_L^{-1}(-\mathsf{I}t,\nabla)(-\mathsf{I}t)^{\alpha-1}\nabla v\,dt \\ & -\frac{1}{2\pi}\int_0^{+\infty} S_L^{-1}(\mathsf{I}t,\nabla)(\mathsf{I}t)^{\alpha-1}\nabla v\,dt \\ =& -\frac{1}{2\pi}\int_0^{+\infty} S_L^{-1}(-\mathsf{I}t,\nabla)t^{\alpha-1}e^{-\mathsf{I}\frac{(\alpha-1)\pi}{2}}\nabla v\,dt \\ & -\frac{1}{2\pi}\int_0^{+\infty} S_L^{-1}(\mathsf{I}t,\nabla)t^{\alpha-1}e^{\mathsf{I}\frac{(\alpha-1)\pi}{2}}\nabla v\,dt, \end{aligned} \tag{11.11}$$

where $f_\alpha(\nabla)v$ is defined if and only if the last two integrals converge in $L^2(\mathbb{R}^3, \mathbb{H})$.

Let us consider $L^2(\mathbb{R}^3, \mathbb{H})$ as a Hilbert space over $\mathbb{C}_{\mathsf{I}}$ as in the proof of Theorem 11.1. If we write $v \in L^2(\mathbb{R}^3, \mathbb{H})$ as $v = v_1 + \mathsf{J}v_2$ with $v_1, v_2 \in L^2(\mathbb{R}, \mathbb{C}_{\mathsf{I}})$ and apply the Fourier-transform componentwise, we obtain an isometric $\mathbb{C}_{\mathsf{I}}$-linear isomorphism $\Psi : v \mapsto (\widehat{v_1}, \widehat{v_2})^T$ between $L^2(\mathbb{R}^3, \mathbb{H})$ and $\widehat{V} := L^2(\mathbb{R}^3, \mathbb{C}_{\mathsf{I}}) \oplus L^2(\mathbb{R}^3, \mathbb{C}_{\mathsf{I}})$. For any quaternionic linear operator T on $L^2(\mathbb{R}^3, \mathbb{H})$, the composition $\Psi T \Psi^{-1}$ is a $\mathbb{C}_{\mathsf{I}}$-linear operator on $\widehat{V}$ with $\mathrm{dom}(\Psi T\Psi^{-1}) = \Psi\,\mathrm{dom}(T)$.

Applying ∇ to $v \in \mathrm{dom}(\nabla) \subset L^2(\mathbb{R}^3, \mathbb{H})$ corresponds to applying the multiplication operator M_G associated with the matrix-valued function $G(\boldsymbol{\xi})$ defined in (11.5) to $\widehat{v}(\boldsymbol{\xi}) = (\widehat{v_1}(\boldsymbol{\xi}), \widehat{v_2}(\boldsymbol{\xi}))^T$. Hence, $\nabla = \Psi^{-1}M_G\Psi$ and

$$\begin{aligned} \Psi\,\mathrm{dom}(\nabla) =& \mathrm{dom}(M_G) = \left\{\widehat{v} \in \widehat{V} : G(\boldsymbol{\xi})\widehat{v}(\boldsymbol{\xi}) \in \widehat{V}\right\} \\ =& \left\{\widehat{v} \in \widehat{V} : |\boldsymbol{\xi}|\widehat{v}(\boldsymbol{\xi}) \in \widehat{V}\right\}. \end{aligned} \tag{11.12}$$

That is last identity holds, as for $\widehat{v}(\boldsymbol{\xi}) = (\widehat{v_1}(\boldsymbol{\xi}), \widehat{v_2}(\boldsymbol{\xi}))^T \in \widehat{V}$ straightforward computations show that

$$\begin{aligned} |G(\boldsymbol{\xi})\widehat{v}(\boldsymbol{\xi})|^2 =& \left|\begin{pmatrix} -\xi_1\widehat{v_1}(\boldsymbol{\xi}) + (-\mathsf{I}\xi_2 + \xi_3)\widehat{v_2}(\boldsymbol{\xi}) \\ (\mathsf{I}\xi_2 + \xi_3)\widehat{v_1}(\boldsymbol{\xi}) + \xi_1\widehat{v_2}(\boldsymbol{\xi}) \end{pmatrix}\right|^2 \\ =& (\xi_1^2 + \xi_2^2 + \xi_3^2)(|\widehat{v_1}(\boldsymbol{\xi})|^2 + |\widehat{v_2}(\boldsymbol{\xi})|^2) = |\boldsymbol{\xi}|^2|\widehat{v}(\boldsymbol{\xi})|^2. \end{aligned} \tag{11.13}$$

Because of (11.11), we have

$$\begin{aligned} f_\alpha(\nabla)v =& -\Psi\frac{1}{2\pi}\int_0^{+\infty}\left(\Psi^{-1}S_L^{-1}(-\mathsf{I}t,\nabla)t^{\alpha-1}e^{-\mathsf{I}\frac{(\alpha-1)\pi}{2}}\nabla\Psi^{-1}\right)\Psi v\,dt \\ & -\Psi\frac{1}{2\pi}\int_0^{+\infty}\left(\Psi^{-1}S_L^{-1}(\mathsf{I}t,\nabla)t^{\alpha-1}e^{\mathsf{I}\frac{(\alpha-1)\pi}{2}}\nabla\Psi^{-1}\right)\Psi v\,dt, \end{aligned} \tag{11.14}$$

Since $\mathsf{I}v = \mathsf{I}(v_1 + \mathsf{J}v_2) = v_1\mathsf{I} - \mathsf{J}(v_2\mathsf{I})$ and Ψ is $\mathbb{C}_{\mathsf{I}}$-linear, we find $\Psi\mathsf{I}\Psi^{-1}(\widehat{v_1}, \widehat{v_2})^T = (\widehat{v_1}\mathsf{I}, \widehat{v_2}(-\mathsf{I}))^T$, i.e. multiplication with I on $L^2(\mathbb{R}^3, \mathbb{H})$ from the left corresponds to the

multiplication with the matrix $E := \operatorname{diag}(\mathsf{I}, -\mathsf{I})$ on $\widehat{V}$. As $\mathcal{Q}_{-\mathsf{I}t}(\nabla)^{-1} = (\nabla^2 + t^2)^{-1} = (-\Delta + t^2)^{-1}$ is a scalar operator and hence commutes with any quaternion, we have

$$S_L^{-1}(-\mathsf{I}t, \nabla) = \mathcal{Q}_{-\mathsf{I}t}(\nabla)^{-1}\mathsf{I}t - \nabla\mathcal{Q}_{-\mathsf{I}t}(\nabla)^{-1} = (\mathsf{I}t - \nabla)\mathcal{Q}_{-\mathsf{I}t}(\nabla)^{-1},$$

and in turn

$$\begin{aligned}
&\Psi^{-1}S_L^{-1}(-\mathsf{I}t, \nabla)t^{\alpha-1}e^{-\mathsf{I}\frac{(\alpha-1)\pi}{2}}\nabla\Psi^{-1}\\
=&\Psi^{-1}\left(\mathsf{I}t\mathcal{Q}_{-\mathsf{I}t}(\nabla)^{-1} - \nabla\mathcal{Q}_{-\mathsf{I}t}(\nabla)^{-1}\right)t^{\alpha-1}e^{-\mathsf{I}\frac{(\alpha-1)\pi}{2}}\nabla\Psi^{-1}\\
=&\left(tM_E\mathcal{Q}_{-\mathsf{I}t}(M_G)^{-1} - M_G\mathcal{Q}_{-\mathsf{I}t}(M_G)^{-1}\right)t^{\alpha-1}M_{\exp\left(-\frac{(\alpha-1)\pi}{2}E\right)}M_G.
\end{aligned}$$

The operator $\mathcal{Q}_{\mathsf{I}t}(M_G)^{-1}$ is

$$\mathcal{Q}_{\mathsf{I}t}(M_G)^{-1} = (M_G^2 + t^2\mathcal{I})^{-1} = M_{(G^2+t^2\mathcal{I})^{-1}} = M_{(t^2+|\boldsymbol{\xi}|^2)^{-1}\mathcal{I}}$$

with $|\boldsymbol{\xi}|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$ and the operator in the first integral of (11.14) equals therefore

$$\begin{aligned}
&\Psi^{-1}S_L^{-1}(-\mathsf{I}t, \nabla)t^{\alpha-1}e^{-\mathsf{I}\frac{(\alpha-1)\pi}{2}}\nabla\Psi^{-1}\\
=&M_{tE(t^2+|\boldsymbol{\xi}|^2)^{-1}-G(t^2+|\boldsymbol{\xi}|^2)^{-1}}t^{\alpha-1}M_{\exp\left(-\frac{(\alpha-1)\pi}{2}E\right)}M_G.
\end{aligned}$$

It is hence the multiplication operator $M_{A_1(t,\boldsymbol{\xi})}$ determined by the matrix-valued function

$$\begin{aligned}
A_1(t,\boldsymbol{\xi}) =& \frac{t^{\alpha-1}}{t^2+|\boldsymbol{\xi}|^2}\left(tE - G(\boldsymbol{\xi})\right)\exp\left(-\frac{(\alpha-1)\pi}{2}E\right)G(\boldsymbol{\xi})\\
=&\frac{t^{\alpha-1}}{t^2+\xi_1^2+\xi_2^2+\xi_3^2}\\
&\cdot\begin{pmatrix} e^{-\mathsf{I}\frac{\alpha\pi}{2}}\xi_1(t-\mathsf{I}\xi_1) + \mathsf{I}e^{\mathsf{I}\frac{\alpha\pi}{2}}(\xi_2^2+\xi_3^2) & \left(e^{\mathsf{I}\frac{\alpha\pi}{2}}\xi_1 + e^{-\mathsf{I}\frac{\alpha\pi}{2}}(\xi_1+\mathsf{I}t)\right)(\xi_2+\mathsf{I}\xi_3)\\ \left(\mathsf{I}e^{-\mathsf{I}\frac{\alpha\pi}{2}}\xi_1 + e^{\mathsf{I}\frac{\alpha\pi}{2}}(-t+\mathsf{I}\xi_1)\right)(\mathsf{I}\xi_2+\xi_3) & e^{\mathsf{I}\frac{\alpha\pi}{2}}(-t+\mathsf{I}\xi_1)\xi_1 - \mathsf{I}e^{-\mathsf{I}\frac{\alpha\pi}{2}}(\xi_2^2+\xi_3^2)\end{pmatrix}.
\end{aligned}$$

Similarly the operator in the second integral of (11.14) is

$$\begin{aligned}
&\Psi^{-1}S_L^{-1}(\mathsf{I}t, \nabla)t^{\alpha-1}e^{\mathsf{I}\frac{(\alpha-1)\pi}{2}}\nabla\Psi^{-1}\\
=&M_{-tE(t^2+|\boldsymbol{\xi}|^2)^{-1}-G(t^2+|\boldsymbol{\xi}|^2)^{-1}}t^{\alpha-1}M_{\exp\left(\frac{(\alpha-1)\pi}{2}E\right)}M_G.
\end{aligned}$$

It is hence the multiplication operator $M_{A_2(t,\boldsymbol{\xi})}$ determined by the matrix-valued function

$$\begin{aligned}
A_2(t,\boldsymbol{\xi}) =& \frac{t^{\alpha}}{t^2+|\boldsymbol{\xi}|^2}\left(-tE - G(\boldsymbol{\xi})\right)\exp\left(\frac{(\alpha-1)\pi}{2}E\right)G(\boldsymbol{\xi})\\
=&\frac{t^{\alpha-1}}{t^2+\xi_1^2+\xi_2^2+\xi_3^2}\\
&\cdot\begin{pmatrix} e^{\mathsf{I}\frac{\alpha\pi}{2}}\xi_1(t+\mathsf{I}\xi_1) - \mathsf{I}e^{-\mathsf{I}\frac{\alpha\pi}{2}}(\xi_2^2+\xi_3^2) & -\left(e^{\mathsf{I}\frac{\alpha\pi}{2}}(-\mathsf{I}t+\xi_1) + e^{-\mathsf{I}\frac{\alpha\pi}{2}}\xi_1\right)(\xi_2+\mathsf{I}\xi_3)\\ \left(e^{\mathsf{I}\frac{\alpha\pi}{2}}\xi_1 + e^{-\mathsf{I}\frac{\alpha\pi}{2}}(-\mathsf{I}t+\xi_1)\right)(\xi_2-\mathsf{I}\xi_3) & -e^{-\mathsf{I}\frac{\alpha\pi}{2}}(t+\mathsf{I}\xi_1)\xi_1 + \mathsf{I}e^{\mathsf{I}\frac{\alpha\pi}{2}}(\xi_2^2+\xi_3^2)\end{pmatrix}.
\end{aligned}$$

Hence, we have $f_\alpha(\nabla)v = \Psi^{-1}f_\alpha(M_G)\Psi v$ with

$$f_\alpha(M_G)\widehat{v} := -\frac{1}{2\pi}\int_0^{+\infty} M_{A_1(t,\boldsymbol{\xi})}\widehat{v}\,dt - \frac{1}{2\pi}\int_0^{+\infty} M_{A_2(t,\boldsymbol{\xi})}\widehat{v}\,dt \tag{11.15}$$

for $\widehat{v} = \Psi v \in \Psi\, \mathrm{dom}(\nabla)$.

We show now that these integrals converge for any $\widehat{v} \in \Psi\, \mathrm{dom}(\nabla)$. As Ψ is isometric, this is equivalent to (11.10) converging for any $v \in \mathrm{dom}(\nabla)$. Since all norms on a finite-dimensional vector space are equivalent, there exists a constant $C > 0$ such that

$$\| M\| \leq C \max_{\ell,\kappa\in\{1,2\}} |m_{\ell,\kappa}| \qquad \forall M = \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{pmatrix} \in \mathbb{C}_{\mathsf{I}}^{2\times 2}. \tag{11.16}$$

The modulus of the $(1,1)$-entry of $A_1(t,\boldsymbol{\xi})$ with $t \geq 0$ is

$$\begin{aligned} &\frac{t^{\alpha-1}}{t^2+\xi_1^2+\xi_2^2+\xi_3^2}\left|e^{-\mathsf{I}\frac{\alpha\pi}{2}}\xi_1(t-\mathsf{I}\xi_1)+\mathsf{I}e^{\mathsf{I}\frac{\alpha\pi}{2}}\left(\xi_2^2+\xi_3^2\right)\right| \\ =&\frac{t^{\alpha-1}}{t^2+\xi_1^2+\xi_2^2+\xi_3^2}\left(|\xi_1 t|+|\xi|^2\right) \leq \frac{t^{\alpha-1}}{t^2+|\xi|^2}\left(|\boldsymbol{\xi}|t+|\boldsymbol{\xi}|^2\right). \end{aligned}$$

Similarly, one sees that also the $(2,2)$-entry of $A_1(t,\boldsymbol{\xi})$ satisfies this estimate. For the $(1,2)$-entry we have on the other hand

$$\begin{aligned} &\frac{t^{\alpha-1}}{t^2+\xi_1^2+\xi_2^2+\xi_3^2}\left|\left(\mathsf{I}e^{-\mathsf{I}\frac{\alpha\pi}{2}}\xi_1+e^{\mathsf{I}\frac{\alpha\pi}{2}}(-t+\mathsf{I}\xi_1)\right)(\mathsf{I}\xi_2+\xi_3)\right| \\ \leq&\frac{t^{\alpha-1}}{t^2+\xi_1^2+\xi_2^2+\xi_3^2}\left(2|\xi_1||\xi_2+\mathsf{I}\xi_3|+t|\xi_2+\mathsf{I}\xi_3|\right) \leq \frac{2t^{\alpha-1}}{t^2+|\boldsymbol{\xi}|^2}\left(|\boldsymbol{\xi}|^2+t|\boldsymbol{\xi}|\right). \end{aligned}$$

Similar computations show that the $(2,1)$-entry does also satisfy this estimate and hence we deduce from (11.16) that

$$\|A_1(t,\boldsymbol{\xi})\| \ \leq 2C\frac{t^{\alpha-1}}{t^2+|\boldsymbol{\xi}|^2}\left(|\boldsymbol{\xi}|t+|\boldsymbol{\xi}|^2\right).$$

Analogous arguments show that this estimate is also satisfied by $\|A_2(t,\boldsymbol{\xi})\|$.

For the integrals in (11.15) we hence obtain

$$\begin{aligned} &\int_0^{+\infty} \|M_{A_1(t,\boldsymbol{\xi})}\widehat{v}\|_{\widehat{V}}dt + \int_0^{+\infty} \|M_{A_2(t,\boldsymbol{\xi})}\widehat{v}\|_{\widehat{V}}\,dt \\ \leq&2\int_0^{+\infty} 2C\left\|\frac{t^{\alpha-1}}{t^2+|\boldsymbol{\xi}|^2}\left(|\boldsymbol{\xi}|t+|\boldsymbol{\xi}|^2\right)|\widehat{v}(\boldsymbol{\xi})|\right\|_{L^2(\mathbb{R}^3)} dt \\ \leq&4C\int_0^1 t^{\alpha-1}\left\|\frac{|\boldsymbol{\xi}|t}{t^2+|\boldsymbol{\xi}|^2}|\widehat{v}(\boldsymbol{\xi})|+\frac{|\boldsymbol{\xi}|^2}{t^2+|\boldsymbol{\xi}|^2}|\widehat{v}(\boldsymbol{\xi})|\right\|_{L^2(\mathbb{R}^3)} dt \\ &+4C\int_1^{+\infty} t^{\alpha-2}\left\|\frac{t^2}{t^2+|\boldsymbol{\xi}|^2}|\boldsymbol{\xi}\widehat{v}(\boldsymbol{\xi})|+\frac{t|\boldsymbol{\xi}|}{t^2+|\boldsymbol{\xi}|^2}|\boldsymbol{\xi}\widehat{v}(\boldsymbol{\xi})|\right\|_{L^2(\mathbb{R}^3)} dt. \end{aligned}$$

Now observe that

$$\frac{t^2}{t^2+|\boldsymbol{\xi}|^2} \leq 1, \qquad \frac{|\boldsymbol{\xi}|^2}{t^2+|\boldsymbol{\xi}|^2} \leq 1, \qquad \frac{t|\boldsymbol{\xi}|}{t^2+|\boldsymbol{\xi}|^2} \leq \frac{1}{2} < 1.$$

Because of (11.12), the relation $\widehat{v} \in \Psi\, \mathrm{dom}(\nabla)$ implies that $|\widehat{v}(\boldsymbol{\xi})|$ and $||\boldsymbol{\xi}|\widehat{v}(\boldsymbol{\xi})|$ both

belong to $L^2(\mathbb{R}^3)$ and hence we finally find

$$\begin{aligned}&\int_0^{+\infty} \|M_{A_1(t,\boldsymbol{\xi})}\widehat{v}\|_{\widehat{V}} dt + \int_0^{+\infty} \|M_{A_2(t,\boldsymbol{\xi})}\widehat{v}\|_{\widehat{V}}\, dt\\ \leq& 8C\|v(\boldsymbol{\xi})\|_{L^2(\mathbb{R}^3)} \int_0^1 t^{\alpha-1}\, dt + 8C\|\boldsymbol{\xi}\widehat{v}(\boldsymbol{\xi})\|_{L^2(\mathbb{R}^3)} \int_1^{+\infty} t^{\alpha-2}\, dt,\end{aligned}$$

which is finite as $\alpha \in (0,1)$. Hence (11.15) converges for any $\widehat{v} \in \Psi\,\mathrm{dom}(\nabla)$ and (11.10) converges in turn for any $v \in \mathrm{dom}(\nabla)$.

□

Theorem 11.7. *The operator $f_\alpha(\nabla)$ extends to a closed operator on $L^2(\mathbb{R}^3, \mathbb{H})$. For $v \in \mathrm{dom}(\nabla^2) = \mathrm{dom}(-\Delta)$, it is moreover given by*

$$f_\alpha(\nabla)v = (-\Delta)^{\frac{\alpha}{2}-1}\left[\frac{1}{2}(-\Delta)^{\frac{1}{2}} + \frac{1}{2}\nabla\right]\nabla v. \tag{11.17}$$

Proof. Let $v \in \mathrm{dom}(\nabla^2) = \mathrm{dom}(-\Delta)$. We have because of (11.7) that

$$\begin{aligned}f_\alpha(\nabla)v =& \frac{1}{2\pi}\int_{-\infty}^{+\infty} (-\mathsf{I}t\mathcal{I} + \nabla)\mathcal{Q}_{c,-\mathsf{I}t}(\nabla)^{-1}(-\mathsf{I})^2(-t\mathsf{I})^{\alpha-1}\nabla v\, dt\\ =&-\frac{1}{2\pi}\int_0^{+\infty} (-\mathsf{I}t\mathcal{I} + \nabla)\mathcal{Q}_{c,-\mathsf{I}t}(\nabla)^{-1}t^{\alpha-1}e^{-\mathsf{I}(\alpha-1)\frac{\pi}{2}}\nabla v\, dt\\ &-\frac{1}{2\pi}\int_0^{+\infty} (\mathsf{I}t\mathcal{I} + \nabla)\mathcal{Q}_{c,\mathsf{I}t}(\nabla)^{-1}t^{\alpha-1}e^{\mathsf{I}(\alpha-1)\frac{\pi}{2}}\nabla v\, dt.\end{aligned} \tag{11.18}$$

Due to (11.9), we have moreover

$$\mathcal{Q}_{c,\mathsf{I}t}(\nabla)^{-1} = (-t^2+\Delta)^{-1} = \mathcal{Q}_{c,-\mathsf{I}t}(\nabla)^{-1}$$

and hence

$$\begin{aligned}f_\alpha(\nabla)v =& -\frac{1}{2\pi}\int_0^{+\infty} t^\alpha\mathcal{Q}_{c,\mathsf{I}t}(\nabla)^{-1}\mathsf{I}\left(e^{\mathsf{I}(\alpha-1)\frac{\pi}{2}} - e^{-\mathsf{I}(\alpha-1)\frac{\pi}{2}}\right)\nabla v\, dt\\ &-\frac{1}{2\pi}\int_0^{+\infty} \nabla\mathcal{Q}_{c,\mathsf{I}t}(\nabla)^{-1}t^{\alpha-1}\left(e^{\mathsf{I}(\alpha-1)\frac{\pi}{2}} + e^{-\mathsf{I}(\alpha-1)\frac{\pi}{2}}\right)\nabla v\, dt\\ =&\frac{\sin\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} t^\alpha\mathcal{Q}_{c,\mathsf{I}t}(\nabla)^{-1}\nabla v\, dt\\ &-\frac{\cos\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} \nabla\mathcal{Q}_{c,\mathsf{I}t}(\nabla)^{-1}t^{\alpha-1}\nabla v\, dt.\end{aligned} \tag{11.19}$$

For the first integral, we obtain

$$\begin{aligned}&\frac{\sin\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} t^\alpha\mathcal{Q}_{c,\mathsf{I}t}(\nabla)^{-1}\nabla v\, dt\\ =&\frac{\sin\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} t^\alpha(-t^2+\Delta)^{-1}\nabla v\, dt\\ =&\frac{\sin\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} \tau^{\frac{\alpha-1}{2}}(-\tau+\Delta)^{-1}\nabla v\, d\tau\\ =&\frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}\nabla v.\end{aligned} \tag{11.20}$$

The last identity follows from the integral representation for the fractional power A^β with $\mathrm{Re}(\beta) \in (0,1)$ of a complex linear sectorial operator A given in Corollary 3.1.4 of [59], namely

$$A^\beta v = \frac{\sin(\pi\beta)}{\pi} \int_0^{+\infty} \tau^\beta \left(\tau + A^{-1}\right)^{-1} v \, d\tau, \qquad v \in \mathrm{dom}(A). \tag{11.21}$$

As $-\Delta$ is an injective sectorial operator on $L^2(\mathbb{R}^3, \mathbb{C}_{\mathsf{I}})$, also its closed inverse $(-\Delta)^{-1}$ is a sectorial operator. Its fractional power $((-\Delta)^{-1}))^{\frac{1-\alpha}{2}}$ is, because of (11.21), given by the last integral in (11.20). Since $(-\Delta)^{\frac{\alpha-1}{2}} = ((-\Delta)^{-1})^{\frac{1-\alpha}{2}}$, we obtain the last equality.

Observe that the expression $\frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}} \nabla v$ is meaningful as we chose $v \in \mathrm{dom}(\nabla^2)$. Indeed, if we consider the operators in the Fourier space $\widehat{V}$ as in the proof of Theorem 11.6, then $-\Delta$ corresponds to the multiplication operator $M_{|\boldsymbol{\xi}|^2}$ generated by the scalar function $|\boldsymbol{\xi}|^2$. The operator $(-\Delta)^{\frac{\alpha-1}{2}}$ is then the multiplication operator $M_{|\boldsymbol{\xi}|^{\alpha-1}}$ generated by the function $(|\boldsymbol{\xi}|^2)^{\frac{\alpha-1}{2}} = |\boldsymbol{\xi}|^{\alpha-1}$. Hence

$$\begin{aligned} \mathrm{dom}(-\Delta)^{-\frac{\alpha-1}{2}} &= \left\{ v \in L^2(\mathbb{R}^3, \mathbb{H}) : \widehat{v} \in \mathrm{dom}(M_{|\boldsymbol{\xi}|^{\alpha-1}}) \right\} \\ &= \left\{ v \in L^2(\mathbb{R}^3, \mathbb{H}) : |\boldsymbol{\xi}|^{\alpha-1} \widehat{v}(\boldsymbol{\xi}) \in \widehat{V} \right\}. \end{aligned}$$

If $G(\boldsymbol{\xi})$ is as in (11.5), then $\widehat{\nabla v}(\boldsymbol{\xi}) = M_G \widehat{v}(\boldsymbol{\xi}) = G(\boldsymbol{\xi})\widehat{v}(\boldsymbol{\xi}) \in \widetilde{V}$ and because of (11.13) we have $|G(\boldsymbol{\xi})\widehat{v}(\boldsymbol{\xi})| = |\boldsymbol{\xi}| \, |\widehat{v}(\boldsymbol{\xi})| \in L^2(\mathbb{R})$. As $\alpha \in (0,1)$, we therefore find that $|\boldsymbol{\xi}|^{\alpha-1} |M_G \widehat{v}(\boldsymbol{\xi})| = |\boldsymbol{\xi}|^\alpha |\widehat{v}(\boldsymbol{\xi})|$ belongs to $L^2(\mathbb{R}^3)$ and so $\widehat{\nabla v} \in \mathrm{dom}(M_{|\boldsymbol{\xi}|^{\alpha-1}})$. This is equivalent to $\nabla v \in \mathrm{dom}\left((-\Delta)^{\frac{\alpha-1}{2}}\right)$.

As $v \in \mathrm{dom}(\nabla^2) = \mathrm{dom}(-\Delta)$, we obtain similarly that the second integral in (11.19) equals

$$\begin{aligned} & -\frac{\cos\left((\alpha-1)\frac{\pi}{2}\right)}{\pi} \int_0^{+\infty} \nabla \mathcal{Q}_{c,\mathsf{I}t}(\nabla)^{-1} t^{\alpha-1} \nabla v \, dt \\ =& \frac{\sin\left((\alpha-2)\frac{\pi}{2}\right)}{\pi} \int_0^{+\infty} \nabla(-t^2 + \Delta)^{-1} t^{\alpha-1} \nabla v \, dt \\ =& \frac{\sin\left((\alpha-2)\frac{\pi}{2}\right)}{2\pi} \int_0^{+\infty} (-\tau + \Delta)^{-1} \tau^{\frac{\alpha-2}{2}} \nabla^2 v \, d\tau \\ =& \frac{1}{2}(-\Delta)^{\frac{\alpha}{2}-1} \nabla^2 v. \end{aligned} \tag{11.22}$$

Again this expression is meaningful as we assumed $v \in \mathrm{dom}(\nabla^2)$. This is equivalent to $|\boldsymbol{\xi}|^2 \widehat{v}(\boldsymbol{\xi}) \in \widehat{V}$ because $\widehat{\nabla^2 v}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2 \widehat{v}(\boldsymbol{\xi})$. Since $\alpha \in (0,1)$ and $\widehat{v} \in \mathrm{dom}(M_{|\boldsymbol{\xi}|^2})$, the function $|\boldsymbol{\xi}|^2 \widehat{v}(\boldsymbol{\xi})$ belongs to the domain of the multiplication operator $M_{|\boldsymbol{\xi}|^{\alpha-2}}$ because $M_{|\boldsymbol{\xi}|^{\alpha-2}} |\boldsymbol{\xi}|^2 \widehat{v}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^\alpha \widehat{v}(\boldsymbol{\xi}) \in \widehat{V}$. Since $(-\Delta)^{\frac{\alpha}{2}-1}$ corresponds to $M_{|\boldsymbol{\xi}|^{\alpha-2}}$ on the Fourier space $\widehat{V}$, we find $\nabla^2 v$ in $\mathrm{dom}\left((-\Delta)^{\frac{\alpha}{2}-1}\right)$.

Altogether, we find

$$f_\alpha(\nabla) v = (-\Delta)^{\frac{\alpha}{2}-1} \left[\frac{1}{2}(-\Delta)^{\frac{1}{2}} + \frac{1}{2}\nabla \right] \nabla v \qquad \forall v \in \mathrm{dom}(\nabla^2). \tag{11.23}$$

Finally, we show that $f_\alpha(\nabla)$ can be extended to a closed operator. We need to show that for any sequence $v_n \in \operatorname{dom}(f_\alpha(\nabla)) = \operatorname{dom}(\nabla)$ that converges to 0 and for which also the sequence $f_\alpha(\nabla)v_n$ converges, we have $z := \lim_{n\to+\infty} f_\alpha(\nabla)v_n = 0$. In order to do this, we write as in (11.19)

$$\begin{aligned} f_\alpha(\nabla)v =& \frac{\sin\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} t^\alpha(t^2+\Delta)^{-1}\nabla v\,dt \\ &- \frac{\cos\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} \nabla(t^2+\Delta)^{-1}t^{\alpha-1}\nabla v\,dt. \end{aligned}$$

If we choose an arbitrary, but fixed $r > 0$, then $(r+\Delta)^{-1}$ commutes with $(t^2+\Delta)^{-1}$ and ∇ and we deduce from the above integral representation that

$$(r+\Delta)^{-1}f_\alpha(\nabla)v = f_\alpha(\nabla)(r+\Delta)^{-1}v \qquad \forall v \in \operatorname{dom}(\nabla).$$

We show now that the mapping $v \mapsto f_\alpha(\nabla)(r+\Delta)^{-1}v$ is a bounded linear operator on $L^2(\mathbb{R}^3,\mathbb{H})$. Since $(r+\Delta)^{-1}$ maps $L^2(\mathbb{R}^3,\mathbb{H})$ to $\operatorname{dom}(\Delta) = \operatorname{dom}(\nabla^2)$, the composition $\nabla^2(r+\Delta)^{-1}$ of the bounded operator $(r+\Delta)^{-1}$ and the closed operator ∇^2 is bounded itself. As we have seen above, ∇^2 and in turn also the bounded operator $\nabla^2(r+\Delta)^{-1}$ map $L^2(\mathbb{R}^3,\mathbb{H})$ into the domain of the closed operator $(-\Delta)^{\frac{\alpha}{2}-1}$. Hence, also their composition $(-\Delta)^{-\frac{\alpha}{2}-1}\nabla^2(r+\Delta)^{-1}$ is therefore bounded. Similarly, $\nabla(r+\Delta)^{-1}$ is a bounded operator that maps $L^2(\mathbb{R}^3,\mathbb{H})$ to $\operatorname{dom}\big((-\Delta)^{\frac{\alpha-1}{2}}\big)$ as we have seen above, and so the composition $(-\Delta)^{\frac{\alpha-1}{2}}\nabla(r+\Delta)^{-1}$ is also bounded. Because of (11.23), the operator

$$f_\alpha(\nabla)(r+\Delta)^{-1} = \frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}\nabla(r+\Delta)^{-1} \ + \frac{1}{2}(-\Delta)^{\frac{\alpha}{2}-1}\nabla^2(r+\Delta)^{-1}$$

is the linear combination of bounded operators and hence bounded itself.

If a sequence $v_n \in \operatorname{dom}(f_\alpha(\nabla))$ converges to 0 and $z = \lim_{n\to+\infty} f_\alpha(\nabla)v_n$ exists in $L^2(\mathbb{R}^3,\mathbb{H})$, then

$$(r+\Delta)^{-1}z = \lim_{n\to+\infty}(r+\Delta)^{-1}f_\alpha(\nabla)v_n = \lim_{n\to+\infty} f_\alpha(\nabla)(r+\Delta)^{-1}v_n = 0.$$

But as $(r+\Delta)^{-1}$ is the inverse of a closed operator, its kernel is trivial and so $z = \lim_{n\to+\infty} f_\alpha(\nabla)v_n = 0$. Hence, $f_\alpha(\nabla)$ can be extended to a closed operator.

□

Remark 11.8. The identity (11.17) might seem surprising at the first glance, but it is actually rather intuitive. By the spectral theorem there exist two spectral measures $E_{(-\Delta)}$ and E_∇ on $[0,+\infty)$ resp. $\mathbb{R}$ such that $-\Delta = \int_{[0,+\infty)} t\,dE_{-\Delta}(t)$ and $\nabla = \int_{\mathbb{R}} r\,dE_\nabla(r)$. As $\nabla^2 = -\Delta$, the spectral measure $E_{(-\Delta)}$ is furthermore the push-forward measure of E_∇ under the mapping $t \mapsto t^2$ such that

$$\int_{[0,+\infty)} f(t)\,dE_{(-\Delta)}(t) = \int_{\mathbb{R}} f\left(t^2\right)\,dE_\nabla(t)$$

for any measurable function f. Hence, we have for $v \in \operatorname{dom}(\nabla^2)$ that

$$\begin{aligned}
f^{\alpha}(\nabla) =& \int_{\mathbb{R}} t^{\alpha}\chi_{[0,+\infty)}(t)\, dE_{\nabla}(t)v\\
=& \int_{\mathbb{R}} t^{\alpha-2}\frac{1}{2}(|t|+t)t\, dE_{\nabla}(t)v\\
=& \int_{\mathbb{R}} t^{\alpha-2}\, dE_{\nabla}(t)\frac{1}{2}\left(\int_{\mathbb{R}}|t|\, dE_{\nabla}(t) + \int_{\mathbb{R}} t\, dE_{\nabla}(t)\right)\int_{\mathbb{R}} t\, dE_{\nabla}(t)v\\
=& \int_{[0,+\infty)} t^{\frac{\alpha}{2}-1}\, dE_{(-\Delta)}(t)\frac{1}{2}\\
&\cdot\left(\int_{[0,+\infty)}|t|^{\frac{1}{2}}\, dE_{(-\Delta)}(t) + \int_{\mathbb{R}} t\, dE_{\nabla}(t)\right)\int_{\mathbb{R}} t\, dE_{\nabla}(t)v\\
=&(-\Delta)^{-\frac{\alpha}{2}-1}\left[\frac{1}{2}(-\Delta)^{\frac{1}{2}} + \frac{1}{2}\nabla\right]\nabla v.
\end{aligned}$$

The vector part of $f_{\alpha}(\nabla)$ is because of (11.17) given by

$$\operatorname{Vec} f_{\alpha}(\nabla)v = \frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}\nabla v.$$

If we apply the divergence to this equation with sufficiently regular v, we find

$$\operatorname{div}\left(\operatorname{Vec} f_{\alpha}(\nabla)v\right) = \frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}\Delta v = -\frac{1}{2}(-\Delta)^{\frac{\alpha+1}{2}}.$$

We can thus reformulate the fractional heat equation (11.2) with $\alpha \in (1/2, 1)$ as

$$\frac{\partial}{\partial t}v - 2\operatorname{div}\left(\operatorname{Vec} f_{\beta}(\nabla)v\right) = 0, \qquad \beta = 2\alpha - 1.$$

11.3.1 An Example with Non-Constant Coefficients

As pointed out before, the advantage of the above procedure is that is does not only apply to the gradient to reproduce the fractional Laplacian. Instead it applies to a large class of vector operators, in particular generalized gradients with non-constant coefficients. As a first example, we consider the operator

$$T := \xi_1\frac{\partial}{\partial \xi_1}e_1 + \xi_2\frac{\partial}{\partial \xi_2}e_2 + \xi_3\frac{\partial}{\partial \xi_3}e_3$$

on the space $L^2(\mathbb{R}^3_+, \mathbb{H}, d\mu)$ of $\mathbb{H}$-valued functions on $\mathbb{R}^3_+ = \{\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^T \in \mathbb{R}^3 : \xi_\ell > 0\}$ that are square integrable with respect to $d\mu(\boldsymbol{\xi}) = \frac{1}{\xi_1,\xi_2,\xi_3}d\lambda(\boldsymbol{\xi})$, where λ denotes the Lebesgue measure on $\mathbb{R}^3$. In order to determine $\mathcal{Q}_s(T)^{-1}$ we observe that the operator given by the change of variables $J : f \mapsto f \circ \iota$ with $\iota(\mathbf{x}) = (e^{x_1}, e^{x_2}, e^{x_3})^T$ is an isometric isomorphism between $L^2(\mathbb{R}^3, \mathbb{H}, d\lambda(\mathbf{x}))$ and $L^2(\mathbb{R}^3_+, \mathbb{H}, d\mu(\boldsymbol{\xi}))$. Moreover, $T = J^{-1}\nabla J$ such that

$$Q_s(T) = (s^2\mathcal{I} + T\overline{T}) = J^{-1}(s^2\mathcal{I} + \Delta)J$$

and in turn

$$Q_s(T)^{-1} := (s^2\mathcal{I} - T\overline{T})^{-1} = J^{-1}(s^2\mathcal{I} + \Delta)^{-1}J.$$

We therefore have for sufficiently regular v with calculations analogue to those in (11.18) and (11.19) that

$$\begin{aligned} f_\alpha(T)v =& \frac{\sin((\alpha-1)\pi)}{\pi}\int_0^{+\infty} t^\alpha(-t^2+T\overline{T})^{-1}T\,dt \\ &+ \frac{\cos((\alpha-1)\pi)}{\pi}\int_0^{+\infty} t^{\alpha-1}T(-t^2+T\overline{T})^{-1}Tv\,dt. \end{aligned}$$

Clearly, the vector part of this operator is again given by the first integral such that

$$\begin{aligned} \operatorname{Vec} f_\alpha(T)v =& \frac{\sin((\alpha-1)\pi)}{\pi}\int_0^{+\infty} t^\alpha(-t^2+T\overline{T})^{-1}Tv\,dt \\ =& \frac{\sin((\alpha-1)\pi)}{\pi}\int_0^{+\infty} t^\alpha J^{-1}(-t^2+\Delta)^{-1}JTv\,dt \\ =& J^{-1}\frac{\sin((\alpha-1)\pi)}{\pi}\int_0^{+\infty} t^\alpha(-t^2+\Delta)^{-1}\,dt\,JTv \\ =& \frac{1}{2}J^{-1}(-\Delta)^{\frac{\alpha-1}{2}}JTv, \end{aligned}$$

where the last equation follows from computations as in (11.23). Choosing $\beta = 2\alpha+1$ we thus find for sufficiently regular v that

$$\begin{aligned} &\operatorname{Vec} f_\beta(T)v(\boldsymbol{\xi}) \\ =& \frac{1}{2}J^{-1}(-\Delta)^\alpha JTv(\xi_1,\xi_2,\xi_3) \\ =& \frac{1}{2}J^{-1}(-\Delta)^\alpha \begin{pmatrix} e^{x_1}v_{\xi_1}(e^{x_1},e^{x_2},e^{x_3}) \\ e^{x_2}v_{\xi_2}(e^{x_1},e^{x_2},e^{x_3}) \\ e^{x_3}v_{\xi_3}(e^{x_1},e^{x_2},e^{x_3}) \end{pmatrix} \\ =& \frac{1}{2}J^{-1}\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} -|\mathbf{y}|^{2\alpha}e^{i\mathbf{z}\cdot\mathbf{y}}e^{-\mathbf{x}\cdot\mathbf{y}} \begin{pmatrix} e^{x_1}v_{\xi_1}(e^{x_1},e^{x_2},e^{x_3}) \\ e^{x_2}v_{\xi_2}(e^{x_1},e^{x_2},e^{x_3}) \\ e^{x_3}v_{\xi_3}(e^{x_1},e^{x_2},e^{x_3}) \end{pmatrix} d\mathbf{x}\,d\mathbf{y} \\ =& \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} -|\mathbf{y}|^{2\alpha}e^{i\sum_{k=1}^3\xi_k y_k}e^{-i\mathbf{x}\cdot\mathbf{y}} \begin{pmatrix} e^{x_1}v_{\xi_1}(e^{x_1},e^{x_2},e^{x_3}) \\ e^{x_2}v_{\xi_2}(e^{x_1},e^{x_2},e^{x_3}) \\ e^{x_3}v_{\xi_3}(e^{x_1},e^{x_2},e^{x_3}) \end{pmatrix} d\mathbf{x}\,d\mathbf{y}. \end{aligned}$$

The above computations are just a first easy example to illustrate that more complicated operators than the nabla operator can be considered with the introduced techniques. In particular, one can hope to define and study new types of fractional evolution equations derived from generalized gradient operators with non-constant coefficients of the form

$$T = a_1(\mathbf{x})\frac{\partial}{\partial x_1}e_1 + a_2(\mathbf{x})\frac{\partial}{\partial x_2}e_2 + a_3(\mathbf{x})\frac{\partial}{\partial x_3}e_3. \tag{11.24}$$

The version of the S-functional calculus for operators with commuting components, which we applied in order to study the nabla operator, simplifies the computations considerably. The true power of the theory however becomes apparent when studying operators of the form (11.24), the components of which do not necessarily commute, and here the field is still totally to explore.

CHAPTER 12

On the Equivalence of Complex and Quaternionic Quantum Mechanics

One approach to operator theory in the quaternionic setting consists in considering the quaternionic right linear space V_R as the quaternionification $V_R = V_{\mathsf{i}} \oplus V_{\mathsf{i}}\mathsf{j}$ with $\mathsf{i}, \mathsf{j} \in \mathbb{S}$ and $\mathsf{i} \perp \mathsf{j}$ of a properly chosen $\mathbb{C}_{\mathsf{i}}$-complex subspace V_{i} of V_R. If this subspace is chosen to suit a certain quaternionic linear operator T, then this operator is simply the quaternionic linear extension of a complex linear operator T_{i} on the complex component space V_{i}. It is then sufficient to study the complex linear operator T_{i} in order to fully understand the quaternionic linear operator T and this can be done using traditional techniques from complex operator theory. This strategy is often not applicable, but in particular for normal operators on quaternionic Hilbert spaces it is very useful. If T is for instance an operator on a quaternionic Hilbert space $\mathcal{H}$ and there exists a unitary and anti-selfadjoint operator J on $\mathcal{H}$ that commutes with T, then we can choose $\mathsf{i} \in \mathbb{S}$ and define

$$\mathcal{H}_{\mathsf{J},\mathsf{i}}^{+} := \{ \mathbf{v} \in \mathcal{H} : \mathsf{J}\mathbf{v} = \mathbf{v}\mathsf{i}\} \quad \text{and} \quad \mathcal{H}_{\mathsf{J},\mathsf{i}}^{-} := \{ \mathbf{v} \in \mathcal{H} : \mathsf{J}\mathbf{v} = \mathbf{v}(-\mathsf{i})\}.$$

The sets $\mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}$ and $\mathcal{H}_{\mathsf{J},\mathsf{i}}^{-}$ are $\mathbb{C}_{\mathsf{i}}$-complex Hilbert spaces with the operations and the scalar product they inherit from $\mathcal{H}$. Furthermore

$$\mathcal{H} = \mathcal{H}_{\mathsf{J},\mathsf{i}}^{+} \oplus \mathcal{H}_{\mathsf{J},\mathsf{i}}^{-}$$

and if $\mathsf{j} \in \mathbb{S}$ with $\mathsf{i} \perp \mathsf{j}$, then $\mathbf{v} \mapsto \mathbf{v}\mathsf{j}$ is a $\mathbb{C}_{\mathsf{i}}$-antilinear isometric bijection from $\mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}$ to $\mathcal{H}_{\mathsf{J},\mathsf{i}}^{-}$ and so any $\mathbf{v} \in \mathcal{H}$ can be written as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2\mathsf{j}$ with $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}$. The operator T leaves the space $\mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}$ invariant as $\mathsf{J}T(\mathbf{v}) = T\mathsf{J}(\mathbf{v}) = (T\mathbf{v})\mathsf{i}$ for any $\mathbf{v} \in \mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}$. The $\mathbb{C}_{\mathsf{i}}$-linear operator $T_{\mathbb{C}_{\mathsf{i}}} := T|_{\mathcal{H}_{\mathsf{J},\mathsf{i}+}}$ on $\mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}$ has then properties that are analogue to the one of T. The spectrum $\sigma(T_{\mathbb{C}_{\mathsf{i}}})$ is a subset of $\sigma_S(T) \cap \mathbb{C}_{\mathsf{i}}$ and $T_{\mathbb{C}_{\mathsf{i}}}$ is bounded, normal, (anti-)

selfadjoint or unitary if and only if T has these properties. These ideas go back to [82] and were used intensively in [49, 51]. The above approach is however not suitable for all quaternionic linear operators—Example 10.18 provides an example of an operator that cannot be treated as the quaternionic linear extension of a complex linear operator on a suitable component space.

In this final chapter we use the fundamental understanding that we gained about quaternionic linear operators in order to argue that there exists a substantial logical flaw in the current formulation of quaternionic quantum mechanics. In particular, we conjecture that any quaternionic quantum system can be reduced to a complex quantum system on a suitable component space, just as certain quaternionic linear operators can be reduced to complex linear operators on a suitable complex component space as described above. We furthermore show that this conjecture holds true for elementary relativistic systems in the sense of [70], which seem to be the only type systems for which also the equivalence of real and complex quantum mechanics is shown properly. Finally, we conclude with a chapter that discusses how the misconception that complex and quaternionic quantum mechanics are not equivalent arose from the wrong assumption that there exists a physically determined quaternionic left multiplication on the Hilbert space that serves as state space. All presented results are published in [46].

12.1 Internal and External Quaternionification

We start with recalling two methods for constructing complex and quaternionic Hilbert spaces starting from a real Hilbert space, the procedures of *internal* and *external* complexification and quaternionification as developed by Sharma in [76].

Let $\mathcal{H}$ be a real Hilbert space. Then we can construct a complex Hilbert space $E_{\mathbb{C}}(\mathcal{H}) = \mathcal{H} \otimes \mathbb{C}$ from $\mathcal{H}$ by considering couples of vectors in $\mathcal{H}$. Precisely, we define $E_{\mathbb{C}}(\mathcal{H}) := \mathcal{H}^2$, set for $(\mathbf{v}_1, \mathbf{v}_2), (\mathbf{u}_1, \mathbf{u}_2) \in E_{\mathbb{C}}(\mathcal{H})$ and $a = a_0 + ia_1 \in \mathbb{C}$

$$\begin{aligned}(\mathbf{v}_1, \mathbf{v}_2) + (\mathbf{u}_1, \mathbf{u}_2) :=& (\mathbf{v}_1 + \mathbf{u}_1, \mathbf{v}_2 + \mathbf{u}_2)\\ a(\mathbf{v}_1, \mathbf{v}_2) :=& (a_1\mathbf{v}_1 - a_2\mathbf{v}_2, a_1\mathbf{v}_2 + a_2\mathbf{v}_1)\end{aligned}$$

and define a complex scalar product on $E_{\mathbb{C}}(\mathcal{H})$ as

$$\langle(\mathbf{v}_1, \mathbf{v}_2), (\mathbf{u}_1, \mathbf{u}_2)\rangle_{E_{\mathbb{C}}(\mathcal{H})} := \langle\mathbf{v}_1, \mathbf{u}_1\rangle_{\mathcal{H}} + \langle\mathbf{v}_2, \mathbf{u}_2\rangle_{\mathcal{H}} + i\left(\langle\mathbf{v}_1, \mathbf{u}_2\rangle_{\mathcal{H}} - \langle\mathbf{v}_2, \mathbf{u}_1\rangle_{\mathcal{H}}\right).$$

This is the standard procedure for complexifying a real Hilbert space and it corresponds to writing the couples $(\mathbf{v}_1, \mathbf{v}_2)$ in $E_{\mathbb{C}}(\mathcal{H})$ as $\mathbf{v}_1 + i\mathbf{v}_2$ and performing formal computations using the structure on $\mathcal{H}$ and the fact that $i^2 = -1$. We shall use this notation in the following since it is more convenient.

Definition 12.1. Let $\mathcal{H}$ be a real Hilbert space. We call the complex Hilbert space $E_{\mathbb{C}}(\mathcal{H}) = \mathcal{H} \otimes \mathbb{C}$ the *external complexification* of $\mathcal{H}$.

Any operator T on $\mathcal{H}$ has a unique complex linear extension $T_{\mathbb{C}}$ to $E_{\mathbb{C}}(\mathcal{H})$ that is obtained by componentwise application, namely

$$T_{\mathbb{C}}(\mathbf{v}_1 + i\mathbf{v}_2) = T(\mathbf{v}_1) + iT(\mathbf{v}_2).$$

Theorem 12.2. *Let $\mathcal{H}$ be a real Hilbert space and let $E_{\mathbb{C}}(\mathcal{H})$ be its external complexification.*

(i) The space $E_{\mathbb{C}}(\mathcal{H})$ is a complex Hilbert space of the same dimension as $\mathcal{H}$ and a set of vectors in $\mathcal{H}$ is an orthonormal basis of $\mathcal{H}$ if and only if it is an orthonormal basis of $E_{\mathbb{C}}(\mathcal{H})$.

(ii) If $T : \operatorname{dom}(T) \subset \mathcal{H} \to \mathcal{H}$ is an $\mathbb{R}$-linear operator, then the domain of its complex linear extension $T_{\mathbb{C}}$ is $\operatorname{dom}(T_{\mathbb{C}}) = \operatorname{dom}(T) + \operatorname{dom}(T)i$. Furthermore, T is bounded if and only if $T_{\mathbb{C}}$ is bounded and in this case $\|T\| = \|T_{\mathbb{C}}\|$. The extension is compatible with the adjoint, that is $(T_{\mathbb{C}})^* = T_{\mathbb{C}}^*$ and consequently $T_{\mathbb{C}}$ is (anti-) selfadjoint, normal or unitary on $E_{\mathbb{C}}(\mathcal{H})$ if and only if T is (anti-)selfadjoint, normal or unitary on $\mathcal{H}$.

Similarly, we can construct a quaternionic Hilbert space $E_{\mathbb{H}}(\mathcal{H})$ from a real Hilbert space $\mathcal{H}$. We can choose $\mathbf{i}_1, \mathbf{i}_2 \in \mathbb{S}$ with $\mathbf{i}_1 \perp \mathbf{i}_2$ and set $\mathbf{i}_0 = 1$ and $\mathbf{i}_3 = \mathbf{i}_1\mathbf{i}_2$, so that any quaternion $a \in \mathbb{H}$ can be written as $a = \sum_{\ell=0}^{3} a_\ell \mathbf{i}_\ell$. We then set

$$E_{\mathbb{H}}(\mathcal{H}) := \mathcal{H} \otimes \mathbb{H} \cong \left\{ \sum_{\ell=0}^{3} \mathbf{v}_\ell \mathbf{i}_\ell : \mathbf{v}_\ell \in \mathcal{H} \right\},$$

and define as in Remark 2.33 for $\mathbf{v} = \sum_{\ell=0}^{3} \mathbf{v}_\ell \mathbf{i}_\ell$ and $\mathbf{u} = \sum_{\ell=0}^{3} \mathbf{u}_\ell \mathbf{i}_\ell$ in $E_{\mathbb{H}}(\mathcal{H})$ and $a = \sum_{\ell=0}^{3} a_\ell \mathbf{i}_\ell \in \mathbb{H}$ the operations

$$\mathbf{v} + \mathbf{u} := \sum_{\ell=0}^{3} (\mathbf{v}_\ell + \mathbf{u}_\ell)\mathbf{i}_\ell, \qquad \mathbf{v}a := \sum_{\ell,\kappa=0}^{3} (a_\kappa \mathbf{v}_\ell)\mathbf{i}_\ell \mathbf{i}_\kappa,$$

and

$$\langle \mathbf{v}, \mathbf{u} \rangle_{E_{\mathbb{H}}(\mathcal{H})} := \sum_{\ell,\kappa=0}^{3} \langle \mathbf{v}_\ell, \mathbf{u}_\kappa \rangle_{\mathcal{H}} \mathbf{i}_\ell \mathbf{i} \kappa$$

This yields a quaternionic right Hilbert space. The choice of $\mathbf{i}_1$ and $\mathbf{i}_2$ in this construction is irrelevant since a different choice yields a Hilbert space that is isometrically isomorphic to $E_{\mathbb{H}}(\mathcal{H})$.

Definition 12.3. Let $\mathcal{H}$ be a real Hilbert space. We call the quaternionic Hilbert space $E_{\mathbb{H}}(\mathcal{H}) = \mathcal{H} \otimes \mathbb{H}$ the *external quaternionification* of $\mathcal{H}$.

Any operator T on $\mathcal{H}$ has a unique quaternionic linear extension $T_{\mathbb{H}}$ to $E_{\mathbb{H}}(\mathcal{H})$, which is obtained by componentwise application, namely

$$T_{\mathbb{H}}(\mathbf{v}) = \sum_{\ell=0}^{3} T(\mathbf{v}_\ell)\mathbf{i}_\ell \qquad \text{for } \mathbf{v} = \sum_{\ell=0}^{3} \mathbf{v}_\ell \mathbf{i}_\ell \in E_{\mathbb{H}}(\mathcal{H}).$$

Theorem 12.4. *Let $\mathcal{H}$ be a real Hilbert space and let $E_{\mathbb{H}}(\mathcal{H})$ be its external complexification.*

(i) The space $E_{\mathbb{H}}(\mathcal{H})$ is a quaternionic Hilbert space of the same dimension as $\mathcal{H}$ and a set of vectors in $\mathcal{H}$ is an orthonormal basis of $\mathcal{H}$ if and only if it is an orthonormal basis of $E_{\mathbb{H}}(\mathcal{H})$.

(ii) If $T : \mathrm{dom}(T) \subset \mathcal{H} \to \mathcal{H}$ is an $\mathbb{R}$-linear operator, then the domain of its quaternionic linear extension $T_{\mathbb{H}}$ is $\mathrm{dom}(T_{\mathbb{H}}) = \mathrm{dom}(T) \otimes \mathbb{H}$. Furthermore, T is bounded if and only if $T_{\mathbb{H}}$ is bounded and in this case $\|T\| = \|T_{\mathbb{H}}\|$. The extension is compatible with the adjoint, that is $(T_{\mathbb{H}})^* = T_{\mathbb{H}}^*$ and so $T_{\mathbb{H}}$ is (anti-) selfadjoint, normal or unitary on $E_{\mathbb{H}}(\mathcal{H})$ if and only if T is (anti-)selfadjoint, normal or unitary on $\mathcal{H}$.

Finally, we can construct in a similar manner a quaternionic Hilbert space from a complex Hilbert space. If $\mathcal{H}$ is a complex Hilbert space over $\mathbb{C}$, then we can choose $\mathbf{i}, \mathbf{j} \in \mathbb{S}$ with $\mathbf{i} \perp \mathbf{j}$ and identify the imaginary unit $\mathbf{i} \in \mathbb{H}$ with the complex imaginary unit $i \in \mathbb{C}$ so that $\mathcal{H}$ becomes a Hilbert space over $\mathbb{C}_{\mathbf{i}}$. We can then set

$$E_{\mathbb{H}}(\mathcal{H}) := \mathcal{H} \oplus \mathcal{H}\mathbf{j} \cong \{\mathbf{v}_1 + \mathbf{v}_2\mathbf{j} : \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{H}\}$$

and define for $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2\mathbf{j}$ and $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2\mathbf{j}$ in $E_{\mathbb{H}}(\mathcal{H})$ and $a = a_1 + a_2\mathbf{j} \in \mathbb{H}$ with $a_1, a_2 \in \mathbb{C}_{\mathbf{i}}$ the operations

$$\mathbf{v} + \mathbf{u} := (\mathbf{v}_1 + \mathbf{u}_1) + (\mathbf{v}_2 + \mathbf{u}_2)\mathbf{j}, \quad \mathbf{v}a := (\mathbf{v}_1 a_1 - \mathbf{v}_2\overline{a_2}) + (\mathbf{v}_1 a_2 + \mathbf{v}_2\overline{a_1})\mathbf{j}$$

and the scalar product

$$\langle \mathbf{v}, \mathbf{u} \rangle_{E_{\mathbb{H}}(\mathcal{H})} := \langle \mathbf{v}_1, \mathbf{u}_1 \rangle + \overline{\langle \mathbf{v}_2, \mathbf{u}_2 \rangle} + \Big(\langle \mathbf{v}_1, \mathbf{u}_2 \rangle - \overline{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle} \Big) \mathbf{j}.$$

This yields again a quaternionic right Hilbert space and the choice of $\mathbf{j} \in \mathbb{S}$ with $\mathbf{i} \perp \mathbf{j}$ in this construction is once more irrelevant since each choice yields a quaternionic Hilbert space that is isomorphically isomorphic to $E_{\mathbb{H}}(\mathcal{H})$.

Definition 12.5. Let $\mathcal{H}$ be a complex Hilbert space. We call the quaternionic Hilbert space $E_{\mathbb{H}}(\mathcal{H})$ the *external quaternionification* of $\mathcal{H}$.

Any complex linear operator T on $\mathcal{H}$ has a unique quaternionic linear extension $T_{\mathbb{H}}$ to $E_{\mathbb{H}}(\mathcal{H})$ that is obtained by component wise application, namely

$$T_{\mathbb{H}}(\mathbf{v}) = T(\mathbf{v}_1) + T(\mathbf{v}_2)\mathbf{j} \qquad \text{for } \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2\mathbf{j}.$$

Theorem 12.6. *Let $\mathcal{H}$ be a complex Hilbert space and let $E_{\mathbb{H}}(\mathcal{H})$ be its external quaternionification.*

(i) The space $E_{\mathbb{H}}(\mathcal{H})$ is a quaternionic Hilbert space of the same dimension as $\mathcal{H}$ and a set of vectors in $\mathcal{H}$ is an orthonormal basis of $\mathcal{H}$ if and only if it is an orthonormal basis of $E_{\mathbb{H}}(\mathcal{H})$.

(ii) If $T : \mathrm{dom}(T) \subset \mathcal{H} \to \mathcal{H}$ is a complex-linear operator, then the domain of its quaternionic linear extension $T_{\mathbb{H}}$ is $\mathrm{dom}(T_{\mathbb{H}}) = \mathrm{dom}(T) + \mathrm{dom}(T)\mathbf{j}$. Furthermore, T is bounded if and only if $T_{\mathbb{H}}$ is bounded and in this case $\|T\| = \|T_{\mathcal{H}}\|$. The extension is compatible with the adjoint, that is $(T_{\mathbb{H}})^* = T_{\mathbb{H}}^*$ and so $T_{\mathbb{H}}$ is (anti-) selfadjoint, normal or unitary on $E_{\mathbb{H}}(\mathcal{H})$ if and only if T is (anti-)selfadjoint, normal or unitary on $\mathcal{H}$.

External complexification or quaternionification happens by enlarging the underlying vector space and defining a complex or quaternionic structure on the enlarged space.

This is always possible. A different strategy is *internal complexification* resp. *quaternionification*, which happens by defining a complex resp. quaternionic linear structure on the existing space.

Let $\mathcal{H}$ be a real Hilbert space and let J be a unitary and anti-selfadjoint operator on $\mathcal{H}$, that is $\mathsf{J}^* = \mathsf{J}^{-1} = -\mathsf{J}$. Then $\mathsf{J}^2 = -\mathcal{I}$ and hence we can define the J-induced multiplication with complex scalars on $\mathcal{H}$ as

$$(a_0 + ia_1)\mathbf{v} := a_0\mathbf{v} + a_1\mathsf{J}\mathbf{v} \qquad \forall a = a_0 + ia_1 \in \mathbb{C}, \mathbf{v} \in \mathcal{H}$$

and the J-induced complex scalar product on $\mathcal{H}$ as

$$\langle \mathbf{v}, \mathbf{u}\rangle_{\mathsf{J}} := \langle \mathbf{v}, \mathbf{u}\rangle - i\langle \mathbf{v}, \mathsf{J}\mathbf{u}\rangle.$$

Since

$$\langle \mathbf{v}, \mathsf{J}\mathbf{v}\rangle = \langle \mathsf{J}^*\mathbf{v}, \mathbf{v}\rangle = -\langle \mathsf{J}\mathbf{v}, \mathbf{v}\rangle = -\langle \mathbf{v}, \mathsf{J}\mathbf{v}\rangle,$$

$\mathbf{v}$ and $\mathsf{J}\mathbf{v}$ are orthogonal in $\mathcal{H}$ for any $\mathbf{v} \in \mathcal{H}$ and so the norm induced by $\langle\cdot,\cdot\rangle_{\mathsf{J}}$ is the norm induced by $\langle\cdot,\cdot\rangle$.

Definition 12.7. We call the complex Hilbert space $\mathcal{H}_{\mathsf{J}} := (\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathsf{J}})$ the internal complexification of $\mathcal{H}$ that is induced by J.

Theorem 12.8. *Let $\mathcal{H}$ be a real Hilbert space and let J be an anti-selfadjoint and unitary operator on $\mathcal{H}$.*

(i) The space $\mathcal{H}_{\mathsf{J}}$ is a complex Hilbert space, the dimension of which is half of the dimension of $\mathcal{H}$, and a subset of $(\mathbf{v}_n)_{n\in\Lambda}$ of $\mathcal{H}$ is an orthonormal basis of $\mathcal{H}_{\mathsf{J}}$ if and only if $(\mathbf{v}_n)_{n\in\Lambda} \cup (\mathsf{J}\mathbf{v}_n)_{n\in\Lambda}$ is an orthonormal basis of $\mathcal{H}$. (In particular this implies that $\mathcal{H}$ has even dimension if its dimension is finite.)

(ii) An $\mathbb{R}$-linear operator $T : \operatorname{dom}(T) \subset \mathcal{H} \to \mathcal{H}$ is complex linear with respect to the J-induced structure if and only if T commutes with J. Such operator is bounded as an operator on $\mathcal{H}$ if and only if it is bounded as an operator on $\mathcal{H}_{\mathsf{J}}$ and in this case $\|T\|_{\mathcal{B}(\mathcal{H})} = \|T\|_{\mathcal{B}(\mathcal{H}_{\mathsf{J}})}$. Moreover the adjoint T^* of T on $\mathcal{H}$ is also the adjoint of T on $\mathcal{H}_J$ and hence T is (anti-)selfadjoint, normal or unitary on $\mathcal{H}$ if and only if it is bounded, (anti-)selfadjoint, normal or unitary on $\mathcal{H}_{\mathsf{J}}$.

Similarly, we can define an internal quaternionification of a real Hilbert space. If I and J are two anti-selfadjoint and unitary operators on $\mathcal{H}$ with $\mathsf{IJ} = -\mathsf{JI}$, then we can choose $\mathbf{i}, \mathbf{j} \in \mathbb{S}$ with $\mathbf{i} \perp \mathbf{j}$ and set $\mathbf{k} := \mathbf{ij}$. We can then define the multiplication of vectors in $\mathcal{H}$ with a quaternionic scalar $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \mathcal{H}$ from the right as

$$\mathbf{v}a := a_0\mathbf{v} + a_1\mathsf{I}\mathbf{v} + a_2\mathsf{J}\mathbf{v} + a_3\mathsf{JI}\mathbf{v}$$

and, with the abbreviation Θ for the quadruple $(\mathsf{I}, \mathsf{J}, \mathbf{i}, \mathbf{j})$, a quaternionic scalar product as

$$\langle \mathbf{v}, \mathbf{u}\rangle_{\Theta} := \langle \mathbf{v}, \mathbf{u}\rangle - \langle \mathbf{v}, \mathsf{I}\mathbf{u}\rangle\mathbf{i} - \langle \mathbf{v}, \mathsf{J}\mathbf{u}\rangle\mathbf{j} - \langle \mathbf{v}, \mathsf{JI}\mathbf{u}\rangle\mathbf{k}.$$

Definition 12.9. We call the quaternionic Hilbert space $\mathcal{H}_\theta := (\mathcal{H}, \langle\cdot,\cdot\rangle_\Theta)$ the internal quaternionification of $\mathcal{H}$ that is induced by the quadruple $\Theta = (\mathsf{I}, \mathsf{J}, \mathbf{i}, \mathbf{j})$.

Theorem 12.10. *Let $\mathcal{H}$ be a real Hilbert space and let $\Theta = (\mathsf{I}, \mathsf{J}, \mathbf{i}, \mathbf{j})$ be a quadruple consisting of two anti-selfadjoint and unitary operators on $\mathcal{H}$ that anticommute and two imaginary units $\mathbf{i}, \mathbf{j} \in \mathbb{S}$ with $\mathbf{i} \perp \mathbf{j}$.*

(i) *The space $\mathcal{H}_\Theta$ is a quaternionic (right) Hilbert space, the dimension of which is a quarter of the dimension of $\mathcal{H}$ and a subset $(\mathbf{v}_n)_{n\in\Lambda}$ of $\mathcal{H}$ is an orthonormal basis for $\mathcal{H}_\Theta$ if and only if $(\mathbf{v}_n, \mathsf{I}\mathbf{v}_n, \mathsf{J}\mathbf{v}_n, \mathsf{J}\mathsf{I}\mathbf{v}_n)_{n\in\Lambda}$ is a an orthonormal basis for $\mathcal{H}$. (In particular this implies that the dimension of $\mathcal{H}$ is a multiple of four if its dimension is finite.)*

(ii) *An $\mathbb{R}$-linear operator $T : \operatorname{dom}(T) \subset \mathcal{H} \to \mathcal{H}$ is quaternionic linear with respect to the Θ-induced structure if and only if it commutes with I and J. Such operator is bounded as an operator on $\mathcal{H}$ if and only if it is bounded as an operator on $\mathcal{H}_\Theta$ and in this case $\|T\|_{\mathcal{B}(\mathcal{H})} = \|T\|_{\mathcal{B}(\mathcal{H}_\Theta)}$. Moreover the adjoint T^* of T on $\mathcal{H}$ is also the adjoint of T on $\mathcal{H}_\Theta$ and hence T is (anti-)selfadjoint, normal or unitary on $\mathcal{H}$ if and only if it is bounded, (anti-)selfadjoint, normal or unitary on $\mathcal{H}_{\mathsf{J}}$.*

We conclude this section with discussing how quaternionic resp. complex Hilbert spaces can be considered as quaternionification or complexifications of their subspaces.

We start with a complex Hilbert space $\mathcal{H}$. A conjugation K on $\mathcal{H}$ is an antilinear and norm-preserving mapping from $\mathcal{H}$ into itself such that $K \circ K = \mathcal{I}$. Given a conjugation, we can define $\mathcal{H}_K := (\mathcal{I}+K)(\mathcal{H})$. We find that $\mathcal{H}_K$ is an $\mathbb{R}$-linear subspace of $\mathcal{H}$ that is even a real Hilbert space with the structure that it inherits from $\mathcal{H}$ and that furthermore $E_{\mathbb{C}}(\mathcal{H}_K) = \mathcal{H}_K \otimes \mathbb{C} \cong \mathcal{H}$. A complex linear operator T on $\mathcal{H}$ is then the complex linear extension of an operator on $\mathcal{H}$ if and only if it commutes with K, that is if and only if $T \circ K = K \circ T$.

A conjugation exists in any complex Hilbert space. We can for instance choose an orthogonal basis $(\mathbf{b}_n)_{n\in\Lambda}$ of $\mathcal{H}$ and define

$$K(\mathbf{v}) = \sum_{n\in\Lambda} \overline{\langle \mathbf{b}_n, \mathbf{v}\rangle_{\mathcal{H}}}\mathbf{b}_n. \tag{12.1}$$

The subspace $\mathcal{H}_K$ is then precisely the $\mathbb{R}$-linear span of $(\mathbf{b}_n)_{n\in\Lambda}$. Conversely, if we start from a conjugation K, then any orthonormal basis of $\mathcal{H}_K$ induces the conjugation K via (12.1).

Let now $\mathcal{H}$ be a quaternionic Hilbert space. If J is a unitary and anti-selfadjoint operator on $\mathcal{H}$, then we can choose $\mathbf{i} \in \mathbb{S}$ and define

$$\mathcal{H}_{\mathsf{J},\mathbf{i}}^{+} := \{\, \mathbf{v} \in \mathcal{H} : \mathsf{J}\mathbf{v} = \mathbf{v}\mathbf{i}\} \quad \text{and} \quad \mathcal{H}_{\mathsf{J},\mathbf{i}}^{-} := \{\, \mathbf{v} \in \mathcal{H} : \mathsf{J}\mathbf{v} = \mathbf{v}(-\mathbf{i})\}.$$

The sets $\mathcal{H}_{\mathsf{J},\mathbf{i}}^{+}$ and $\mathcal{H}_{\mathsf{J},\mathbf{i}}^{-}$ are $\mathbb{C}_{\mathbf{i}}$-complex Hilbert spaces with the operations and the scalar product they inherit from $\mathcal{H}$. Furthermore

$$\mathcal{H} = \mathcal{H}_{\mathsf{J},\mathbf{i}}^{+} \oplus \mathcal{H}_{\mathsf{J},\mathbf{i}}^{-} = \mathcal{H}_{\mathsf{J},\mathbf{i}}^{+} \oplus \mathcal{H}_{\mathsf{J},\mathbf{i}}^{+}\mathbf{j}$$

and so $E_{\mathbb{H}}(\mathcal{H}_{\mathsf{J},\mathbf{i}}^{+}) \cong \mathcal{H}$. An operator T on $\mathcal{H}$ is the quaternionic linear extension of a $\mathbb{C}_{\mathbf{i}}$-linear operator on $\mathcal{H}_{\mathsf{J},\mathbf{i}}^{+}$ if and only if T and J commute.

Finally, if I, J are two anti-selfadjoint and unitary operators on $\mathcal{H}$ with $\mathsf{I}\mathsf{J} = -\mathsf{J}\mathsf{I}$, then we can choose $\mathbf{i}, \mathbf{j} \in \mathbb{S}$ with $\mathbf{i} \perp \mathbf{j}$ and define

$$\mathcal{H}_{\mathbb{R}} = \{\mathbf{v} \in \mathcal{H} : \mathsf{I}\mathbf{v} = \mathbf{i}, \mathsf{J}\mathbf{v} = \mathbf{v}\mathbf{j}\}$$

and find that $\mathcal{H}_{\mathbb{R}}$ is a real Hilbert space such that $E_{\mathbb{H}}(\mathcal{H}_{\mathbb{R}}) = \mathcal{H}$. An operator T on $\mathcal{H}$ is the quaternionic linear extension of an $\mathbb{R}$-linear operator on $\mathcal{H}_{\mathbb{R}}$ if and only if T commutes with I and J.

If we consider the left multiplication $\mathcal{L}$ that is generated on $\mathcal{H}$ by interpreting I and J as the multiplication with a **i** and **j**, respectively, then $\mathcal{H}_{\mathbb{R}}$ is the real Hilbert space of all vectors that commute with any quaternionic scalar and any orthogonal basis $(\mathbf{b}_\ell)_{\ell\in\Lambda}$ of $\mathcal{H}_{\mathbb{R}}$ generates the left scalar multiplication via

$$a\mathbf{v} = \sum_{\ell\in\Lambda} \mathbf{b}_\ell \langle \mathbf{b}_\ell, \mathbf{v}\rangle_{\mathcal{H}}.$$

Observe how defining a left multiplication on a quaternionic Hilbert space is the analogue of defining a conjugation on a complex Hilbert space. They both determine a subspace that serves for writing each vector in terms of components in an $\mathbb{R}$-linear subspace, which is similar to writing the scalars in $\mathbb{C}$ resp. $\mathbb{H}$ in terms of their real components.

12.2 A Conjecture About the Equivalence of Complex and Quaternionic Quantum Systems

An experimental proposition about a physical system is the statement that the outcome of an experiment belongs to a certain subset of all possible outcomes. The set of all such experimental propositions and their relations determine the logical structure of the system, which is called its propositional calculus. The propositional calculus of a classical mechanical system has the structure of a Boolean algebra. The propositional calculus of a quantum mechanical system on the other hand has a different structure. The distributive identity, which is valid in Boolean algebras, cannot hold in this setting due to the existence of incompatible observables, which cannot be observed simultaneously.

Birkhoff and von Neumann argued in [15] based on some very plausible physical assumptions that the propositional calculus of a quantum mechanical system carries instead the structure of an orthomodular lattice, which initiated the research interest in the field of quantum logics.

Definition 12.11. A partially ordered system $(L, <)$ is called a lattice if for any $x, y \in L$ there exists

- a meet $x \wedge y$ such that $x \wedge y < x$ and $x \wedge y < y$ and such that $z < x$ and $z < y$ implies $z < x \wedge y$ and
- a join $x \vee y$ such that $x < x \vee y$ and $y < x \vee y$ and such that $x < z$ and $y < z$ implies $x \vee y < z$.

A lattice is called bounded if it has a least element 0 and greatest element 1 such that $0 < x$ and $x < 1$ for all $x \in L$ and a bounded lattice is called orthocomplemented if every element $x \in L$ has a unique orthocomplement $\neg x$ such that

$$\neg(\neg x) = x \qquad x \wedge \neg x = 0, \qquad x \vee \neg x = 1$$

and such that

$$x < y \quad \text{implies} \quad \neg y < \neg x.$$

A lattice L is called complete if every subset $A \subset L$ has a greatest lower bound $\bigwedge A$ and a least upper bound $\bigvee A$ and it is called σ-complete if this holds true for any countable

subset A of L. Finally, an orthocomplemented lattice is called modular, if it satisfies for all $x, y \in L$ the orthomodular law

$$\text{if} \quad x < y, \quad \text{then} \quad y = x \vee (\neg x \wedge y). \tag{12.2}$$

Remark 12.12. Birkhoff and von Neumann did actually not arrive at an orthomodular lattice, but at a orthocomplement lattice in which the modular identity

$$\text{if} \quad x < z, \quad \text{then} \quad x \vee (y \wedge z) = (x \vee y) \wedge z$$

holds. The weaker form (12.2) is however the version used today. In particular it is the one used in the paper [70], the argumentation of which we follow in this section.

Birkhoff and von Neumann showed that such orthomodular lattice can be realised as a lattice of closed subspaces of a Hilbert space over the real numbers, the complex numbers or over the quaternions [15]. The relation $<$ corresponds then to the usual subset relation, the operation $\wedge$ to the intersection and the operation $\vee$ to the closed sum of two subspaces and the orthocomplement $\neg$ corresponds to taking the orthogonal complement of a subspace. Equivalently, we can also consider the lattice of orthogonal projections onto these subspaces instead of the subspaces themselves.

The possibility of formulating quantum mechanics on a real Hilbert space was soon discarded due to the analysis in [79, 80]. In these papers, Stueckelberg argues that any such quantum system admits an internal complexification—otherwise Heisenberg's uncertainty principle cannot hold. There must exist an imaginary anti-selfadjoint operator J on the real Hilbert space $\mathcal{H}$, that commutes with any observable and the unitary group that describes the time development of the system. Hence, observables are complex linear operators on the internal complexification $\mathcal{H}_{\mathsf{J}}$ of $\mathcal{H}$ that is induced by J, cf. Theorem 12.8 so that one is actually dealing with a complex quantum system. (The analysis in [79, 80] is not correct as [78] showed. However, it is still assumed that any real quantum system admits an internal complexification and a formally correct argument at least for elementary relativistic systems is given in [70].)

Quantum mechanics on a quaternionic Hilbert space $\mathcal{H}$ on the other hand was developed by several authors starting with [41] and it seemed that such formulation of quantum mechanics was not equivalent to the formulation on a complex Hilbert space [1]. However, as we shall see in the following, this seems to be a misconception. Instead, we argue that any quaternionic quantum system is the external quaternionification of a complex quantum system on a suitably chosen complex subspace of $\mathcal{H}$ and that the belief that the two theories are inequivalent arose from a logical mistake that was made from the very beginning of quaternionic quantum mechanics.

Let us consider a quantum system on a quaternionic Hilbert space and let us assume that there exists a unitary and anti-selfadjoint operator J that commutes with every observable of the system and with the unitary semigroup $U(t), t \in \mathbb{R}$, that describes the time evolution. In this case, we can choose $\mathsf{i} \in \mathbb{S}$ and reduce the quaternionic quantum system to a $\mathbb{C}_{\mathsf{i}}$-complex quantum system on the complex subspace

$$\mathcal{H}^{+}_{\mathsf{J},\mathsf{i}} = \{\mathbf{v} \in \mathcal{H} : \mathsf{J}\mathbf{v} = \mathbf{v}\mathbf{i}\}.$$

Since all observable and all time translations $U(t)$ commute with J, they are quaternionic linear extensions of operators on $\mathcal{H}^{+}_{\mathsf{J},\mathsf{i}}$. The spectral measures of observables

also commute with the operators J because the observables themselves do. Hence, the range K of such projection, which corresponds to an experimental proposition in the propositional calculus of the system, is actually the quaternionification $K = E_{\mathbb{H}}(K_{\mathbf{i}}) = K_{\mathbf{i}} \oplus K_{\mathbf{i}}\mathbf{j}$ with $\mathbf{j} \in \mathbb{S}, \mathbf{j} \perp \mathbf{i}$ of a closed complex linear subspaces on $K_{\mathbf{i}}$ of $\mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}$. The projection itself is in turn the quaternionic linear extension of a projection on $\mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}$. In particular, this holds true for one-dimensional subspaces in the propositional calculus and which correspond to pure states of the system. Any such subspace K_0 is of the form $K_0 = K_{0,\mathbf{i}} \oplus K_{0,\mathbf{i}}\mathbf{j}$ with a one-dimensional subspace $K_{0,\mathbf{i}}$ of $\mathcal{H}_{\mathbf{j}}$. In other words

$$K_0 = \operatorname{span}_{\mathbb{C}_{\mathbf{i}}}(\mathbf{v}) \oplus \operatorname{span}_{\mathbb{C}_{\mathbf{i}}}(\mathbf{v})\mathbf{j} = \operatorname{span}_{\mathbb{H}}\{\mathbf{v}\}$$

with some $\mathbf{v} \in K_{0,\mathbf{i}}$ and hence any pure state of the system can be represented by a vector in $\mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}$. Finally, if we represent a state of the system by a vector $\mathbf{v} \in \mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}$, then the time evolution of the system can be entirely described by vector in $\mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}$. The state of the system at time $t > 0$ is given by $U(t)\mathbf{v}$, which belongs again to $\mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}$ as

$$\mathsf{J}(U(t)\mathbf{v}) = U(t)(\mathsf{J}\mathbf{v}) = U(t)(\mathbf{v}\mathbf{i}) = (U(t)\mathbf{v})\mathbf{i}.$$

The quaternionic quantum system on $\mathcal{H}$ is therefore simply the external quaternionification of a $\mathbb{C}_{\mathbf{i}}$-complex quantum system on $\mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}$, which contains all the physically relevant information. We conjecture that this relation is always true.

Conjecture 12.13. *Any quaternionic quantum system is the external quaternionification of a complex quantum system on a complex subspace of the underlying quaternionic Hilbert space.*

12.3 Classification of Elementary Quantum Systems

We cannot prove Conjecture 12.13 for any arbitrary quantum systems, but we are able to show that it holds true at least for elementary relativistic quantum systems. We show this in Section 12.4 where we apply the arguments of [70] where the equivalence of real and complex quantum theories are shown for such systems. We furthermore stress that also the equivalence of real and complex quantum mechanics is only known for this type of system because the argumentation of Stueckelberg is not correct as pointed out in [77].

In order to show the equivalence of complex and quaternionic quantum theory in this special case, we shall further formalise the ideas in Section 12.2. We consider a quantum system and represent its propositional calculus by a lattice $\mathcal{L}$ of orthogonal projections on a real, complex or quaternionic Hilbert space $\mathcal{H}$. If $\mathcal{L}$ is the lattice of all orthogonal projections on $\mathcal{H}$, then we write $\mathcal{L}(\mathcal{H})$. We recall several important concepts that are shared in all three settings with the aim of defining a proper notion of quantum system. (We follow the summary of results in [66, 84] given in [70] in order to prepare for the arguments in Section 12.4).

1) The orthogonal projections in $\mathcal{L}$ are called *elementary observables*. Such elementary observables correspond to experimental propositions and they have only two outcomes: 1 if the proposition is true, 0 if it is wrong.

2) Observables are modelled by possibly unbounded self-adjoint operators on $\mathcal{H}$. Any self-adjoint operator A on $\mathcal{H}$ determines via the spectral theorem a unique spectral measure $E_A : \mathsf{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ and conversely any such operator is uniquely determined by its spectral measure. Hence, we can consider the spectral measure E_A instead of the operator A itself and we henceforth call a spectral measure defined on the Borel sets $\mathsf{B}(\mathbb{R})$ of $\mathbb{R}$, the values of which are orthogonal projections in $\mathcal{L}$, an observable of the quantum system. If A is an observable modelled by the spectral measure $E_A : \mathsf{B}(\mathbb{R}) \to \mathcal{L}$, then the interpretation of the elementary proposition $E_A(\Delta)$ with $\Delta \in \mathsf{B}(\mathbb{R})$ is that the outcome of the measurement of A belongs to Δ.

   Two observables are said to be *compatible* if they are made of mutually commuting orthogonal projections.

3) A *quantum state* is a σ-additive probability measure over the lattice $\mathcal{L}$. More precisely, a quantum state is a map $\mu : \mathcal{L} \to [0,1]$ such that $\mu(\mathcal{I}) = 1$ and such that for any sequence $(E_\ell)_{\ell\in\mathbb{N}}$ in $\mathcal{L}$ with $E_\ell E_\kappa = 0$ for $\ell \neq \kappa$ one has

$$\mu\left(s\text{-}\sum_{\ell\in\mathbb{N}} E_\ell\right) = \sum_{\ell\in\mathbb{N}} \mu(E_\ell),$$

   where $s\text{-}\sum_{\ell\in\mathbb{N}}$ indicates that the series converges in the strong operator topology. The value of $\mu(E)$ is the probability that the outcome of measuring the proposition E equals 1 if the state of the system is μ.

   Pure states are extremal points in the convex set of probability measures and they are in one-to-one correspondence with one-dimensional rays in the Hilbert space. If $\mathbf{v}$ is a unit vector in the ray associated with the pure state μ, then $\mu(E) = \|E\mu\|^2$.

4) *Lüders-von Neumann's post measurement axiom* is in this setting formulated in the following way: If the outcome of the ideal measurement of $F \in \mathcal{L}$ in the state μ is 1, then the post measurement state is

$$\mu_F(E) := \frac{\mu(FEF)}{\mu(F)}, \qquad \forall E \in \mathcal{L}.$$

5) A *symmetry* is an automorphism $h : \mathcal{L} \to \mathcal{L}$ of the lattice of elementary propositions and we shall denote the set of all such automorphisms by $\mathrm{Aut}(\mathcal{L})$. A subclass of symmetries are those induced by unitary (or in the complex case also anti-unitary) operators $U \in \mathcal{B}(\mathcal{H})$ by means of $h_U(E) := UEU^{-1}$.

6) A *continuous symmetry* is a one-parameter group of lattice automorphisms $(h_s)_{s\in\mathbb{R}}$ such that $s \mapsto \mu(h_s(E))$ is continuous for every $E \in \mathcal{L}$ and every quantum state μ. The time evolution of the system $(\tau_t)_{t\in\mathbb{R}}$ is a preferred continuous symmetry.

7) A *dynamical symmetry* is a continuous symmetry $(h_s)_{s\in\mathbb{R}}$ that commutes with the time evolution so that $h_s \circ \tau_t = \tau_t \circ h_s$ for $s, t \in \mathbb{R}$.

Since computations with observables are meaningful, a quantum system should permit an algebraic structure. This structure is the one of an von Neumann algebra.

Definition 12.14. A Banach algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ is a Banach space $(\mathcal{A}, \|\cdot\|)$ over $\mathbb{K}$ endowed with a bilinear and associative product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that

$$\|\mathbf{x}\mathbf{y}\| \leq \|\mathbf{x}\| \, \|\mathbf{y}\| \qquad \forall \mathbf{x}, \mathbf{y} \in \mathcal{A}.$$

A $*$-algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ is a Banach algebra over $\mathbb{K}$ endowed with an involution $* : \mathcal{A} \to \mathcal{A}$ such that

$$(\mathbf{x}^*)^* = \mathbf{x} \quad (\mathbf{x}\mathbf{y})^* = \mathbf{y}^*\mathbf{x}^* \quad (a\mathbf{x} + b\mathbf{y})^* = \zeta(a)\mathbf{x}^* + \zeta(b)\mathbf{y}^*$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{A}$ and all $a, b \in \mathbb{K}$, where ζ is the identity if $\mathbb{K} = \mathbb{R}$ and the complex conjugation if $\mathbb{K} = \mathbb{C}$.

Finally, a $*$-algebra of $\mathbb{K}$ is called a C^*-algebra if in addition

$$\|\mathbf{x}^*\mathbf{x}\| = \|\mathbf{x}\|^2$$

and such C^*-algebra is called unital if it contains a neutral element $\mathbf{e}$.

It is well-known that the space of bounded operators on a complex Hilbert space forms together with the composition and the adjoint conjugation a complex unital C^*-algebra and similarly the space of bounded operators on a real Hilbert space forms together with the composition and the adjoint conjugation a real unital C^*-algebra. The space of bounded operators on a quaternionic Hilbert space however forms together with the composition and the adjoint conjugation again a real unital C^*-algebra and not a quaternionic one, because it is only a real Banach space, cf. Remark 2.45.

Definition 12.15. Let $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ be a set of bounded operators on a real, complex or quaternionic Hilbert space $\mathcal{H}$. We define the commutant of $\mathfrak{M}$ as

$$\mathfrak{M}' := \{ T \in \mathcal{B}(\mathcal{H}) : [T, A] := TA - AT = 0 \quad \text{for all } A \in \mathfrak{M} \} .$$

If $\mathfrak{M}$ is closed under the adjoint conjugation, then $\mathfrak{M}'$ is a $*$-algebra with unit. Since the product in $\mathcal{B}(\mathcal{H})$ is continuous, $\mathfrak{M}'$ is closed in the uniform operator topology. Hence, if $\mathfrak{M}$ is closed under the adjoint conjugation, then $\mathfrak{M}'$ is a C^*-subalgebra in $\mathcal{B}(\mathcal{H})$. One can furthermore easily show that $\mathfrak{M}'$ is closed in both the weak and the strong operator topology.

We furthermore have $\mathfrak{M} \subset (\mathfrak{M}')' =: \mathfrak{M}''$ and $\mathfrak{M}_1' \subset \mathfrak{M}_2'$ if $\mathfrak{M}_1 \supset \mathfrak{M}_2$ so that $\mathfrak{M}' = (\mathfrak{M}'')'$. We can therefore not reach beyond the second commutant by iteration. We recall the following important theorem due to von Neumann, the proof of which can be found in any book about operator algebras, cf. for instance Theorem 5.3.1 in [61] (the proof is only formulated for the real or complex setting, but it also holds in the quaternionic one).

Theorem 12.16. *Let $\mathcal{H}$ be a real, complex or quaternionic Hilbert space and let $\mathcal{A}$ be a unital $*$-sub-algebra of $\mathcal{B}(\mathcal{H})$. The following statements are equivalent*

(i) $\mathcal{A} = \mathcal{A}''$.

(ii) $\mathcal{A}$ *is weakly closed.*

(iii) $\mathcal{A}$ *is strongly closed.*

Hence, if $\mathcal{C}$ is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, then $\mathcal{C}'' = cl_w(\mathcal{C}) = cl_s(\mathcal{C})$, where $cl_w(\mathcal{C})$ and $cl_s(\mathcal{C})$ denote the closure with respect to the weak and strong operator topology, respectively.

Definition 12.17. A *von Neumann algebra* $\mathfrak{R}$ in the space $\mathcal{B}(\mathcal{H})$ of bounded operators on a real, complex or quaternionic Hilbert space is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that satisfies the three equivalent conditions (i) to (iii) in Theorem 12.16. The *center* $\mathcal{C}(\mathfrak{R})$ of $\mathfrak{R}$ is the abelian von Neumann algebra $\mathcal{C}(\mathfrak{R}) := \mathfrak{R} \cap \mathfrak{R}'$.

Corollary 12.18. *If a set $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ of bounded operators on a real, complex or quaternionic Hilbert space is closed under the adjoint conjugation, then $\mathfrak{M}''$ is the smallest von Neumann-algebra that contains $\mathfrak{M}$. It is called the von Neumann algebra generated by $\mathfrak{M}$.*

We shall in the following mainly deal with von Neumann-algebras that are irreducible.

Definition 12.19. Let $\mathcal{H}$ be a real, complex or quaternionic Hilbert space. A family of operators $\mathfrak{A} \subset \mathcal{B}(U)$ is called reducible if there exists a non-trivial closed subspace $K \subset \mathcal{H}$ such that $A(K) \subset K$ for all $A \in \mathfrak{A}$. The family $\mathfrak{A}$ is called irreducible, if it is not reducible.

Remark 12.20. If $\mathfrak{A}$ is irreducible, then it is easy to see that

$$\{E \in \mathcal{L}(\mathcal{H}) : [E, A] = 0 \quad \forall A \in \mathfrak{A}\} = \{0, \mathcal{I}\}.$$

The opposite implication holds true if $\mathcal{A}$ is closed under adjoint conjugation. In this case, we have for any closed subspace $K \subset \mathcal{H}$ with $A(K) \subset K$ for all $A \in \mathcal{A}$ that $\langle A\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, A^*\mathbf{v}\rangle = 0$ for $\mathbf{u} \in K^\perp$ and $\mathbf{v} \in K$. Hence, also $A(K^\perp) \subset K^\perp$ for all $A \in \mathcal{A}$. If $\mathfrak{A}$ is reducible, then we can find a nontrivial subspace K and the orthogonal projection onto K does then belong to $\{E \in \mathcal{L}(\mathcal{H}) : [E, A] = 0 \quad \forall A \in \mathfrak{A}\}$.

The differences between von Neumann algebras on a quaternionic Hilbert space and von Neumann algebras on a complex Hilbert space are the same as the differences between von Neumann algebras on a real Hilbert space and von Neumann algebras on a complex Hilbert space stated in [70, Theorem 2.29]. (We do not recall the proof here, because it is the same for the quaternionic and the real case.)

Theorem 12.21. *Let $\mathfrak{R}$ be a von Neumann algebra over a real, complex or quaternionic Hilbert space $\mathcal{H}$, let $\mathcal{L}(\mathfrak{R})$ be the lattice of orthogonal projectors in $\mathfrak{R}$ and let*

$$\mathfrak{J}(\mathfrak{R}) := \left\{ J \in \mathfrak{R} : J^* = -J, -J^2 \in \mathcal{L}(\mathfrak{R}) \right\}.$$

(i) A bounded self-adjoint operator A belongs to $\mathfrak{R}$ if and only if the projections of the the spectral measure of A belong to $\mathfrak{R}$.

(ii) The set $\mathcal{L}(\mathfrak{R})$ is a complete (in particular σ-complete) orthomodular sublattice of $\mathcal{L}(\mathcal{H})$.

(iii) $\mathfrak{R}$ is irreducible if and only if $\mathcal{L}(\mathfrak{R}') = \{0, \mathcal{I}\}$.

(iv) If $\mathcal{H}$ is a real or quaternionic Hilbert space, then

(a) $\mathcal{L}(\mathfrak{R})''$ *contains all selfadjoint operators in* $\mathfrak{R}$.

(b) $(\mathcal{L}(\mathfrak{R}) \cup \mathfrak{J}(\mathfrak{R}))'' = \mathfrak{R}$

(c) $(\mathcal{L}(\mathfrak{R}))'' \subsetneq \mathfrak{R}$ *if and only if there exists* $J \in \mathfrak{J}(\mathfrak{R}) \setminus \mathcal{L}(\mathfrak{R})''$.

(v) If $\mathcal{H}$ *is a complex Hilbert space, then* $\mathcal{L}(\mathfrak{R})'' = \mathfrak{R}$.

In order to be able to calculate with observables, it seems reasonable to assume that the set of observables of a quantum mechanical system is embedded in a von Neumann algebra.

Definition 12.22. A real, complex or quaternionic quantum system is a von Neumann algebra $\mathfrak{R}$ on a real, complex resp. quaternionic Hilbert space $\mathcal{H}$.

Remark 12.23. We call $\mathfrak{R}$ also the von Neumann algebra of observables. The proper observables are precisely the self-adjoint operators whose spectral measures take values in $\mathfrak{R}$ and the lattice of elementary propositions corresponds to the lattice of orthogonal projections in $\mathcal{L}(\mathfrak{R})$. If we consider a complex Hilbert space, then the von Neumann algebra $\mathfrak{R}$ of observables is by Theorem 12.21 precisely the von Neumann algebra that is generated by the lattice $\mathcal{L}(\mathfrak{R})$, which represents the propositional calculus of the system.

Moretti and Oppio argued in [70] that the symmetry under the Poincaré-group defines on any elementary relativistic quantum system that is defined on a real Hilbert space an up to sign unique unitary and anti-selfadjoint operator J. This operator induces an internal complexification of the real Hilbert space $\mathcal{H}$ that turns the real quantum system into a complex quantum one. We show now that their arguments can also be used to show that the symmetry under the Poincaré-group induces also on any elementary relativistic quantum system a complex structure so that the quaternionic quantum system turns out to be the external quaternionification of a complex elementary relativistic quantum system.

We first give a formal definition of the term *elementary quantum system* following the arguments of [70, Section 5] and show that such systems admit a classification that is analogue to the classification of real elementary quantum systems in Theorem 5.3 of [70].

An elementary quantum system must not allow super-selection rules—otherwise we could work separately on the super-selection sectors. Mathematically, this condition is expressed by requiring that the center of $\mathcal{L}(\mathfrak{R})$ is trivial. Furthermore, we assume that there does not exist any non-trivial orthogonal projection in $\mathfrak{R}'$, that is $\mathfrak{R}$ is irreducible. Such projection could be interpreted as an elementary observable of another external system, whereas we want to be the elementary system to be the entire system we are dealing with. Under the assumption that the center of $\mathcal{L}(\mathfrak{R})$ is trivial, this condition can also be interpreted as the existence of a maximal set of compatible observables.

Definition 12.24. An elementary real, complex or quaternionic quantum system is an irreducible von Neumann algebra $\mathfrak{R}$ on a separable real, complex resp. quaternionic Hilbert space $\mathcal{H}$.

Remark 12.25. A complex quantum system is irreducible if and only if one has $\mathfrak{R}' = \{a\mathcal{I} : a \in \mathbb{C}\}$ or equivalently if and only if $\mathfrak{R} = \mathcal{B}(\mathcal{H})$. Just as in the real case, this is not true in the quaternionic setting, cf. [70, Remark 5.2].

The essential tool we need for showing the equivalence of complex and quaternionic quantum systems, is a precise classification of irreducible von Neumann-algebras on a quaternionic Hilbert space. We first recall the corresponding result for irreducible von Neumann-algebras on a real Hilbert space [70, Theorem 5.3].

Theorem 12.26. *If $\mathfrak{R}$ is an irreducible von Neumann algebra on a real Hilbert space $\mathcal{H}$, then precisely one of the following statements holds true.*

(i) The commutant $\mathfrak{R}'$ of $\mathfrak{R}$ is isomorphic to the real numbers. Precisely, we have

$$\mathfrak{R}' = \{a\mathcal{I} : a \in \mathbb{R}\}.$$

In this case

$$\mathfrak{R} = \mathcal{B}(\mathcal{H}), \quad \mathcal{C}(\mathfrak{R}) = \{a\mathcal{I} : a \in \mathbb{R}\} \quad \textit{and} \quad \mathcal{L}(\mathfrak{R}) = \mathcal{L}(\mathcal{H})$$

and we call $\mathfrak{R}$ of real-real type.

(ii) The commutant $\mathfrak{R}'$ of $\mathfrak{R}$ is isomorphic to the field of complex numbers. Precisely, we have

$$\mathfrak{R}' = \{\, a\mathcal{I} + b\mathsf{J} : a, b \in \mathbb{R}\},$$

where J is an up to sign unique unitary and anti-selfadjoint operator on $\mathcal{H}$. Any operator in $\mathfrak{R}$ is complex linear on the internal complexification $\mathcal{H}_{\mathsf{J}}$ of $\mathcal{H}$ induced by J and we have

$$\mathfrak{R} \cong \mathcal{B}\left(\mathcal{H}_{\mathsf{J}}\right), \quad \mathcal{C}(\mathfrak{R}) = \{a\mathcal{I} + b\mathsf{J} : a, b \in \mathbb{R}\} \quad \textit{and} \quad \mathcal{L}(\mathfrak{R}) \cong \mathcal{L}(\mathcal{H}_{\mathsf{J}}).$$

In this case, we call $\mathfrak{R}$ of real-complex-type.

(iii) The commutant $\mathfrak{R}'$ of $\mathfrak{R}$ is isomorphic to the skew-field of quaternions. Precisely, we have

$$\mathfrak{R}' = \{a\mathcal{I} + b\mathsf{I} + c\mathsf{J} + d\mathsf{K} : a, b, c, d \in \mathbb{R}\},$$

where I, J, and K are mutually anti-commuting unitary and anti-selfadjoint operators on $\mathcal{H}$ that do not belong to $\mathfrak{R}$ such that $\mathsf{IJ} = \mathsf{K}$. If we choose $\mathbf{i}, \mathbf{j} \in \mathbb{S}$ with $\mathbf{i} \perp \mathbf{j}$ and set $\Theta = (\mathsf{I}, \mathsf{J}, \mathbf{i}, \mathbf{j})$, then any operator in $\mathfrak{R}$ is quaternionic right linear on the internal quaternionification $\mathcal{H}_{\Theta}$ of $\mathcal{H}$ induced by Θ. Moreover, we have

$$\mathfrak{R} \cong \mathcal{B}\left(\mathcal{H}_{\Theta}\right), \quad \mathcal{C}(\mathfrak{R}) = \{a\mathcal{I} : a \in \mathbb{R}\} \quad \textit{and} \quad \mathcal{L}(\mathfrak{R}) \cong \mathcal{L}(\mathcal{H}_{\Theta})$$

and we call $\mathfrak{R}$ of real-quaternionic type.

An analogous result holds for irreducible von Neumann algebras on a quaternionic Hilbert space. In this case, we can however not introduce additional structure on $\mathcal{H}$ by finding an *internal* complexification resp. quaternionification of $\mathcal{H}$ so that $\mathfrak{R}$ consists of all the linear operators on the more structured space. Instead, we can find a subspace with less structure, so that $\mathfrak{R}$ is the *external* quaternionification of all bounded linear operators on this subspace.

Theorem 12.27. *If $\mathfrak{R}$ is an irreducible von Neumann algebra on a quaternionic Hilbert space $\mathcal{H}$, then precisely one of the following statements hold true.*

(i) *The commutant $\mathfrak{R}'$ of $\mathfrak{R}$ is isomorphic to the real numbers. Precisely, we have*

$$\mathfrak{R}' = \{a\mathcal{I} : a \in \mathbb{R}\}.$$

In this case

$$\mathfrak{R} = \mathcal{B}(\mathcal{H}), \quad \mathcal{C}(\mathfrak{R}) = \{a\mathcal{I} : a \in \mathbb{R}\} \quad \text{and} \quad \mathcal{L}(\mathfrak{R}) = \mathcal{L}(\mathcal{H})$$

and we call $\mathfrak{R}$ proper quaternionic.

(ii) *The commutant $\mathfrak{R}'$ of $\mathfrak{R}$ is isomorphic to the field of complex numbers. Precisely, we have*

$$\mathfrak{R}' = \{\, a\mathcal{I} + b\mathsf{J} : a, b \in \mathbb{R}\},$$

where J is an up to sign unique unitary and anti-selfadjoint operator on $\mathcal{H}$. If we choose $\mathbf{i} \in \mathbb{S}$, then $\mathcal{H}^{+}_{\mathsf{J},\mathbf{i}} := \{\mathbf{v} \in \mathcal{H} : \mathsf{J}\mathbf{v} = \mathbf{v}\mathbf{i}\}$ is a complex Hilbert space over $\mathbb{C}_{\mathbf{i}}$ and

$$\mathfrak{R} \cong \mathcal{B}\left(\mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}\right), \quad \mathcal{C}(\mathfrak{R}) = \{a\mathcal{I} + \mathsf{J} : a, b \in \mathbb{R}\} \quad \text{and} \quad \mathcal{L}(\mathfrak{R}) \cong \mathcal{L}(\mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}),$$

where we identify an operator in $\mathcal{B}\left(\mathcal{H}^{+}_{J,\mathbf{i}}\right)$ with its quaternionic linear extension to $\mathcal{H}$. In this case, we call $\mathfrak{R}$ complex-induced.

(iii) *The commutant $\mathfrak{R}'$ of $\mathfrak{R}$ is isomorphic to the skew-field of quaternions. Precisely, we have*

$$\mathfrak{R}' = \{a\mathcal{I} + b\mathsf{I} + c\mathsf{J} + d\mathsf{K} : a, b, c, d \in \mathbb{R}\},$$

where I, J, and K are mutually anti-commuting unitary and anti-selfadjoint operators on $\mathcal{H}$ that do not belong to $\mathfrak{R}$ such that $\mathsf{IJ} = \mathsf{K}$. If we choose $\mathbf{i}, \mathbf{j} \in \mathbb{S}$ with $\mathbf{i} \perp \mathbf{j}$ and set $\mathbf{k} := \mathbf{ij}$, then $\mathcal{H}_{\mathbb{R}} := \{\mathbf{v} \in \mathcal{H} : \mathsf{I}\mathbf{v} = \mathbf{v}\mathbf{i}, \mathsf{J}\mathbf{v} = \mathbf{v}\mathbf{i}\}$ is a real Hilbert space and

$$\mathfrak{R} \cong \mathcal{B}\left(\mathcal{H}_{\mathbb{R}}\right), \quad \mathcal{C}(\mathfrak{R}) = \{a\mathcal{I} : a \in \mathbb{R}\} \quad \text{and} \quad \mathcal{L}(\mathfrak{R}) \cong \mathcal{L}(\mathcal{H}_{\mathbb{R}}),$$

where we identify an operator in $\mathcal{B}\left(\mathcal{H}_{\mathbb{R}}\right)$ with its quaternionic linear extension to $\mathcal{H}$. In this case, we call $\mathfrak{R}$ real-induced.

Proof. If $T \in \mathfrak{R}'$ is self-adjoint, then its spectral measure takes values in $\mathfrak{R}'$. Since $\mathfrak{R}' = \{0, \mathcal{I}\}$ by (iii) in Theorem 12.21 because $\mathfrak{R}$ is irreducible, we have $E(\Delta) = 0$ or $E(\Delta) = \mathcal{I}$ for any $\Delta \in \mathsf{B}(\mathbb{R})$. Now observe that there exists precisely one number $n_0 \in \mathbb{Z}$ such that $E((n_0 - 1, n_0]) = \mathcal{I}$. If there existed two such numbers $n_0, n_1 \in \mathbb{Z}$, then we would have $E((n_0 - 1, n_0]) + E((n_1 - 1, n_1]) = \mathcal{I} + \mathcal{I} = 2\mathcal{I}$, which is not an orthogonal projections. If on the other hand $E((n-1, n]) = 0$ for all $n \in \mathbb{Z}$, we would obtain the contradiction

$$\mathcal{I}\mathbf{v} = E(\mathbb{R})\mathbf{v} = \sum_{n \in \mathbb{Z}} E((n-1, n])\mathbf{v} = \sum_{n \in \mathbb{Z}} 0\mathbf{v} = \mathbf{0}$$

for all $\mathbf{v} \in \mathcal{H}$. Let hence $\Delta_0 = (a_0, b_0]$ with $a_0 := n - 1$ and $b_0 = n$ be such that $E(\Delta_0) = \mathcal{I}$. We define now inductively a sequence of Borel sets $\Delta_n = (a_n, b_n]$ with $E(\Delta_n) = \mathcal{I}$. Precisely, if $\Delta_n = (a_n, b_n]$ with $E(\Delta_n) = \mathcal{I}$ is given, then the same argument as before shows that either

$$E\left(a_n, (a_n + b_n)/2]\right) = \mathcal{I} \quad \text{or} \quad E\left((a_n + b_n)/2, b_n]\right) = \mathcal{I}.$$

In the first case we set $a_{n+1} = a_n$ and $b_{n+1} = (a_n + b_n)/2$ and it the latter case we set $a_{n+1} = (a_n + b_n)/2$ and $b_{n+1} = b_n$. Then $E(\Delta_{n+1}) = \mathcal{I}$ for $\Delta_{n+1} = (a_{n+1}, b_{n+1}]$. Now let $a = \lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n$. Due to the continuity of the the spectral measure with respect to monotone limits in the strong operator topology, we find that

$$E(\{a\}) = E\left(\bigcap_{n\in\mathbb{N}} \Delta_n\right) = \lim_{n\to\infty} E(\Delta_n) = \mathcal{I}.$$

Thus $E(\mathbb{R} \setminus \{a\}) = 0$ and we conclude that

$$T = \int_{\mathbb{R}} s\, dE(s) = \int_{\{a\}} s\, dE(s) + \int_{\mathbb{R}\setminus\{a\}} s\, dE(s) = aE(\{a\}) = a\mathcal{I}.$$

Let now T be an arbitrary operator in $\mathfrak{R}'$. Then also $T^* \in \mathfrak{R}'$ and

$$T = \frac{1}{2}(T + T^*) + \frac{1}{2}(T - T^*).$$

The operator $T_1 := \frac{1}{2}(T + T^*)$ is a self adjoint operator and belongs to $\mathfrak{R}'$ and hence $T_1 = a\mathcal{I}$ for some $a \in \mathbb{R}$ by the above argumentation. The operator $T_2 := \frac{1}{2}(T - T^*)$ on the other hand is anti-self adjoint and so the operator T_2^2 is selfadjoint and belongs to $\mathfrak{R}'$. By the above arguments, we find again that there exists some $c \in \mathbb{R}$ such that $T_2^2 = c\mathcal{I}$. We even have $c \leq 0$ as

$$c\|\mathbf{v}\|^2 = \langle \mathbf{v}, c\mathcal{I}\mathbf{v}\rangle = \langle \mathbf{v}, T_2^2\mathbf{v}\rangle = -\langle T_2\mathbf{v}, T_2\mathbf{v}\rangle = -\|T_2\mathbf{v}\|^2$$

for any $\mathbf{v} \in \mathcal{H}$ due to the anti-selfadjointness of T_2. We also see that $c = 0$ if and only if $T_2 = 0$ so that in this case $T = \frac{1}{2}(T + T^*) = a\mathcal{I}$. If $c \neq 0$, then we set $\mathsf{J} := \frac{1}{\sqrt{-c}} T_2$. Then $\mathsf{J}^* = -\mathsf{J}$ because T_2 is anti-selfadjoint and $\mathsf{J}^2 = \frac{1}{-c} T_2^2 = -\mathcal{I}$. Setting $b = \sqrt{-c}$, we find that

$$T = a\mathcal{I} + b\mathsf{J}$$

with $a, b \in \mathbb{R}$.

Let now T and S be operators in $\mathfrak{R}'$. Then $T = a\mathcal{I} + b\mathsf{J}$ and $S = c\mathcal{I} + d\mathsf{I}$ with $a, b, c, d \in \mathbb{R}$ and two unitary and anti-selfadjoint operators I and J. Since

$$\begin{aligned}\|T\mathbf{v}\|^2 =& \langle T\mathbf{v}, T\mathbf{v}\rangle = \langle (a\mathcal{I} + b\mathsf{J})\mathbf{v}, (a\mathcal{I} + b\mathsf{J})\mathbf{v}\rangle \\ =& a^2\langle \mathbf{v}, \mathbf{v}\rangle + ba\langle \mathsf{J}\mathbf{v}, \mathbf{v}\rangle + ab\langle \mathbf{v}, \mathsf{J}\mathbf{v}\rangle + b^2\langle \mathsf{J}\mathbf{v}, \mathsf{J}\mathbf{v}\rangle \\ =& a^2\langle \mathbf{v}, \mathbf{v}\rangle - ab\langle \mathbf{v}, \mathsf{J}\mathbf{v}\rangle + ab\langle \mathbf{v}, \mathsf{J}\mathbf{v}\rangle + b^2\langle \mathbf{v}, \mathbf{v}\rangle = (a^2 + b^2)\|\mathbf{v}\|^2,\end{aligned}$$

for any $\mathbf{v} \in \mathcal{H}$ and so in particular $\|T\| = \sqrt{a^2 + b^2}$. Similarly, we see that $\|S\mathbf{v}\|^2 = (c^2 + d^2)\|\mathbf{v}\|^2$ and $\|S\| = \sqrt{c^2 + d^2}$. Finally, we deduce from these relations that

$$\|ST\mathbf{v}\|^2 = (c^2 + d^2)\|T\mathbf{v}\|^2 = (c^2 + d^2)(a^2 + b^2)\|\mathbf{v}\|^2$$

and so

$$\|ST\| = \sqrt{(c^2 + d^2)(a^2 + b^2)} = \sqrt{(c^2 + d^2)}\sqrt{(a^2 + b^2)} = \|S\|\|T\|.$$

The commutant $\mathfrak{R}'$ is therefore a normed real associative algebra with unit such that $\|TS\| = \|T\|\|S\|$ for all $T, S \in \mathfrak{R}'$. By [83] any such algebra is isomorphic to either

the field of real numbers $\mathbb{R}$, to the field of complex numbers $\mathbb{C}$ ore the skew-field of quaternions $\mathbb{H}$. Let h be an isomorphism of $\mathfrak{R}$ to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, respectively.

If $\mathfrak{R}'$ is isomorphic to $\mathbb{R}$, then simply $\mathfrak{R}' = h^{-1}(\mathbb{R}) = \{a\mathcal{I} : a \in \mathbb{R}\}$ and we find that $\mathfrak{R} = \mathfrak{R}'' = \mathcal{B}(\mathcal{H})$, because any quaternionic linear operator commutes with any operator $a\mathcal{I}$ with $a \in \mathbb{R}$. Hence, we also find $\mathcal{C}(\mathfrak{R}) = \mathfrak{R} \cap \mathfrak{R}' = \{a\mathcal{I} : a \in \mathbb{R}\}$ and $\mathcal{L}(\mathfrak{R}) = \mathcal{L}(\mathcal{H})$.

If $\mathfrak{R}'$ is isomorphic to $\mathbb{C}$, then $\mathfrak{R}' = \{\, a\mathcal{I} + b\mathsf{J} : a, b \in \mathbb{R}\}$ with $\mathsf{J} = h^{-1}(i)$. Let us show that J is unitary and anti-selfadjoint. Since h is an isomorphism, we have $\mathsf{J}^2 = h(i^2) = h(-1) = -\mathcal{I}$. Since $\mathfrak{R}'$ is a $*$-algebra, not only J but also the operator J^* and in turn even $\mathsf{J}\mathsf{J}^*$ belong to $\mathfrak{R}'$. Since $\mathsf{J}\mathsf{J}^*$ is selfadjoint, the arguments at the beginning of the proof imply that

$$\mathsf{J}\mathsf{J}^* = a\mathcal{I}$$

for some $a \in \mathbb{R}$. Moreover, $a > 0$ because

$$a\|\mathbf{v}\|^2 = \langle \mathbf{v}, a\mathcal{I}\mathbf{v}\rangle = \langle \mathbf{v}, \mathsf{J}\mathsf{J}^*\mathbf{v}\rangle = \langle \mathsf{J}^*\mathbf{v}, \mathsf{J}^*\mathbf{v}\rangle = \|\mathsf{J}\mathbf{v}\|^2.$$

Since $\mathsf{J}^2 = -\mathcal{I}$, we have $\mathsf{J}^* = (-\mathsf{J}\mathsf{J})\mathsf{J}^* = -\mathsf{J}(\mathsf{J}\mathsf{J}^*) = -\mathsf{J}a$ and so $\mathsf{J}^* = -\frac{1}{a}\mathsf{J}$. Taking the adjoint, we find that $\mathsf{J} = -\frac{1}{a}\mathsf{J}^*$ and so also $\mathsf{J}^* = -a\mathsf{J}$. Finally, the identity $0 = \mathsf{J}^* - \mathsf{J}^* = \left(a - \frac{1}{a}\right)\mathsf{J}$ implies $a = 1$ and so $\mathsf{J}^* = -\mathsf{J}$. Hence, J is actually unitary and anti-selfadjoint.

An operator $T \in \mathcal{B}(\mathcal{H})$ belongs to $\mathfrak{R} = \mathfrak{R}''$ if and only if it commutes with any operator in $\mathfrak{R}' = \{a\mathcal{I} + b\mathsf{J} : a, b \in \mathbb{R}\}$. Since any operator $\mathcal{B}(\mathcal{H})$ commutes with real multiples of the identity, an operator commutes with $\mathfrak{R}'$ if and only if it commutes with J. This in turn is the case if and only if the operator T is the quaternionic linear extension of an operator in $\mathcal{B}(\mathcal{H}^+_{\mathsf{J},\mathbf{i}})$ and so $\mathfrak{R} \cong \mathcal{B}(\mathcal{H}^+_{\mathsf{J},\mathbf{i}})$. In particular, this implies $\mathsf{J} \in \mathfrak{R}$, because it is the quaternionic linear extension of the multiplication with $\mathbf{i}$ on $\mathcal{H}^+_{\mathsf{J},\mathbf{i}}$, and $\mathcal{C}(\mathfrak{R}) = \mathfrak{R}'$ and $\mathcal{L}(\mathfrak{R}) \cong \mathcal{L}(\mathcal{H}^+_{\mathsf{J},\mathbf{i}})$.

Finally, if $\mathfrak{R}'$ is isomorphic to $\mathbb{H}$, then

$$\mathfrak{R}' = \{\, a\mathcal{I} + b\mathsf{I} + c\mathsf{J} + d\mathsf{K} : a, b, c, d \in \mathbb{R}\}.$$

with $\mathsf{I} = h^{-1}(e_1)$, $\mathsf{J} = h^{-1}(e_2)$ and $\mathsf{K} = h^{-1}(e_3)$, where e_1, e_2, and e_3 are the generating units of $\mathbb{H}$. As above, one can see that I, J, and K are anti-selfadjoint and unitary. Since an operator belongs to $\mathfrak{R} = \mathfrak{R}''$ if and only if it commutes with I and J and in turn also with $\mathsf{K} = \mathsf{I}\mathsf{J}$, the operators I, J and K themselves do not belong to $\mathfrak{R}$ because they anticommute mutually. If we choose $\mathbf{i}, \mathbf{j} \in \mathbb{S}$ with $\mathbf{i} \perp \mathbf{j}$, then we can define a left-multiplication on $\mathcal{H}$ by setting $\mathsf{I}\mathbf{v} = \mathbf{v}\mathbf{i}$ and $\mathsf{J}\mathbf{v} = \mathbf{v}\mathbf{j}$ for all $\mathbf{v} \in \mathcal{H}$ and turn $\mathcal{H}$ into a two-sided Banach space. The space $\mathcal{H}_{\mathbb{R}}$ consists then of those vectors that commute with all quaternions and it is a real Hilbert space, cf. the discussion in Section 2.4.

An operator belongs to $\mathfrak{R} = \mathfrak{R}''$ if and only if it commutes with any operator in $\mathfrak{R}'$, that is with any operator of the form $a\mathcal{I} + b\mathsf{I} + c\mathsf{J} + d\mathsf{K}$ or—equivalently—with any quaternion when we consider the left multiplication induced by I and J on $\mathcal{H}$. These operators are however precisely those that are quaternionic linear extensions of real-linear operators on $\mathcal{H}_{\mathbb{R}}$, cf. Definition 2.46 and the discussion before.

□

Wigner's theorem states that any symmetry of an elementary complex quantum system can be represented by a unitary linear or an anti-linear anti-unitary operator on $\mathcal{H}$.

Similarly, any symmetry of a real or a quaternionic quantum system can be represented by a unitary linear operator on $\mathcal{H}$ [84, Theorem 4.27]. This statement is specified for elementary real systems in [70, Proposition 5.5], which we want to recall now.

Theorem 12.28. *We consider an elementary system described by an irreducible von Neumann algebra $\mathfrak{R}$ on a real Hilbert space. If h is a symmetry of the system—that is $h : \mathcal{L}(\mathfrak{R}) \to \mathcal{L}(\mathfrak{R})$ is a lattice automorphism—then there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that*

$$h(E) = UEU^{-1} \qquad \forall E \in \mathcal{L}(\mathfrak{R}). \tag{12.3}$$

Furthermore the following facts hold true.

a) If $\mathfrak{R}$ is of real-real or real-quaternionic type, then $U \in \mathfrak{R}$.

b) If $\mathfrak{R}$ is real-complex with $\mathfrak{R}' = \{a\mathcal{I} + b\mathsf{J} : a, b \in \mathbb{R}\}$, then U either commutes with J (and hence $U \in \mathfrak{R}$) or it anticommutes with J (and hence $U \notin \mathfrak{R}'$ but $U^2 \in \mathfrak{R}$).

c) If $\mathfrak{R}$ is of real-real or real-quaternionic type, then every unitary operator U in $\mathfrak{R}$ defines a symmetry via (12.3). *Similarly, if $\mathfrak{R}$ is of real-complex type, then every unitary operator U that either commutes or anticommutes with J defines a symmetry via* (12.3). *Two such unitary operators U and U' define the same symmetry if and only if $U'U^{-1} \in \mathcal{C}(\mathfrak{R})$.*

Again we find a similar result for elementary quaternionic quantum systems.

Theorem 12.29. *We consider an elementary system described by an irreducible von Neumann algebra $\mathfrak{R}$ on a quaternionic Hilbert space $\mathcal{H}$. If h is a symmetry—that is $h : \mathcal{L}(\mathfrak{R}) \to \mathcal{L}(\mathfrak{R})$ is a lattice automorphism—then there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that*

$$h(E) = UEU^{-1} \qquad \forall E \in \mathcal{L}(\mathfrak{R}). \tag{12.4}$$

Furthermore the following facts hold true.

a) If $\mathfrak{R}$ is proper quaternionic or real induced, then $U \in \mathfrak{R}$.

b) If $\mathfrak{R}$ is complex induced with $\mathfrak{R}' = \{a\mathcal{I} + bJ : a, b \in \mathbb{R}\}$, then U either commutes with J (and hence $U \in \mathfrak{R}$) or J anticommutes with J (and hence $U \notin \mathfrak{R}'$ but $U^2 \in \mathfrak{R}$).

c) If $\mathfrak{R}$ is proper quaternionic or real induced, then every unitary operator U in $\mathfrak{R}$ defines a symmetry via (12.4). *Similarly, if $\mathfrak{R}$ is complex induced, then every unitary operator U that either commutes or anticommutes with J defines a symmetry via* (12.4). *Two such unitary operators U and U' define the same symmetry if and only if $U'U^{-1} \in \mathcal{C}(\mathfrak{R})$.*

Proof. We recall that the lattice $\mathcal{L}(\mathfrak{R})$ is isomorphic to $\mathcal{L}(\mathcal{H})$, to $\mathcal{L}(\mathcal{H}^+_{\mathsf{J},\mathsf{i}})$ or to $\mathcal{L}(\mathcal{H}_{\mathbb{R}})$ because of Theorem 12.27. Any isomorphism lattice automorphism on $\mathcal{L}(\mathfrak{R})$ hence induces a lattice automorphism on $\mathcal{L}(\mathcal{H})$, $\mathcal{L}(\mathcal{H}^+_{\mathsf{J},\mathsf{i}})$ resp. $\mathcal{L}(\mathcal{H}_{\mathbb{R}})$.

If $\mathcal{L}(\mathfrak{R}) = \mathcal{L}(\mathcal{H})$, then the quaternionic version of Wigner's theorem, Theorem 4.27 in [84], implies for any symmetry h the existence of a bijective function $S : \mathcal{H} \to \mathcal{H}$ with the properties that

(A) S is additive and

(B) there exists $q \in \mathbb{H}$ with $|q| = 1$ such that $S(\mathbf{v}a) = \mathbf{v}q^{-1}aq$ and $\langle S\mathbf{v}, S\mathbf{u}\rangle = q^{-1}\langle \mathbf{v}, \mathbf{u}\rangle q$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{H}$ and all $a \in \mathbb{H}$

such that

$$h(E) = SES^{-1}. \tag{12.5}$$

Furthermore any function $S_p : \mathcal{H} \to \mathcal{H}$ of the form $S_p(\mathbf{v}) = S\mathbf{v}p$ with $p \in \mathbb{H}$ and $|p| = 1$ also satisfies (12.5). If we choose $p = q^{-1}$, then we obtain a quaternionic right linear unitary operator $U = S_p \in \mathcal{B}(\mathcal{H})$ such that (12.4) holds true. Furthermore, again by Wigner's theorem, any other unitary operator in $\mathcal{B}(\mathcal{H})$ satisfies (12.4) if and only if $U'\mathbf{v} = S_r(\mathbf{v}) = S(\mathbf{v})r = U(\mathbf{v})p^{-1}r$ for some $r \in \mathbb{H}$ with $|r| = 1$. An operator of this form is however quaternionic right linear and unitary if and only if $p^{-1}r \in \{\pm 1\}$, which is equivalent to $U'U^{-1} = \mp\mathcal{I}$ and hence to $U'U^{-1}$ being a unitary operator in $\mathcal{C}(\mathfrak{R})$. Finally, Wigner's theorem also implies that any bijective function $S : \mathcal{H} \to \mathcal{H}$ that satisfies (A) and (B) induces a symmetry on $\mathcal{L}(\mathcal{H})$ via (12.4). Hence, in particular, any unitary operator on $\mathcal{H}$ induces a symmetry.

If $\mathcal{L}(\mathfrak{R}) = \mathcal{L}(\mathcal{H}_{\mathbb{R}})$, then any symmetry h on $\mathcal{L}(\mathfrak{R})$ defines a symmetry $h_{\mathbb{R}}$ on $\mathcal{L}(\mathcal{H}_{\mathbb{R}})$ via

$$h_{\mathbb{R}}(E_{\mathbb{R}}) = h(E)|_{\mathcal{H}_{\mathbb{R}}}, \qquad \text{if} \quad E_{\mathbb{R}} = E|_{\mathbb{R}}. \tag{12.6}$$

Hence, the real version of Wigner's theorem [84, Theorem 4.27] implies the existence of a unitary operator $U_{\mathbb{R}}$ on $\mathcal{H}_{\mathbb{R}}$ such that $h_{\mathbb{R}}(E_{\mathbb{R}}) = U_{\mathbb{R}}E_{\mathbb{R}}U_{\mathbb{R}}^{-1}$. If we denote the quaternionic linear extension of $U_{\mathbb{R}}$ to all of $\mathcal{H}$ by U, then U is a unitary operator on $\mathcal{H}$. For any $E \in \mathcal{L}(\mathfrak{R})$ we have after setting $E_{\mathbb{R}} := E|_{\mathcal{H}_{\mathbb{R}}}$ that

$$h(E)|_{\mathcal{H}_{\mathbb{R}}} = h_{\mathbb{R}}(E_{\mathbb{R}}) = U_{\mathbb{R}}E_{\mathbb{R}}U_{\mathbb{R}}^{-1}.$$

Extending these operators to quaternionic linear operators on $\mathcal{H}$, we find $h(E) = UEU^{-1}$. It follows also from Wigner's theorem that the operator $U_{\mathbb{R}}$ is unique up to sign so that a unitary operator $U'_{\mathbb{R}}$ induces h via (12.4) if and only if $U'_{\mathbb{R}}U_{\mathbb{R}}^{-1} = \pm\mathcal{I}$. Thus, a unitary operator $U' \in \mathcal{B}(\mathcal{H})$ induces h if and only of $(U'U)|_{\mathcal{H}_{\mathbb{R}}} = U'_{\mathbb{R}}U_{\mathbb{R}}^{-1} = \pm\mathcal{I}$, which is equivalent to $U'U = \pm\mathcal{I}$ and in turn to $U'U^{-1}$ being a unitary operator in $\mathcal{C}(\mathfrak{R})$. Finally, Wigner's theorem also states that any unitary operator on $\mathcal{H}_{\mathbb{R}}$ induces a symmetry on $\mathcal{B}(\mathcal{H}_{\mathbb{R}})$. Since the unitary operators in $\mathfrak{R}$ are exactly the operators that are quaternionic linear extensions of unitary operators on $\mathcal{H}_{\mathbb{R}}$, any such operator induces a symmetry on $\mathcal{L}(\mathfrak{R})$ via (12.6).

If finally $\mathcal{L}(\mathfrak{R}) = \mathcal{L}(\mathcal{H}_{\mathsf{J},\mathsf{i}}^{+})$, then any symmetry h on $\mathcal{L}(\mathfrak{R})$ defines a symmetry $h_{\mathbb{C}_{\mathsf{i}}}$ on $\mathcal{L}(\mathcal{H}_{\mathbb{C}_{\mathsf{i}}^{+}})$ via

$$h_{\mathbb{C}_{\mathsf{i}}}(E_{\mathbb{C}_{\mathsf{i}}}) = h(E)|_{\mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}}, \qquad \text{if} \quad E_{\mathbb{C}_{\mathsf{i}}} = E|_{\mathcal{H}_{\mathsf{J},\mathsf{i}}}^{+}. \tag{12.7}$$

Hence, the complex linear version of Wigner's theorem [84, Theorem 4.28] implies the existence of a bijective mapping $S : \mathcal{H}_{\mathsf{J},\mathsf{i}}^{+} \to \mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}$ such that either

(I) S is $\mathbb{C}_{\mathsf{i}}$-complex linear and $\langle S\mathbf{v}, S\mathbf{u}\rangle_{\mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}} = \langle \mathbf{v}, \mathbf{u}\rangle_{\mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}}$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}$, i.e. S is a bounded unitary operator on $\mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}$ or

(II) S is $\mathbb{C}_{\mathsf{i}}$-complex anti-linear and $\langle S\mathbf{v}, S\mathbf{u}\rangle_{\mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}} = \overline{\langle \mathbf{v}, \mathbf{u}\rangle_{\mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}}}$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{H}_{\mathsf{J},\mathsf{i}}^{+}$

and such that

$$h_{\mathbb{C}_{\mathsf{i}}}(E_{\mathbb{C}_{\mathsf{i}}}) = S \circ E_{\mathbb{C}_{\mathsf{i}}} \circ S^{-1} \qquad \text{for } E_{\mathbb{C}_{\mathsf{i}}} \in \mathcal{L}(\mathcal{H}^+_{\mathsf{J},\mathsf{i}}). \tag{12.8}$$

If (I) holds true, then the quaternionic linear extension U of S to $\mathcal{H}$ is a unitary operator on $\mathcal{H}$ that commutes with J such that (12.4) holds true. If on the other hand (II) holds true, then we can choose $\mathbf{j} \in \mathbb{S}$ with $\mathbf{i} \perp \mathbf{j}$ and find that the operator $U_{\mathbb{C}_{\mathsf{i}}} : \mathcal{H}^+_{\mathsf{J},\mathsf{i}} \to \mathcal{H}^-_{\mathsf{J},\mathsf{i}}$ given by $U_{\mathbb{C}_{\mathsf{i}}}(\mathbf{v}) = S(\mathbf{v})\mathbf{j}$ is a $\mathbb{C}_{\mathsf{i}}$-linear unitary operator. Indeed, for $\mathbf{u}, \mathbf{v} \in \mathcal{H}$ and $a \in \mathbb{C}_{\mathsf{i}}$, we have

$$U_{\mathbb{C}_{\mathsf{i}}}(\mathbf{v}a) = S(\mathbf{v}a)\mathbf{j} = S(\mathbf{v})\overline{a}\mathbf{j} = S(\mathbf{v})\mathbf{j}a$$

and

$$\begin{aligned} \langle U_{\mathbb{C}_{\mathsf{i}}}\mathbf{v}, U_{\mathbb{C}_{\mathsf{i}}}\mathbf{u}\rangle_{\mathcal{H}^-_{\mathsf{J},\mathsf{i}}} &= \langle S(\mathbf{v})\mathbf{j}, S(\mathbf{u})\mathbf{j}\rangle_{\mathcal{H}} = -\mathbf{j}\langle S(\mathbf{v}), S(\mathbf{u})\rangle_{\mathcal{H}}\mathbf{j} \\ &= -\mathbf{j}\langle S(\mathbf{v}), S(\mathbf{u})\rangle_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}}\mathbf{j} = (-\mathbf{j})\overline{\langle \mathbf{v}, \mathbf{u}\rangle_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}}}\mathbf{j} = \langle \mathbf{u}, \mathbf{v}\rangle_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}} \end{aligned}$$

because $\overline{\langle \mathbf{v}, \mathbf{u}\rangle_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}}} \in \mathbb{C}_{\mathsf{i}}$. Since $S : \mathcal{H}^+_{\mathsf{J},\mathsf{i}} \to \mathcal{H}^+_{\mathsf{J},\mathsf{i}}$ and $\mathbf{v} \mapsto \mathbf{v}\mathbf{j} : \mathcal{H}^+_{\mathsf{J},\mathsf{i}} \to \mathcal{H}^-_{\mathsf{J},\mathsf{i}}$ are bijective, also their composition $U_{\mathbb{C}_{\mathsf{i}}}$ is bijective. If we write $\mathbf{v} \in \mathcal{H}$ as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2\mathbf{j}$ with $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{H}^+_{\mathsf{J},\mathsf{i}}$, then the quaternionic linear extension U of $U_{\mathbb{C}_{\mathsf{i}}}$ to all of $\mathcal{H}$ is given by $U(\mathbf{v}) = U_{\mathbb{C}_{\mathsf{i}}}\mathbf{v}_1 + U_{\mathbb{C}_{\mathsf{i}}}\mathbf{v}_2\mathbf{j}$. This operator is obviously also bijective and moreover unitary since

$$\begin{aligned} \langle U\mathbf{u}, U\mathbf{v}\rangle_{\mathcal{H}} &= \langle U(\mathbf{u}_1 + \mathbf{u}_2\mathbf{j}), U(\mathbf{v}_1 + \mathbf{v}_2\mathbf{j})\rangle_{\mathcal{H}} \\ =&\langle U_{\mathbb{C}_{\mathsf{i}}}\mathbf{u}_1, U_{\mathbb{C}_{\mathsf{i}}}\mathbf{v}_1\rangle_{\mathcal{H}^-_{\mathsf{J},\mathsf{i}}} + \langle U_{\mathbb{C}_{\mathsf{i}}}\mathbf{u}_1, U_{\mathbb{C}_{\mathsf{i}}}\mathbf{v}_2\rangle_{\mathcal{H}^-_{\mathsf{J},\mathsf{i}}}\mathbf{j} \\ &- \mathbf{j}\langle U_{\mathbb{C}_{\mathsf{i}}}\mathbf{u}_2, U_{\mathbb{C}_{\mathsf{i}}}\mathbf{v}_1\rangle_{\mathcal{H}^-_{\mathsf{J},\mathsf{i}}} - \mathbf{j}\langle U_{\mathbb{C}_{\mathsf{i}}}\mathbf{u}_2, U_{\mathbb{C}_{\mathsf{i}}}\mathbf{v}_2\rangle_{\mathcal{H}^-_{\mathsf{J},\mathsf{i}}}\mathbf{j} \\ =&\langle \mathbf{u}_1, \mathbf{v}_1\rangle_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}} + \langle \mathbf{u}_1, \mathbf{v}_2\rangle_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}}\mathbf{j} - \mathbf{j}\langle \mathbf{u}_2, \mathbf{v}_1\rangle_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}} - \mathbf{j}\langle \mathbf{u}_2, \mathbf{v}_2\rangle_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}}\mathbf{j} \\ =&\langle \mathbf{u}_1 + \mathbf{u}_2\mathbf{j}, \mathbf{v}_1 + \mathbf{v}_2\mathbf{j}\rangle_{\mathcal{H}} = \langle \mathbf{u}, \mathbf{v}\rangle_{\mathcal{H}}. \end{aligned}$$

The inverse of U is the quaternionic linear extension of $U_{\mathbb{C}_{\mathsf{i}}}^{-1}$, which is given by

$$U^{-1}(\mathbf{v}) = U_{\mathbb{C}_{\mathsf{i}}}^{-1}(\mathbf{v}_1\mathbf{j})(-\mathbf{j}) + U_{\mathbb{C}_{\mathsf{i}}}^{-1}(\mathbf{v}_2\mathbf{j}).$$

On the other hand $U_{\mathbb{C}_{\mathsf{i}}}^{-1}(\tilde{\mathbf{v}}) = S^{-1}(\tilde{\mathbf{v}}(-\mathbf{j}))$ for $\tilde{\mathbf{v}} \in \mathcal{H}^-_{\mathsf{J},\mathsf{i}}$ and so

$$U^{-1}(\mathbf{v}) = U_{\mathbb{C}_{\mathsf{i}}}^{-1}(\mathbf{v}_1\mathbf{j})(-\mathbf{j}) + U_{\mathbb{C}_{\mathsf{i}}}^{-1}(\mathbf{v}_2\mathbf{j}) = S^{-1}(\mathbf{v}_1)(-\mathbf{j}) + S^{-1}(\mathbf{v}_2).$$

For $E \in \mathcal{L}(\mathfrak{R})$, we therefore find

$$\begin{aligned} &UEU^{-1}\mathbf{v} = UE\left(S^{-1}(\mathbf{v}_1)(-\mathbf{j}) + S^{-1}(\mathbf{v}_2)\right) = \\ =&U\left(E_{\mathbb{C}_{\mathsf{i}}}(S^{-1}(\mathbf{v}_1))(-\mathbf{j}) + E_{\mathbb{C}_{\mathsf{i}}}S^{-1}(\mathbf{v}_2)\right) \\ =&U\left(S^{-1}\left(h_{\mathbb{C}_{\mathsf{i}}}(E_{\mathbb{C}_{\mathsf{i}}})(\mathbf{v}_1)\right)(-\mathbf{j}) + S^{-1}\left(h_{\mathbb{C}_{\mathsf{i}}}(E_{\mathbb{C}_{\mathsf{i}}})(\mathbf{v}_2)\right)\right) = \\ =&U_{\mathbb{C}_{\mathsf{i}}}\left(S^{-1}\left(h_{\mathbb{C}_{\mathsf{i}}}(E_{\mathbb{C}_{\mathsf{i}}})(\mathbf{v}_1)\right)\right)(-\mathbf{j}) + U_{\mathbb{C}_{\mathsf{i}}}\left(S^{-1}\left(h_{\mathbb{C}_{\mathsf{i}}}(E_{\mathbb{C}_{\mathsf{i}}})(\mathbf{v}_2)\right)\right) \\ =&S\left(S^{-1}\left(h_{\mathbb{C}_{\mathsf{i}}}(E_{\mathbb{C}_{\mathsf{i}}})(\mathbf{v}_1)\right)\right)\mathbf{j}(-\mathbf{j}) + S\left(S^{-1}\left(h_{\mathbb{C}_{\mathsf{i}}}(E_{\mathbb{C}_{\mathsf{i}}})(\mathbf{v}_2)\right)\right)\mathbf{j} \\ =&h_{\mathbb{C}_{\mathsf{i}}}(E_{\mathbb{C}_{\mathsf{i}}})(\mathbf{v}_1)(-\mathbf{j})\mathbf{j} + h_{\mathbb{C}_{\mathsf{i}}}(E_{\mathbb{C}_{\mathsf{i}}})(\mathbf{v}_2)\mathbf{j} = h(E)\mathbf{v} \end{aligned}$$

and so also in this case (12.4) holds true.

If U is a unitary operator on $\mathcal{H}$ that commutes with J, then its restriction $S := U|_{\mathcal{H}_{\mathbb{C}_{\mathsf{i}}}}$ to $\mathcal{H}^+_{\mathsf{J},\mathsf{i}}$ is a unitary operator on $\mathcal{H}^+_{\mathsf{J},\mathsf{i}}$, that is it satisfies (I). Hence, Wigner's theorem

implies that S induces a symmetry $h_{\mathbb{C}_\mathsf{i}}$ on $\mathcal{L}(\mathcal{H}^+_{\mathsf{J},\mathbf{i}})$ via (12.8) and so U induces the symmetry h on $\mathcal{L}(\mathfrak{R})$ that is characterized by (12.8) via (12.4). If on the other hand U is a unitary operator that anticommutes with J, then $U_{\mathbb{C}_\mathsf{i}} := U|_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}}$ is a unitary operator from $\mathcal{H}^+_{\mathsf{J},\mathbf{i}}$ to $\mathcal{H}^-_{\mathsf{J},\mathbf{i}}$. Consequently, $S(\mathbf{v}) := (U_{\mathbb{C}_\mathsf{i}}\mathbf{v})(-\mathbf{j})$ is a bijective $\mathbb{C}_\mathbf{i}$-anti-linear mapping from $\mathcal{H}^+_{\mathsf{J},\mathbf{i}}$ into itself that satisfies

$$\begin{aligned}\langle S(\mathbf{v}), S(\mathbf{u})\rangle_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}} &= \langle (U_{\mathbb{C}_\mathsf{i}}\mathbf{v})(-\mathbf{j}), (U_{\mathbb{C}_\mathsf{i}}\mathbf{u})(-\mathbf{j})\rangle_{\mathcal{H}} = \mathbf{j}\langle U_{\mathbb{C}_\mathsf{i}}\mathbf{v}, U_{\mathbb{C}_\mathsf{i}}\mathbf{u}\rangle_{\mathcal{H}}(-\mathbf{j})\\ &= \mathbf{j}\langle U_{\mathbb{C}_\mathsf{i}}\mathbf{v}, U_{\mathbb{C}_\mathsf{i}}\mathbf{u}\rangle_{\mathcal{H}^-_{\mathsf{J},\mathsf{i}}}(-\mathbf{j}) = \mathbf{j}\langle \mathbf{v},\mathbf{u}\rangle_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}}(-\mathbf{j}) = \overline{\langle \mathbf{v},\mathbf{u}\rangle_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}}}\end{aligned}$$

for any $\mathbf{u},\mathbf{v} \in \mathcal{H}^+_{\mathsf{J},\mathbf{i}}$. Hence, S satisfies (II). Wigner's theorem implies again that S induces a symmetry $h_{\mathbb{C}_\mathsf{i}}$ on $\mathcal{L}(\mathcal{H}^+_{\mathsf{J},\mathbf{i}})$ via (12.8). If E is the quaternionic linear extension of $E_{\mathbb{C}_\mathsf{i}}$ to all of $\mathcal{H}$, then the quaternionic linear extension of $h_{\mathbb{C}_\mathsf{i}}(E_{\mathbb{C}_\mathsf{i}}) = S \circ E_{\mathbb{C}_\mathsf{i}} \circ S^{-1}$ to all of $\mathcal{H}$ is due to $S^{-1}(\mathbf{v}) = U^{-1}_{\mathbb{C}_\mathsf{i}}(\mathbf{v}\mathbf{j})$ given by

$$\begin{aligned} h_{\mathbb{C}_\mathsf{i}}(E_{\mathbb{C}_\mathsf{i}})(\mathbf{v}) &= S\circ E_{\mathbb{C}_\mathsf{i}} \circ S^{-1}(\mathbf{v}_1) + S\circ E_{\mathbb{C}_\mathsf{i}}\circ S^{-1}(\mathbf{v}_2)\mathbf{j} =\\ =& S\circ E_{\mathbb{C}_\mathsf{i}}\left(U^{-1}_{\mathbb{C}_\mathsf{i}}(\mathbf{v}_1\mathbf{j})\right) + S\circ E_{\mathbb{C}_\mathsf{i}}\left(U^{-1}_{\mathbb{C}_\mathsf{i}}(\mathbf{v}_2\mathbf{j})\right)\mathbf{j}\\ =& S\left(EU^{-1}(\mathbf{v}_1\mathbf{j})\right) + S\left(EU^{-1}(\mathbf{v}_2\mathbf{j})\right)\mathbf{j}\\ =& U_{\mathbb{C}_\mathsf{i}}\left(EU^{-1}(\mathbf{v}_1\mathbf{j})(-\mathbf{j})\right) + U_{\mathbb{C}_\mathsf{i}}\left(EU^{-1}\mathbf{v}_2\mathbf{j}(-\mathbf{j})\right)\mathbf{j}\\ =& UEU^{-1}\mathbf{v}_1 + UEU^{-1}\mathbf{v}_2\mathbf{j} = UEU^{-1}\mathbf{v}.\end{aligned}$$

We conclude that U induces the symmetry h on $\mathcal{L}(\mathfrak{R})$ that is characterized by (12.8) via (12.4).

Finally, Wigner's theorem also states that two bijective functions S and S' that satisfy (I) or (II) induce the same symmetry $h_{\mathbb{C}_\mathsf{i}}$ on $\mathcal{B}(\mathcal{H}^+_{\mathsf{J},\mathbf{i}})$ if and only if $S' = \alpha S$ with $\alpha \in \mathbb{C}_\mathbf{i}$ and $|\alpha| = 1$. In particular either S and S' both satisfy (I) or they both satisfy (II). Now observe that U is a unitary operator on $\mathcal{H}$ that commutes with J if and only if $S = U|_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}}$ satisfies (I) and that U is a unitary operator that anticommutes with J if and only if the operator $S\mathbf{v} = U|_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}}\mathbf{v}\mathbf{j}$ satisfies (II). Since the a unitary operator U that commutes or anticommutes with J induces a symmetry h on $\mathcal{L}(\mathfrak{R})$ if and only if the respective operator S induces the symmetry $h_{\mathbb{C}_\mathsf{i}}$ determined by (12.8) on $\mathcal{L}(\mathcal{H}^+_{\mathsf{J},\mathbf{i}})$, we find that two unitary operators U and U' that induce the same symmetry h either both commute or both anticommute with J.

In the first case, the respective operators S and S' are simply the restrictions $S = U|_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}}$ and $S' = U|'_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}}$. We find due to Wigner's theorem that U and U' induce the same symmetry h if and only if $S = S'\alpha$ with $\alpha \in \mathbb{C}_\mathbf{i}$ and $|\alpha| = 1$, or equivalently

$$(U'U^{-1})|_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}} = S'S^{-1} = \alpha SS^{-1} = \alpha\mathcal{I}.$$

Since the quaternionic linear extension of the multiplication $\alpha\mathcal{I}$ with the complex number $\alpha = \alpha_0 + \mathbf{i}\alpha_1 \in \mathbb{C}_\mathbf{i}$ on $\mathcal{H}^+_{\mathsf{J},\mathbf{i}}$ to all of $\mathcal{H}$ is the operator $\alpha_0\mathcal{I} + \alpha_1 J \in \mathcal{C}(\mathfrak{R})$, we find that U and U' induce the same symmetry on $\mathcal{L}(\mathfrak{R})$ if and only if $U'U^{-1} = \alpha_0\mathcal{I} + \alpha_1\mathsf{J}$ with $\alpha_0, \alpha_1 \in \mathbb{R}$ so that $\alpha_0^2 + \alpha_1^2 = 1$. But operators of this type are precisely the unitary operators in $\mathcal{C}(\mathfrak{R})$.

In the second case, in which U and U' anticommute with J, we have that $S\mathbf{v} = (U_{\mathbb{C}_\mathsf{i}}\mathbf{v})(-\mathbf{j})$ and $S'\mathbf{v} = (U'_{\mathbb{C}_\mathsf{i}}\mathbf{v})(-\mathbf{j})$ for $\mathbf{v} \in \mathcal{H}^+_{\mathsf{J},\mathbf{i}}$ with $U_{\mathbb{C}_\mathsf{i}} := U|_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}}$ and $U_{\mathbb{C}_\mathsf{i}} := U|'_{\mathcal{H}^+_{\mathsf{J},\mathsf{i}}}$.

Again Wigner's theorem implies that U and U' induce the same symmetry h if and only if $S' = \alpha S$ with $\alpha \in \mathbb{C}_{\mathbf{i}}$ and $|\alpha| = 1$. This is equivalent to

$$\alpha \mathcal{I}\mathbf{v} = S^{-1} \circ S'(\mathbf{v}) = S^{-1}\left((U'_{\mathbb{C}_{\mathbf{i}}}\mathbf{v})\mathbf{j}\right) = U_{\mathbb{C}_{\mathbf{i}}}^{-1}\left((U'_{\mathbb{C}_{\mathbf{i}}}\mathbf{v})\mathbf{j}(-\mathbf{j})\right) = U_{\mathbb{C}_{\mathbf{i}}}^{-1}U'_{\mathbb{C}_{\mathbf{i}}}(\mathbf{v})$$

for all $\mathbf{v} \in \mathcal{H}^{+}_{\mathsf{J},\mathbf{i}}$. Taking the quaternionic linear extension to all of $\mathcal{H}$, we conclude as before that U and U' induce the same symmetry if and only if $U'U^{-1} \in \mathcal{C}(\mathfrak{R})$.

□

12.4 Reduction of Quaternionic to Complex Relativistic Systems

We recall that the Poincaré group is the Lie-group of Minkowski-spacetime symmetries and we recall that a unitary representation of a Lie-group G on a real, complex or quaternionic Hilbert space $\mathcal{H}$ is a group homomorphism $h : G \to \mathcal{B}(\mathcal{H})$ such that the mapping $(g, \mathbf{v}) \mapsto h(g)\mathbf{v}$ for $g \in G$ and $\mathbf{v} \in \mathcal{H}$ is continuous and such that $h(g)$ is unitary for all $g \in G$. Such representation is called locally faithful, if it is injective in a neighborhood of the neutral element of G.

An elementary relativistic system is an elementary quantum system that supports a representation h of the Poincaré group $\mathcal{P}$ viewed as a maximal symmetry group of the system. Since the system should be the realisation of the physical symmetries, h must contain all the information about the variables of the system and since the system should be elementary, the representation of $\mathcal{P}$ must be irreducible.

Definition 12.30. A real, complex or quaternionic *relativistic elementary system* (RES for short) is an elementary quantum system $\mathfrak{R}$ on a real, complex or quaternionic separable Hilbert space $\mathcal{H}$ that is equipped with a representation of the Poincaré group $h : \mathcal{P} \to \mathrm{Aut}(\mathcal{L}(\mathfrak{R})), g \mapsto h_g$ which is locally faithful and satisfies the following requirements.

(i) h is irreducible, in the sense that if $E \in \mathcal{L}(\mathfrak{R})$ then $h_g(E) = E$ for all $g \in \mathcal{P}$ implies either $E = 0$ or $E = \mathcal{I}$.

(ii) h is continuous in the sense that $g \mapsto \mu(h_g(E))$ is continuous for every fixed $E \in \mathcal{L}(\mathfrak{R})$ and every fixed quantum state μ.

(iii) h defines the observables of the system. If we represent the symmetry h_g for any $g \in \mathcal{P}$ by a unitary operator $U_g \in \mathfrak{R}$ so that $h_g(E) = U_g E U_g^{-1}$, this condition reads as

$$\mathcal{L}(\mathfrak{R}) \subset (\{U_g : g \in \mathcal{P}\} \cup \mathcal{C}(\mathfrak{R}))''. \tag{12.9}$$

Remark 12.31. The above notion of relativistic elementary system was introduced in Definition 5.7 of [70]. As the authors point out Remark 5.8, it coincides with the usual definition of relativistic elementary systems in the sense of Wigner if the quantum system is complex.

Remark 12.32. We implicitly assume in the above definition that we can represent any Poincaré-induced symmetry h_g by a unitary operator in $\mathfrak{R}$. This is obviously true for real quantum systems of real-real or real-quaternionic type and for real-induced or proper quaternionic quantum systems due to Theorems 12.26 and 12.27. For the other cases, this follows from the polar decomposition $\mathcal{P} = SL(2, \mathbb{C}) \ltimes \mathbb{R}^4$ of the Poincaré

group, which allows to write every $g \in P$ as the product $g = rrbb$, where r is a spatial rotation and b is a boost. Even if $\mathfrak{R}$ is a real quantum system of real-complex type or a complex-induced quaternionic quantum system and r or b are represented by unitary operators U_r and U_b that do not belong to $\mathfrak{R}$, then nevertheless U_r^2 and U_b^2 belong to $\mathfrak{R}$ and so also their product $U_g = U_r^2 U_b^2$ belong to $\mathfrak{R}$. Similarly, if $\mathfrak{R}$ is a complex quantum system and U_r or U_g are complex anti-unitary operators, then U_r^2 and U_b^2 and in turn also $U_g = U_r^2 U_g^2$ are complex linear unitary operators in $\mathfrak{R}$. Cf. also [70, Remark 5.8].

Our aim in this section is to show that any quaternionic RES is the external quaternionification of a complex RES, if the operator M_U^2 associated with the squared mass of the system is positive. In order to construct this operator, we choose a Minkowskian reference frame in Minkowski space time and consider the one-parameter Lie-subgroups $\mathfrak{p}_\ell : \mathbb{R} \to \mathcal{P}, \ell = 0, \ldots, 3$ of spacetime-displacements along the four Minkowskian axes. If $h : \mathcal{P} \to \mathcal{B}(\mathcal{H})$ is a unitary representation of the Poincaré group such that $h_g(E) = U_g E U_g^{-1}$ for all $E \in \mathcal{L}(\mathfrak{R})$, where $h : g \to h_g$ is the representation of $\mathcal{P}$ on $\operatorname{Aut}(\mathcal{L}(\mathfrak{R}))$ in Definition 12.30, then $U_\ell(t) := h(\mathfrak{p}_\ell(t)) = U_{h_{\mathfrak{p}_\ell(t)}}$ is for any $\ell = 0, \ldots, 3$ a strongly continuous group of unitary operators on $\mathcal{H}$. We define P_ℓ as the infinitesimal generator of the group $U_\ell(t)$, which is a densely defined anti-selfadjoint operator on $\mathcal{H}$. The operator associated with the squared mass of the system is the operator

$$M_U^2 := -P_0^2 + \sum_{\ell}^{3} P_\ell^2,$$

and we say that M_U^2 is positive if $\langle \mathbf{v}, M_U^2 \mathbf{v} \rangle \geq 0$ for all $\mathbf{v} \in \operatorname{dom}(M_U^2)$.

Remark 12.33. The above definition of M_U^2 is not very rigorous, in particular because it is not immediate that $\operatorname{dom}(M_U^2) = \bigcap_{\ell=0}^{3} \operatorname{dom}(P_\ell)$ is nonempty. A common core for these operators, which is even dense in $\mathcal{H}$, is the Gårding space. It consists of all vectors $\mathbf{v}_f \in \mathcal{H}$ generated by

$$\mathbf{v}_f := \int_{\mathcal{P}} f(g)\, U_g(\mathbf{v})\, dg \quad \text{with} \quad f \in C_0^\infty(\mathcal{P}, \mathbb{R}),$$

where dg denotes the left-invariant Haar measure on $\mathcal{P}$. The construction of M_U^2 for a real RES in [70] is based on a representation of the Lie algebra of $\mathcal{P}$ in terms of operators on the Gårding space. We can follow the same procedure in the quaternionic setting. Since the properties of the Gårding space depend only on the topology on $\mathcal{H}$ and not on the field of scalars, we can simply choose $\mathsf{i} \in \mathbb{S}$ and consider the quaternionic Hilbert space $\mathcal{H}$ as a complex Hilbert space $\mathcal{H}_{\mathsf{i}}$ over $\mathbb{C}_{\mathsf{i}}$ as in Remark 2.40. We then obtain the results concerning well-definedness, density etc. of the Gårding space and the existence of a representation of the Lie algebra of $\mathcal{P}$ in terms of operators on this space simply by applying the complex results on $\mathcal{H}_{\mathsf{i}}$. (This is similar to [70], where the proofs of the properties of the Gårding space in the case $\mathcal{H}$ is a real Hilbert space consist essentially in applying the results from the complex case to the external complexification of the real space $\mathcal{H}$.) We do not recall these arguments in detail since this would go beyond the scope of this thesis, while it would also not seem to add a lot of mathematical value.

Finally, we recall Theorem 5.11 in [70], which shows that any real quantum system is of real-complex type, provided that the operator M_U^2 associated with the squared mass of the system is positive.

Theorem 12.34. *Let $\mathfrak{R}$ be a real RES on a real Hilbert space $\mathcal{H}$ and let $h : \mathcal{P} \to \mathcal{B}(\mathcal{H})$ be the locally-faithful strongly-continuous unitary representation of the Poincaré group on $\mathcal{H}$. If the operator M_U^2 that is associated with the squared mass of the system is positive, then the following facts hold.*

(i) The von Neumann-algebra $\mathfrak{R}$ is of real-complex type and so $\mathfrak{R} = \mathcal{B}(\mathcal{H}_{\mathsf{J}})$, where $\mathcal{H}_{\mathsf{J}} := (\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathsf{J}})$ is the internal complexification of $\mathcal{H}$ induced by the unitary and anti-selfadjoint operator J such that $\mathfrak{R}' = \{\, a\mathcal{I} + b\mathsf{J} : a, b \in \mathbb{R}\}$, which is unique up-to-sign.

(ii) The representation h of $\mathcal{P}$ is irreducible on $\mathcal{H}_{\mathsf{J}}$ and defines a complex RES that is equivalent to $\mathfrak{R}$.

(iii) The operator J is a Poincaré invariant and coincides up to sign with the unitary factor of the polar decomposition of the anti-selfadjoint generator of the group of temporal translations, that is either $P_0 = \mathsf{J}|P_0|$ or $P_0 = -\mathsf{J}|P_0|$.

Remark 12.35. Observe that this theorem in particular implies that a real RES can neither be of real-real nor of real-quaternionic type.

The proof of the above theorem requires quite deep and lengthy physical arguments, which are beyond the scope of this thesis. Instead of replicating them in the quaternionic case in order to show that any quaternionic RES is equivalent to a complex one, we therefore first show that any such system is equivalent to a real RES and then apply the above theorem.

Theorem 12.36. *Let $\mathfrak{R}$ be a quaternionic RES on a quaternionic Hilbert space $\mathcal{H}$ and let $h : \mathcal{P} \to \mathcal{B}(\mathcal{H})$ be a locally-faithful strongly-continuous unitary representation of the Poincaré group as in Definition 12.30. If the operator M_U^2 associated with the squared mass of the system is positive, then the following facts hold.*

(i) The von Neumann-algebra $\mathfrak{R}$ is complex induced and so $\mathfrak{R}$ is the external quaternionification of $\mathcal{B}(\mathcal{H}^+_{\mathsf{J},\mathsf{i}})$, where $\mathsf{i} \in \mathbb{S}$ and J is the unitary and anti-selfadjoint operator J such that $\mathfrak{R}' = \{\, a\mathcal{I} + b\mathsf{J} : a, b \in \mathbb{R}\}$ that is unique up-to-sign.

(ii) The representation g of $\mathcal{P}$ is irreducible on $\mathcal{H}^+_{\mathsf{J},\mathsf{i}}$ and defines a complex RES that is equivalent to $\mathfrak{R}$.

(iii) The operator J is a Poincaré invariant and coincides up to sign with the unitary factor of the polar decomposition of the anti-selfadjoint generator of the group of temporal translations, that is either $P_0 = \mathsf{J}|P_0|$ or $P_0 = -\mathsf{J}|P_0|$.

Proof. Since the von Neumann-algebra $\mathfrak{R}$ is by definition irreducible, it is due to Theorem 12.27 either real-induced, complex induced or proper quaternionic.

We assume that $\mathfrak{R}$ is real induced. In this case, there exist unitary and anti-self-adjoint operators $\mathsf{I}, \mathsf{J}, \mathsf{K} := \mathsf{IJ}$ in $\mathcal{B}(\mathcal{H})$ that anti-commute mutually such that

$$\mathfrak{R}' = \{a\mathcal{I} + b\mathsf{I} + c\mathsf{J} + d\mathsf{K} : a, b, c, d \in \mathbb{R}\}. \tag{12.10}$$

The von Neumann-algebra $\mathfrak{R}$ is then the external quaternionification of the real von Neumann-algebra $\mathfrak{R}_{\mathbb{R}} := \mathfrak{R}|_{\mathcal{H}_{\mathbb{R}}} = \mathcal{B}(\mathcal{H}_{\mathbb{R}})$ of real-real type, which is obviously irreducible. Hence, $\mathfrak{R}_{\mathbb{R}}$ is an elementary real quantum system that is logically equivalent

to $\mathfrak{R}$. If h is the locally faithful representation of the poincare group $\mathcal{P}$, then it induces a locally faithful representation $h_{\mathbb{R}} : g \mapsto h_{\mathbb{R},g}$ of $\mathcal{P}$ on $\mathcal{H}_{\mathbb{R}}$ because $\mathcal{L}(\mathfrak{R})$ and $\mathcal{L}(\mathfrak{R}_{\mathbb{R}})$ are isomorphic lattices. Precisely, we have for all $E_{\mathbb{R}} \in \mathcal{L}(\mathfrak{R}_{\mathbb{R}})$ and all $g \in \mathcal{P}$ that

$$h_{\mathbb{R},g}(E_{\mathbb{R}}) = h_g(E)|_{\mathcal{H}_{\mathbb{R}}}, \tag{12.11}$$

where E is the quaternionic linear extension of $E_{\mathbb{R}}$ to all of E so that $E_{\mathbb{R}} = E|_{\mathcal{H}_{\mathbb{R}}}$. This representation is obviously irreducible: if $h_{\mathbb{R},g}(E_{\mathbb{R}}) = E_{\mathbb{R}}$ for all $g \in \mathcal{P}$ then (12.11) implies for the projection $E \in \mathcal{L}(\mathfrak{R})$ with $E_{\mathbb{R}} = E|_{\mathcal{H}_{\mathbb{R}}}$ that $h_g(E) = E$ for all $g \in \mathcal{P}$. Due to the irreducibility of h, we find that either $E = 0$ or $E = \mathcal{I}_{\mathcal{H}}$ and in turn also either $E_{\mathbb{R}} = E|_{\mathcal{H}_{\mathbb{R}}} = 0$ or $E_{\mathbb{R}} = E|_{\mathcal{H}_{\mathbb{R}}} = \mathcal{I}_{\mathcal{H}_{\mathbb{R}}}$. Similarly, any quantum state $\mu_{\mathbb{R}} : \mathcal{L}(\mathfrak{R}_{\mathbb{R}}) \to [0, +\infty)$, is equivalent to a quantum state $\mu : \mathcal{L}(\mathfrak{R}) \to [0, +\infty)$ and we have

$$\mu_{\mathbb{R}}(E_{\mathbb{R}}) = \mu(E), \quad \text{if} \quad E_{\mathbb{R}} = E|_{\mathcal{H}_{\mathbb{R}}}.$$

Due to the continuity of h, the mapping $g \mapsto \mu_{\mathbb{R}}(h_{\mathbb{R},g}(E_{\mathbb{R}})) = \mu(h_g(E))$ is continuous for any fixed $E_{\mathbb{R}} \in \mathcal{L}(\mathfrak{R}_{\mathbb{R}})$ and any quantum state $\mu_{\mathbb{R}}$ on $\mathcal{L}(\mathfrak{R}_{\mathbb{R}})$ and so h is continuous.

Finally, let $\mathfrak{U}_{\mathbb{R}} := \{U_{\mathbb{R},g} : g \in \mathcal{P}\}$ be a set of unitary operators on $\mathcal{H}_{\mathbb{R}}$ such that $h_{\mathbb{R},g}(E_{\mathbb{R}}) = U_{\mathbb{R},g} E_{\mathbb{R}} U_{\mathbb{R},g}^{-1}$ for all $E_{\mathbb{R}} \in \mathcal{L}(\mathfrak{R}_{\mathbb{R}})$. The set

$$\mathfrak{U} := \{\, U_g \in \mathcal{B}(\mathcal{H}) : U_g|_{\mathcal{H}_{\mathbb{R}}} \in \mathfrak{U}_{\mathbb{R}}\}$$

of quaternionic linear extensions of operators in $\mathfrak{U}_{\mathbb{R}}$ is then a set of unitary operators on $\mathcal{H}$ such that $h_g(E) = U_g E U_g^{-1}$ for all $E \in \mathcal{L}(\mathfrak{R})$ and all $g \in \mathcal{P}$.

Since the operators I, J and K commute with any $U \in \mathfrak{U}$, they belong to $\mathfrak{U}'$. Any operator $A \in \mathfrak{U}''$ commutes therefore with I, J and K and is hence the quaternionic linear extension of an operator $A_{\mathbb{R}} \in \mathcal{B}(\mathcal{H}_{\mathbb{R}})$. If $B_{\mathbb{R}} \in \mathfrak{U}_{\mathbb{R}}'$, then its quaternionic linear extension B to all of $\mathcal{H}$ belongs to $\mathfrak{U}'$ and so we have $AB = BA$ for any $A \in \mathfrak{U}''$. Taking the restriction to $\mathcal{H}_{\mathbb{R}}$, we find that $A_{\mathbb{R}} B_{\mathbb{R}} = B_{\mathbb{R}} A_{\mathbb{R}}$ for any $A \in \mathfrak{U}''$ and any $B_{\mathbb{R}} \in \mathfrak{U}_{\mathbb{R}}'$ and so

$$\mathcal{L}(\mathfrak{R}_{\mathbb{R}}) = \mathcal{L}(\mathfrak{R})|_{\mathcal{H}_{\mathbb{R}}} \subset \mathfrak{U}''|_{\mathcal{H}_{\mathbb{R}}} \subset \mathfrak{U}_{\mathbb{R}}''.$$

Altogether, we find that $\mathfrak{R}_{\mathbb{R}}$ is a real RES of real-real type. The operator $M_{\mathbb{R},U}^2$ associated with the squared mass in the real quantum system $\mathfrak{R}_{\mathbb{R}}$ is the restriction of the operator M_U^2 associated with the squared mass in the quaternionic quantum system $\mathcal{H}$ to $\mathcal{H}_{\mathbb{R}}$, that is $M_{\mathbb{R},U}^2 = M_U^2|_{\mathcal{H}_{\mathbb{R}}}$. Moreover M_U^2 is positive if and only if $M_{\mathbb{R},U}^2$ is positive. We conclude from Theorem 12.34 and Remark 12.35 that the quaternionic RES $\mathfrak{R}$ is not real-induced if M_U^2 is positive.

If $\mathfrak{R}$ is proper quaternionic, that is $\mathfrak{R} = \mathcal{B}(\mathcal{H})$, then we can argue similarly. We can consider the real Hilbert space $\mathcal{H}_{\mathbb{R}} := (\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathbb{R}})$, which we obtain if we restrict the scalar multiplication on $\mathcal{H}$ to $\mathbb{R}$ and endow this space with the real scalar product $\langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{R}} := \operatorname{Re}\langle \mathbf{u}, \mathbf{v}\rangle$. If we consider the operators in $\mathfrak{R}$ as $\mathbb{R}$-linear operators on $\mathcal{H}_{\mathbb{R}}$, we find that $\mathfrak{R}$ is a real Banach-subalgebra of $\mathcal{B}(\mathcal{H}_{\mathbb{R}})$. Since both scalar products $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ generate the same topology on $\mathcal{H}$, the set $\mathfrak{R}$ is strongly closed not only as a sub-algebra of $\mathcal{B}(\mathcal{H})$ but also as a sub-algebra of $\mathcal{B}(\mathcal{H}_{\mathbb{R}})$ and hence it is a real von Neumann-algebra of operators on $\mathcal{H}_{\mathbb{R}}$.

We show now that $\mathfrak{R}$ is also irreducible as a subset of $\mathcal{B}(\mathcal{H}_{\mathbb{R}})$ and hence assume that K is a closed $\mathbb{R}$-linear subspace of $\mathcal{H}$ such that $T(K) \subset K$ for all $T \in \mathfrak{R}$. The $\mathbb{H}$-linear span

$$\tilde{K} = \operatorname{span}_{\mathbb{H}} K = \{\mathbf{v}a : \mathbf{v} \in K, a \in \mathbb{H}\}$$

is then a closed quaternionic linear subspace of $\mathcal{H}$. Since $T\mathbf{v} \in K$ for any $\mathbf{v} \in K$ and any $T \in \mathfrak{R}$, we find $T(\mathbf{v}a) = (T\mathbf{v})a \in \tilde{K}$ for any $\mathbf{v}a \in \tilde{K}$ and any $T \in \mathfrak{R}$, we find that $\tilde{K}$ is a closed quaternionic linear subspace of $\mathcal{H}$ such that $T(\tilde{K}) \subset \tilde{K}$ for any $T \in \mathfrak{R}$. Since $\mathfrak{R}$ is an irreducible von Neumann-algebra on $\mathcal{H}$, we conclude that either $\tilde{K} = \{\mathbf{0}\}$ or $\tilde{K} = \mathcal{H}$. In the first case, we immediately conclude $K = \{\mathbf{0}\}$. In the second case, we have $\mathcal{H}_{\mathbb{R}} = \mathcal{H} = \mathrm{span}_{\mathbb{H}} K = \{\mathbf{v}a : \mathbf{v} \in K, a \in \mathbb{H}\}$, but not immediate that $\mathcal{H} = K$. Hence, let us assume that $K \neq \mathcal{H}_{\mathbb{R}}$. Then there exist $\mathbf{v} \in K$ and $a \in \mathbb{H}$ such that $\mathbf{v}a \notin K$. (Without loss of generality, we can even assume $\|\mathbf{v}\| = 1$.) The operator $T\mathbf{u} := \mathbf{v}a\langle \mathbf{v}, \mathbf{u}\rangle$ is then a bounded quaternionic right linear operator on $\mathcal{H}$ and hence belongs to $\mathfrak{R}$, but it does not leave K invariant as $T\mathbf{v} = \mathbf{v}a \notin K$. We conclude that such $\mathbf{v}$ cannot exist and so $\tilde{K} = \mathcal{H}$ implies $K = \mathcal{H} = \mathcal{H}_{\mathbb{R}}$. Altogether, we find that if a closed subspace K of $\mathcal{H}_{\mathbb{R}}$ satisfies $T(K) \subset K$ for all $T \in \mathfrak{R}$, then either $K = \{\mathbf{0}\}$ or $K = \mathcal{H}_{\mathbb{R}}$. Hence, $\mathfrak{R}$ is an irreducible real von Neuman-subalgebra of $\mathcal{B}(\mathcal{H}_{\mathbb{R}})$ and therefore an elementary real quantum system on $\mathcal{H}_{\mathbb{R}}$ that is equivalent to the elementary quaternionic quantum system $\mathfrak{R}$ on $\mathcal{H}$. We denote this quantum system by $\mathfrak{R}_{\mathbb{R}}$ in order to stress that we consider the operators as $\mathbb{R}$-linear operators on $\mathcal{H}_{\mathbb{R}}$. It is obviously of real-quaternionic type according to the classification in Theorem 12.26.

Let $h : \mathcal{P} \to \mathrm{Aut}(\mathcal{L}(\mathfrak{R}))$ be the locally faithful representation satisfying condition (i) to (iii) in Definition 12.30. Since $\mathcal{L}(\mathfrak{R}) = \mathcal{L}(\mathfrak{R}_{\mathbb{R}})$ and in turn $\mathrm{Aut}(\mathcal{L}(\mathfrak{R})) = \mathrm{Aut}(\mathcal{L}(\mathfrak{R}_{\mathbb{R}}))$, we find that h is also a locally faithful representation of the Poincaré group on $\mathrm{Aut}(\mathcal{L}(\mathfrak{R}_{\mathbb{R}}))$. We shall denote h also by $h_{\mathbb{R}}$, if we consider it as a representation $h : \mathcal{P} \to \mathrm{Aut}(\mathcal{L}(\mathfrak{R}_{\mathbb{R}}))$. Since h is irreducible, we find that $h_{\mathbb{R},g}(E) = h_g(E) = E$ for all $g \in \mathcal{P}$ implies either $E = 0$ or $E = \mathcal{I}$ for all $E \in \mathcal{L}(\mathcal{H}_{\mathbb{R}}) = \mathcal{L}(\mathcal{H})$ and hence also $h_{\mathbb{R}}$ is irreducible. It obviously also inherits the property of being continuous from h.

What remains to show in order for $\mathfrak{R}_{\mathbb{R}}$ to be a real RES is that $h_{\mathbb{R}}$ determines the observables of $\mathfrak{R}_{\mathbb{R}}$. Let therefore $U_g \in \mathcal{B}(\mathcal{H})$ for $g \in \mathcal{P}$ be unitary operators so that $h_g(E) = U_g E U_g^{-1}$ and denote $\mathfrak{U} := \{U_g : g \in \mathcal{P}\}$ and let us denote U_g considered as an $\mathbb{R}$-linear operator on $\mathcal{H}_{\mathbb{R}}$ by $U_{\mathbb{R},g}$. Then $U_{\mathbb{R},g}$ is an $\mathbb{R}$-linear unitary operator on $\mathcal{H}_{\mathbb{R}}$ so that $h_{\mathbb{R},g}(E) = U_g E U_g^{-1}$. We denote $\mathfrak{U}_{\mathbb{R}} = \{U_{\mathbb{R},g} : g \in \mathcal{P}\}$, that is $\mathfrak{U}_{\mathbb{R}}$ equals $\mathfrak{U}$ where we consider its elements as operators on $\mathcal{H}_{\mathbb{R}}$ instead of $\mathcal{H}$. Since $\mathfrak{R}_{\mathbb{R}}$ is a real von Neumann-algebra of real-quaternionic type, we have $\mathcal{C}(\mathfrak{R}_{\mathbb{R}}) = \{a\mathcal{I} : a \in \mathbb{R}\}$ by Theorem 12.26 and hence we need to show that

$$\mathcal{L}(\mathfrak{R}_{\mathbb{R}}) = \mathcal{L}(\mathfrak{R}) \subset (\mathfrak{U}_{\mathbb{R}} \cup \mathcal{C}(\mathfrak{R}_{\mathbb{R}}))'' = \mathfrak{U}_{\mathbb{R}}'',$$

where $\mathfrak{U}_{\mathbb{R}}''$ denotes the bicommutant of $\mathfrak{U}_{\mathbb{R}} = \mathfrak{U}$ in $\mathcal{B}(\mathcal{H}_{\mathbb{R}})$. We know that the bicommutant $\mathfrak{U}''$ of $\mathfrak{U}$ in $\mathcal{B}(\mathcal{H})$ satisfies

$$\mathcal{L}(\mathfrak{R}) \subset (\mathfrak{U} \cup \mathcal{C}(\mathfrak{R}))'' = \mathfrak{U}'',$$

as $\mathcal{C}(\mathfrak{R}) = \{a\mathcal{I} : a \in \mathbb{R}\}$ by Theorem 12.27 because $\mathfrak{R}$ is a proper quaternionic von Neumann algebra. Moreover, as $\mathfrak{R} = \mathcal{B}(\mathcal{H})$, we have $\mathcal{L}(\mathfrak{R}) = \mathcal{L}(\mathcal{H}) = \{a\mathcal{I} : a \in \mathbb{R}\}$, cf. [70, Lemma 5.16], and so

$$\{a\mathcal{I} : a \in \mathbb{R}\} = \mathcal{L}(\mathfrak{R})' \supset (\mathfrak{U}'')' = \mathfrak{U}' \supset \{a\mathcal{I} : a \in \mathbb{R}\}.$$

Hence, $\mathfrak{U}' = \{a\mathcal{I} : a \in \mathbb{R}\}$.

Let now $\mathbf{i}, \mathbf{j} \in \mathbb{S}$ with $\mathbf{i} \perp \mathbf{j}$ and set $\mathbf{k} := \mathbf{ij}$. For any quaternionic $a \in \mathbb{H}$, the operator $M_a\mathbf{v} := \mathbf{v}a$ is a bounded $\mathbb{R}$-linear operator on $\mathcal{H}_\mathbb{R} = \mathcal{H}$ and hence it belongs to $\mathcal{B}(\mathcal{H}_\mathbb{R})$. Moreover, an arbitrary operator in $\mathcal{B}(\mathcal{H}_\mathbb{R})$ is quaternionic right-linear if and only if it commutes with $M_\mathbf{i}$, $M_\mathbf{j}$ and $M_\mathbf{k}$. As $\mathfrak{U}_\mathbb{R} = \mathfrak{U}$ consists of quaternionic linear operators, we find $M_\mathbf{i}, M_\mathbf{j}, M_\mathbf{k} \in \mathfrak{U}'_\mathbb{R}$ and so

$$\{M_a : a \in \mathcal{H}\} = \{\, a_0\mathcal{I} + a_1 M_\mathbf{i} + a_2 M_\mathbf{j} + a_3 M_\mathbf{k} : a_\ell \in \mathbb{R}\} \subset \mathfrak{U}'_\mathbb{R}. \tag{12.12}$$

If on the other hand $A \in \mathfrak{U}'_\mathbb{R}$, then the operator

$$\begin{aligned} A_\mathbb{H}\mathbf{v} :=& A\mathbf{v} - M_\mathbf{i} A M_\mathbf{i}\mathbf{v} - M_\mathbf{j} A M_\mathbf{j}\mathbf{v} - M_\mathbf{k} A M_\mathbf{k}\mathbf{v} \\ =& A\mathbf{v} - (A(\mathbf{v}\mathbf{i}))\mathbf{i} - (A(\mathbf{v}\mathbf{j}))\mathbf{j} - (A(\mathbf{v}\mathbf{k})\mathbf{k}) \end{aligned}$$

obviously also belongs to $\mathfrak{U}'_\mathbb{R}$ because it consists of the sum of compositions of operators in $\mathfrak{U}'_\mathbb{R}$. We moreover have

$$\begin{aligned} A_\mathbb{H}(\mathbf{v}\mathbf{i}) =& A(\mathbf{v}\mathbf{i}) - (A(\mathbf{v}\mathbf{i}^2))\mathbf{i} - (A(\mathbf{v}\mathbf{i}\mathbf{j}))\mathbf{j} - (A(\mathbf{v}\mathbf{i}\mathbf{k})\mathbf{k}) \\ =& \left(-A(\mathbf{v}\mathbf{i})\mathbf{i} + A(\mathbf{v}) - (A(\mathbf{v}\mathbf{k}))\mathbf{k} - (A(\mathbf{v}\mathbf{j})\mathbf{j})\right)\mathbf{i} = (A_\mathbb{H}\mathbf{v})\mathbf{i} \end{aligned}$$

and similarly also $A_\mathbb{H}(\mathbf{v}\mathbf{j}) = (A_\mathbb{H}\mathbf{v})\mathbf{j}$ and $A_\mathbb{H}(\mathbf{v}\mathbf{k}) = (A_\mathbb{H}\mathbf{v})\mathbf{k}$. Hence, $A_\mathbb{H}$ is a quaternionic right linear operator in $\mathfrak{U}'_\mathbb{R}$. Therefore it belongs to $\mathfrak{U}'$ and we conclude

$$A_\mathbb{H}(\mathbf{v}\mathbf{i}) = \mathbf{v}a_0$$

with $a_0 \in \mathbb{R}$. The operators $AM_\mathbf{i}$, $AM_\mathbf{j}$ and $AM_\mathbf{k}$ also belong to $\mathfrak{U}'_\mathbb{R}$ because they are compositions of operators in $\mathfrak{U}'_\mathbb{R}$. By the above argument, there exist real numbers a_1, a_2, and a_3 such that $(AM_\mathbf{i})_\mathbb{H} = a_1\mathcal{I}$, $(AM_\mathbf{j}) = a_2\mathcal{I}$, and $(AM_\mathbf{k})_\mathbb{H} = a_3\mathcal{I}$. Straightforward computations show that

$$\begin{aligned} A_\mathbb{H}\mathbf{v} = A(\mathbf{v}) - (A(\mathbf{v}\mathbf{i}))\mathbf{i} - (A(\mathbf{v}\mathbf{j}))\mathbf{j} - (A(\mathbf{v}\mathbf{k}))\mathbf{k} =& \mathbf{v}a_0 \\ ((AM_\mathbf{i})_\mathbb{H}\mathbf{v})(-\mathbf{i}) = -A(\mathbf{v}\mathbf{i})\mathbf{i} + A(\mathbf{v}) + (A(\mathbf{v}\mathbf{k}))\mathbf{k} + (A(\mathbf{v}\mathbf{j}))\mathbf{j} =& -\mathbf{v}a_1\mathbf{i} \\ ((AM_\mathbf{j})_\mathbb{H}\mathbf{v})(-\mathbf{j}) = -(A(\mathbf{v}\mathbf{j}))\mathbf{j} + (A(\mathbf{v}\mathbf{k}))\mathbf{k} + A(\mathbf{v}) + (A(\mathbf{v}\mathbf{i}))\mathbf{i} =& -\mathbf{v}a_2\mathbf{j} \\ ((AM_\mathbf{k})_\mathbb{H}\mathbf{v})(-\mathbf{k}) = -A(\mathbf{v}\mathbf{k})\mathbf{k} + (A(\mathbf{v}\mathbf{j}))\mathbf{j} + (A(\mathbf{v}\mathbf{i}))\mathbf{i} + A(\mathbf{v}) =& -\mathbf{v}a_3\mathbf{k}. \end{aligned}$$

If we add these four equations, we are left with

$$4A(\mathbf{v}) = \mathbf{v}a_0 - \mathbf{v}a_1\mathbf{i} - \mathbf{v}a_2\mathbf{j} - \mathbf{v}a_3\mathbf{k}$$

and hence $A = M_a$ with $a = \frac{1}{4}(a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k})$. Hence, the relation in (12.12) is actually an equality and $\mathfrak{U}'_\mathbb{R} = \{\, M_a : a \in \mathbb{H}\}$. An operator $T \in \mathcal{B}(\mathcal{H}_\mathbb{R})$ commutes with M_a if and only if

$$T(\mathbf{v}a) = TM_a\mathbf{v} = M_aT\mathbf{v} = (T\mathbf{v})a$$

and hence the set of operators that commute with M_a for any $a \in \mathbb{H}$ is exactly the set of quaternionic linear operators in $\mathcal{B}(\mathcal{H}_\mathbb{R})$. We conclude $\mathfrak{U}''_\mathbb{R} = \mathcal{B}(\mathcal{H}) = \mathfrak{U}''$ and so in particular $\mathcal{L}(\mathfrak{R}_\mathbb{R}) = \mathcal{L}(\mathfrak{R}) \subset \mathfrak{U}'' = \mathfrak{U}''_\mathbb{R}$.

Althogether, we find that $\mathfrak{R}_\mathbb{R}$ is a real RES of real-quaternionic type on $\mathcal{H}_\mathbb{R}$. However, if M_U^2 is the operator associated with the squared mass of the system in $\mathfrak{R}$, then M_U^2 is also the operator associated with the squared mass of the system in $\mathfrak{R}_\mathbb{R}$ and it is

positive on $\mathcal{H}$ if and only if it is positive as an operator on $\mathcal{H}_{\mathbb{R}}$. Since the real RES cannot be of real-quaternionic type if $M_U^2 \geq 0$, we conclude that positivity of M_U^2 implies that $\mathfrak{R}$ is not proper quaternionic.

The von Neumann-algebra $\mathfrak{R}$ must hence be complex induced. If J is the unitary anti-selfadjoint operator on $\mathcal{H}$ such that $\mathfrak{R}' = \{ a\mathcal{I} + \mathsf{J}b : a, b \in \mathbb{R}\}$ and $\mathbf{i} \in \mathbb{S}$, then $\mathfrak{R}$ is the external quaternionification of the $\mathbb{C}_{\mathbf{i}}$-complex von Neumann-algebra $\mathfrak{R}_{\mathbf{i}} := \mathcal{B}(\mathcal{H}^+_{\mathsf{J},\mathbf{i}})$, which is obviously irreducible, and $\mathcal{L}(\mathfrak{R})$ is the external quaternionification of and hence isomorphic to $\mathcal{L}(\mathfrak{R}_{\mathbf{i}}) = \mathcal{L}(\mathcal{H}^+_{\mathsf{J},\mathbf{i}})$. The representation $h : \mathcal{P} \to \mathrm{Aut}(\mathcal{L}(\mathfrak{R}))$ induces also in this case a representation $h_{\mathbf{i}} : \mathcal{P} \to \mathrm{Aut}(\mathcal{L}(\mathfrak{R}_{\mathbf{i}}))$ of the Poincaré group, namely

$$h_{\mathbf{i},g}(E_{\mathbf{i}}) := h(E)|_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}} \quad \text{if } E_{\mathbf{i}} = E|_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}}.$$

The same arguments that we applied in the real-induced and proper quaternionic case show that $h_{\mathbf{i}}$ inherits irreducibility and continuity from h. Finally, let $U_{\mathbf{i},g}$ for $g \in \mathcal{P}$ be a unitary operator on $\mathcal{H}^+_{\mathsf{J},\mathbf{i}}$ so that $h_{\mathbf{i},g}(E_{\mathbf{i}}) = U_{\mathbf{i},g}E_{\mathbf{i}}U_{\mathbf{i},g}^{-1}$ for all $E_{\mathbf{i}} \in \mathcal{L}(\mathfrak{R}_{\mathbf{i}})$ and set $\mathfrak{U}_{\mathbf{i}} = \{ U_{\mathbf{i},g} : g \in \mathcal{P}\}$. The quaternionic linear extension U_g of $U_{g,\mathbf{i}}$ to all of $\mathcal{H}$ is then a unitary operator on $\mathcal{H}^+_{\mathsf{J},\mathbf{i}}$ such that $h_g(E) = U_gEU_g^{-1}$ for any $E \in \mathcal{L}(\mathfrak{R})$. Since $\mathfrak{R}$ is a complex-induced quaternionic RES, we have

$$\mathcal{L}(\mathfrak{R}) \subset (\mathfrak{U} \cup \mathcal{C}(\mathfrak{R}))'' = (\mathfrak{U} \cup \{\mathsf{J}\})''$$

because $\mathcal{C}(\mathfrak{R}) = \{a\mathcal{I} + b\mathsf{J} : a, b \in \mathbb{R}\}$. An operator $A \in \mathcal{B}(\mathcal{H})$ belongs to $(\mathfrak{U} \cap \{\mathsf{J}\})'$ if and only if it commutes with every operator $U \in \mathfrak{U}$ and with the operator J. This is equivalent to A being the quaternionic linear extension of an operator $A_{\mathbf{i}} = A|_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}}$ in $\mathcal{B}(\mathcal{H}^+_{\mathsf{J},\mathbf{i}})$ that commutes with the restriction $U_{\mathbf{i}} = U|_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}}$ of any $U \in \mathfrak{U}$, in other words to $A_{\mathbf{i}}$ being an element of $\mathfrak{U}_{\mathbf{i}}'$. Therefore

$$(\mathfrak{U} \cup \mathcal{C}(\mathfrak{R}))' = (\mathfrak{U}_{\mathbf{i}}')_{\mathbb{H}} := \left\{ A \in \mathcal{B}(\mathcal{H}) : A|_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}} \in \mathfrak{U}_{\mathbf{i}}' \right\}. \tag{12.13}$$

In particular $\mathsf{J} \in (\mathfrak{U} \cup \mathcal{C}(\mathfrak{R}))'$ and so also any operator $A \in (\mathfrak{U} \cup \mathcal{C}(\mathfrak{R}))''$ is the quaternionic linear extension of an operator in $\mathcal{B}(\mathcal{H}^+_{\mathsf{J},\mathbf{i}})$. Therefore

$$(\mathfrak{U} \cup \mathcal{C}(\mathfrak{R}))'' = \{ A \in \mathcal{B}(\mathcal{H}) : A|_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}} \in \mathfrak{U}_{\mathbf{i}}''\} \tag{12.14}$$

because two operators $A, B \in \mathcal{B}(\mathcal{H})$ so that $A_{\mathbf{i}} = A|_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}}$ and $B_{\mathbf{i}} = B|_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}}$ belong to $\mathcal{B}(\mathcal{H}^+_{\mathsf{J},\mathbf{i}})$ commute if and only if $A_{\mathbf{i}}$ and $B_{\mathbf{i}}$ commute. Combining (12.13) and (12.14) and taking the restrictions to $\mathcal{H}^+_{\mathsf{J},\mathbf{i}}$, we obtain

$$\mathcal{L}(\mathfrak{R}_{\mathbf{i}}) = \left\{ E|_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}} : E \in \mathcal{L}(\mathfrak{R}) \right\} \subset \left\{ A|_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}} : A \in (\mathfrak{U} \cup \{\mathsf{J}\})'' \right\} = \mathfrak{U}_{\mathbf{i}}''$$

and so $\mathfrak{R}_{\mathbf{i}}$ is a complex RES, which concludes the proofs of (i) and (ii).

The operator J is Poincaré invariant because it commutes with every operator in $\mathfrak{R}$ and so in particular with U_g for any $g \in \mathcal{P}$. Hence $U_g\mathsf{J}U_g^{-1} = \mathsf{J}U_gU_g^{-1} = \mathsf{J}$. Finally, if P_0 is the infinitesimal generator of the unitary group of temporal translations in $\mathfrak{R}$ and $P_0 = J|P_0|$ is the polar decomposition of P_0, then $P_{0,\mathbf{i}} := P_0|_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}}$ is the infinitesimal generator of the unitary group of temporal translations in $\mathfrak{R}_{\mathbf{i}}$ and its polar

decomposition is given by $P_{0,\mathbf{i}} = J_{\mathbf{i}}|P_{0,\mathbf{i}}|$ with $J_{\mathbf{i}} := J|_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}}$ and $|P_{0,\mathbf{i}}| = |P_0||_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}}$. Since $\mathfrak{R}_{\mathbf{i}}$ is a complex RES, we have by Theorem 4.3 in [70] however that either

$$J_{\mathbf{i}} = \mathbf{i}\mathcal{I}_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}} = \mathsf{J}|_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}} \quad \text{or} \quad J_{\mathbf{i}} = -\mathbf{i}\mathcal{I}_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}} = -\mathsf{J}|_{\mathcal{H}^+_{\mathsf{J},\mathbf{i}}}$$

and so in turn $J = \mathsf{J}$ or $J = -\mathsf{J}$.

□

Remark 12.37. If we set $\tilde{H} := cP_0$, where c is the speed of light, then the polar decomposition of $\tilde{H}$ is $\tilde{H} = \mathsf{J}H$ with $H = |\tilde{H}| = c|P_0|$. As pointed on in [70, p. 34], the operator H can then be interpreted as a the energy operator, that is the Hamiltonian, of the system.

12.5 The Fundamental Logical Mistake in Quaternionic Quantum Mechanics

The above results suggest that quaternionic quantum mechanics is actually equivalent to classical complex quantum mechanics despite the fact that researchers in this field claimed the incompatibility of the two theories [1]. This misconception is caused by one fundamental logical mistake that was made in quaternionic quantum mechanics from its very beginning in the foundational paper [41]. It is the assumption that there exists a privileged left multiplication that is compatible with the physical theory.

We recall that the initial motivation for developing a quaternionic version of quantum mechanics was the fact that the propositional calculus of a quantum system consists of an orthomodular lattice, which can be realised as a lattice of subspaces on such Hilbert space [15]. A left multiplication is however not determined by calculus. Indeed, the argument in [41] for the existence of a privileged left multiplication is not correct. The authors argue that the physical properties of a quantum system should not depend on the concrete realisation of the number field that one uses so that the system should be invariant under automorphisms of the number field. If one considers a quantum system on a Hilbert space $\mathcal{H}$ over $\mathbb{F}$, where $\mathbb{F}$ is one of the fields $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and ϕ is an automorphism of the $\mathbb{F}$, then there exists a an associated co-unitary transformation of $\mathcal{H}$, that is an additive mapping $U_\phi : \mathcal{H} \to \mathcal{H}$ such that

$$U_\phi(\mathbf{v}a) = U_\phi(\mathbf{v})\phi(a) \qquad \text{and} \qquad \langle (U_\phi(\mathbf{v}), U_\phi(\mathbf{u})\rangle = \phi(\langle \mathbf{v}, \mathbf{u}\rangle). \tag{12.15}$$

All laws of the quantum system must be covariant under the transformation U_ϕ. Any automorphism ϕ of $\mathbb{H}$ is however of the form $\phi(x) = hxh^{-1}$ with $h \in \mathbb{H}$ and $|h| = 1$. The authors hence argue that the mapping $\mathbf{v} \mapsto U_\phi(\mathbf{v})h$ is then quaternionic right linear because

$$U_\phi(\mathbf{v}a)h = U_\phi(\mathbf{v})\phi(a)h = U_\phi(\mathbf{v})hah^{-1}h = U_\phi(\mathbf{v})ha$$

and hence they define a left multiplication on $\mathcal{H}$ via

$$h\mathbf{v} := U_\phi(\mathbf{v})h.$$

However, the co-unitary mapping U_ϕ is not well-defined. Any symmetry U defines via $\mathbf{v} \mapsto U\mathbf{v}h^{-1}$ a co-unitary mapping that satisfies (12.15) under which the laws of the system are covariant. If we choose the same symmetry U for any automorphism ϕ, then

$$h\mathbf{v} = U_\phi(\mathbf{v})h = U\mathbf{v}h^{-1}h = U\mathbf{v}$$

for any ϕ so that $h\mathbf{v}$ is independent of h. We could even choose $U = \mathcal{I}$, so that we define $h\mathbf{v} = \mathbf{v}$, which is obviously nonsense.

We conclude this chapter with an examples of a seeming inconsistency between the complex and the quaternionic theory that arises from the assumption of the existence of a left multiplication. This inconsistency is however resolved if only the existence of a compatible unitary and anti-selfadjoint operator J that commutes with any observable and the unitary group of time translations is assumed. We point out that also other discrepancies such as the difficulty of defining a proper momentum operator, showing Heisenberg's uncertainty principle or developing a framework for composite systems can be resolved if the existence of such operator J is assumed. In order to explain the discrepancy we want to resolve, we quickly recall the main concepts of quaternionic quantum mechanics introduced in [1]. We shall furthermore adopt the Bra-Ket-notation used by physicists in order to make the comparison for the reader easier.

Quaternionic quantum mechanics is in [1] formulated on an abstract Hilbert space $\mathcal{H}$ over the quaternions $\mathbb{H} = \{x_0 + \mathsf{i}x_1 + \mathsf{j}x_2 + \mathsf{k}x_3 : x_\ell \in \mathbb{R}\}$ consisting of ket-vectors $|f\rangle$, where the same vector considered as an element of the dual space via the Riesz-representation theorem is denoted by the bra-vector $\langle f|$. (The dimension of $\mathcal{H}$ is assumed to be greater than two in order for Wigner's theorem to hold.) The scalar product on $\mathcal{H}$ is denoted by $\langle g|f\rangle$ and (pure) states of the system correspond to one-dimensional rays of the form $|f\omega\rangle$ with $\omega \in \mathbb{H}$. Observables are self-adjoint operators A on $\mathcal{H}$, that is $A^\dagger = A$ where $A^\dagger$ denotes the adjoint. Any observable A has a representation of the form $A = \sum |a\rangle a\langle a|$ in terms of an orthonormal eigenbasis of eigenvector $|a\rangle$ associated with eigenvalues a and the expectation value of measuring the observable A if the system is in the state $|f\rangle$ is given by $\langle f|A|a\rangle = \sum_a \langle f|a\rangle a \langle a|f\rangle = \sum a|\langle a|f\rangle|^2$. Symmetries of the systems are unitary operators on $\mathcal{H}$. A continuous one-parameter group of symmetries is of the form $U(t) = e^{tA}$, where A is an anti-selfadjoint operator satisfying $A^\dagger = -A$. The eigenvalues of such A are (equivalence classes $[a]$ of) purely imaginary quaternions and the eigenspaces associated with different eigenvalues are mutually orthogonal. Choosing for any eigenvalue the representative a that belongs to the upper complex halfplane $\mathbb{C}_{\mathsf{i}}^{\geq}$, we can find a representation of A of the form

$$A = \sum |a\rangle a\langle a| = \sum |a\rangle \mathsf{i}|a|\langle a|.$$

We can furthermore set $\mathsf{I}_A := \sum |a\rangle\mathsf{i}\langle a|$ and $|A| = \sum |a\rangle|a|\langle a|$ and find $A = \mathsf{I}_A|A|$, which corresponds to the polar decomposition of A. If we set $\mathsf{J}_A := \sum |a\rangle\mathsf{j}\langle a|$ and $\mathsf{K}_A := \sum |a\rangle\mathsf{k}\langle a|$, then we obtain a left multiplication that commutes with $|A|$ and we can write any state $|f\rangle$ in terms of its four real components

$$|f\rangle = |f_0\rangle + \mathsf{I}_A|f_1\rangle + \mathsf{J}_A|f_2\rangle + \mathsf{K}_A|f_3\rangle = |f_0\rangle + |f_1\rangle\mathsf{i} + |f_2\rangle\mathsf{j} + |f_3\rangle\mathsf{k}.$$

(Note however that only I_A is determined by A, the operators J_A and K_A depend on the eigenbasis of A that we choose.)

We consider the position operator X that has a (continuous) eigenbasis $|x\rangle$ such that $X|x\rangle = |x\rangle x$ on $\mathcal{H}$ and such that

$$\mathcal{I} = \int dx^3\, |x\rangle\langle x|.$$

Adler uses this position operator in order to define the quaternion-valued wave function associated with a state $|f\rangle$ as

$$f(x) = \langle x|f\rangle$$

and finds that

$$\langle g, f\rangle = \int dx^3 \,\langle g|x\rangle\langle x|f\rangle = \int dx^3\, \overline{g(x)} f(x).$$

We assume further more the left multiplication on $\mathcal{H}$ to be the left multiplication induced by

$$\mathsf{I} := \mathsf{I}_x = \int |x\rangle \mathbf{i} \langle x|, \quad \mathsf{J} := \mathsf{J}_x = \int |x\rangle \mathbf{j} \langle x|, \quad \text{and} \quad \mathsf{K} := \mathsf{K}_x = \int |x\rangle \mathbf{k} \langle x| \tag{12.16}$$

and find that this corresponds to the natural pointwise multiplication of the wave function, that is $(af)(x) = \langle x|af\rangle = a(f(x))$ for all $a \in \mathbb{H}$. We once more stress that this choice is however made by the author because it is convenient when working with wave functions, but it is not determined by the physical system and hence does not carry physical information.

The time evolution of the system is described by symmetries $U(t, \delta t)$ mapping the state of the system at time t to the state of the system at time $t + \delta t$. We define $-H$ to be the first coefficient of the Taylor series expansion of $U(t, \delta t) = \mathcal{I} + \delta t(-H)$, that is

$$U(t, \delta t)|f(t)\rangle = \mathcal{I}|f(t) - H|f(t)\rangle + o(\delta t^2)$$

Together with

$$|f(t + \delta t)\rangle = |f(t)\rangle + \delta t \frac{\partial}{\partial t}|f(t)\rangle + o(\delta t^2),$$

we find the dynamics of the system being described by the Schrödinger equation

$$\frac{\partial}{\partial t}|f(t)\rangle = -H(t)|f(t)\rangle. \tag{12.17}$$

An equation of this type is however only possible as long as we stick with a certain representative of the ray that describes the physical state. If we consider a general ray representative $|f(t)\omega_f(t)\rangle$ with a quaternionic phase $\omega_f(t) \in \mathbb{H}$ and $|\omega_f(t)| = 1$, then the corresponding dynamical equation is

$$\frac{\partial}{\partial t}|f(t)\omega_f(t)\rangle = -H(t)|f(t)\omega_f(t)\rangle + |f(t)\omega_f(t)\rangle h_f(t) \tag{12.18}$$

with

$$h_f(t) = \omega_f(t)\omega_f'(t) = \overline{\omega_f(t)}\omega_f'(t).$$

If we differentiate $\overline{\omega_f(t)}\omega_f(t) = 1$, we find that $h_f(t) = \overline{\omega_f(t)}\omega_f'(t) = -\overline{\omega_f'(t)}\omega_f(t) = -\overline{h_f(t)}$ and hence $h_f(t)$ is a purely imaginary quaternion. In contrast to the complex case, it can hence not be commuted with $\omega_f(t)$ in order to integrate it into a modified Hamiltonian and in order to obtain again an equation of the form (12.17).

Adler finally argues in [1, pp. 41] that complex and quaternionic quantum mechanics are inequivalent because they have different transition probabilities and vectors that represent the same state in quaternionic quantum mechanics can be orthogonal and

hence represent different states in complex quantum mechanics. He writes the wave function

$$f(x) = \langle x, f\rangle = f_0(x) + f_1(x)\mathbf{i} + f_2(x)\mathbf{j} + f_3(x)\mathbf{k}$$

of a state $|f(x)\rangle$ in position coordinates, in terms of two symplectic components as

$$f(x) = F_1(x) + \mathbf{j}F_2(x)$$

with

$$F_1(x) = f_0(x) + f_1(x) \qquad F_2(x) = f_2(x) - f_3(x)\mathbf{i}.$$

He then assumes the Hamiltonian H to be written in terms of real components

$$H = H_0 + H_1\mathsf{I} + H_2\mathsf{J} + H_3\mathsf{K} = H_0 + \mathbf{i}H_1 + \mathbf{j}H_2 + \mathbf{k}H_3.$$

(Note that this implicitly assumes that H is the quaternionic linear extension of an $\mathbb{R}$-linear operator from the real component space to $\mathcal{H}$.) The coordinate representation of H in the position basis is then

$$H_\ell(x)\langle x| = \langle x|H_\ell \qquad \ell \in \{0, \ldots, 3\}.$$

If we define $\mathcal{H}_1(x) \ = H_0(x) + \mathbf{i}H_1(x)$ and $\mathcal{H}_2(x) := H_2(x) - \mathbf{i}H_3(x)$ and the matrix-valued function

$$\mathcal{H}(x) := \begin{pmatrix} \mathcal{H}_1(x) & -\overline{\mathcal{H}_2(x)} \\ \mathcal{H}_2(x) & \overline{\mathcal{H}_1(x)} \end{pmatrix},$$

then the Schrödinger equation can be rewritten as

$$\frac{\partial}{\partial t}\begin{pmatrix} F_1(x,t) \\ F_2(x,t) \end{pmatrix} = \mathcal{H}(x)\begin{pmatrix} F_1(x,t) \\ F_2(x,t) \end{pmatrix}.$$

Endowing the space of all component functions with the usual scalar product on the direct sum $L^2(\mathbb{C}_\mathbf{i}) \oplus L^2(\mathbb{C}_\mathbf{i})$, namely

$$\langle f(x), g(x)\rangle_\mathbb{C} := \int \overline{F_1(x)}G_1(x) + \overline{F_2(x)}G_2(x)\,dx^3, \tag{12.19}$$

we find that this describes a complex quantum system since the only imaginary unit that appears in the above equations is the unit $\mathbf{i}$. The units $\mathbf{j}$ and $\mathbf{k}$ disappeared.

Adler argues now that this system is inequivalent to the original quaternionic linear system. The quaternionic scalar product of the states $|f\rangle$ and $|g\rangle$ resp. of their wave functions $f(x)$ and $g(x)$ is

$$\langle f(x), g(x)\rangle = \int \overline{f(x)}g(x)\,dx^3 = \langle f(x), g(x)\rangle_\mathbb{C} \ + \mathbf{j}\langle f(x), g(x)\rangle_\mathcal{S},$$

where $\langle f(x), g(x)\rangle_\mathbb{C}$ is as in (12.19) and $\langle f(x), g(x)\rangle_\mathcal{S}$ is the symplectic scalar product

$$\langle f(x), g(x)\rangle_\mathcal{S} := \int F_1(x)G_2(x) - F_2(x)G_1(x)\,dx^3.$$

Therefore the transition probabilities are

$$|\langle f(x), g(x)\rangle_\mathbb{C}|^2$$

in the complex, but

$$|\langle f(x), g(x)\rangle|^2 = |\langle f(x), g(x)\rangle_{\mathbb{C}}|^2 + |\langle f(x), g(x)\rangle_{\mathcal{S}}|^2$$

in the quaternionic system. Furthermore, an orthonormal basis $|h_\ell\rangle$ of the quaternionic system is not complete in the complex one. Instead one has to extend it and consider the set $|h_\ell\rangle, |h_\ell \mathbf{j}\rangle$ in order to obtain an eigenbasis of the complex system. In particular, $|h_\ell\rangle$ and $|h_\ell \mathbf{j}\rangle$ belong to the same ray and hence they represent the same state in the quaternionic system. However, they are orthogonal in the complex system and hence represent different states (in general even associated with different eigenvalues) in the complex system.

Adler was obviously right, these two systems are not equivalent. However, as we know from our preceding analysis, he compared the wrong systems. The quaternionic system cannot be equivalent to the system that we obtain if we consider the entire quaternionic Hilbert space $\mathcal{H}$ as the complex system by simply taking the $\mathbb{C}_{\mathbf{i}}$-linear part of the quaternionic scalar product. As pointed out by Adler, this introduces new orthogonality relations so that the vectors $|h_\ell\rangle$ and $|h_\ell \mathbf{j}\rangle$, which describe the same state in the quaternionic system, correspond to different states in the complex one. The phase space $\mathcal{H}_{\mathbb{C}}$ of a complex system that is equivalent to a quaternionic one can hence not be the entire quaternionic Hilbert space $\mathcal{H}$. It must be a subspace that does not contain $|h_\ell \mathbf{j}\rangle$ whenever $|h_\ell\rangle \in \mathcal{H}_{\mathbb{C}}$. The natural candidate for such space consists of all vectors that have complex wave functions such that $F_2(x) \equiv 0$. It is however not clear whether any state has a representative in this space, whether it is invariant under time translations etc. The reason for this is that the left multiplication (12.16), which determines the component functions of the wave function and in turn this set of vectors, is not motivated by physical arguments but by the fact that it is convenient to work with.

Instead we have to consider a complex system on the complex subspace

$$\mathcal{H}^+_{\mathsf{J}_H,\mathbf{i}} := \{\mathbf{v} \in \mathcal{H}_H : \mathsf{J}_H \mathbf{v} = \mathbf{v}\mathbf{i}\},$$

where J_H is the unitary anti-selfadjoint operator that appears in the polar decomposition of the anti-selfadjoint Hamiltonian H. From our previous discussion we already know that this system is equivalent to the system on the entire quaternionic space if J_H commutes with any observable. The scalar product of any two vectors in this subspace is moreover naturally complex, so that we do not have to remove information by discarding the symplectic part in the quaternionic scalar product and hence no additional orthogonality relations are introduced.

We conclude by observing that quaternionic quantum mechanics as developed in [1] is not actually a quaternionic theory. A proper quaternionic theory would consider equivalence classes of eigenvalues—spectral spheres as they are determined by the S-eigenvalue operator—and then admit arbitrary vectors in the associated eigenspaces to represent states of the system. In particular it would admit arbitrary quaternionic phases in (12.18). The current version of quaternionic quantum mechanics however chooses eigenvectors associated with eigenvalues in one fixed complex plane $\mathbb{C}_{\mathbf{i}}$ and then only works with these vectors. (This was of course also done because a proper theory of quaternionic linear operators and in particular the definition of the S-spectrum were not known when the theory was developed.) From our perspective, this does however correspond to considering the quaternionic system actually as a complex system over the complex field $\mathbb{C}_{\mathbf{i}}$ and unconsciously working only in the Hilbert space $\mathcal{H}^+_{\mathsf{J}_H,\mathbf{i}}$.

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